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HIGH REYNOLDS NUMBER FLOW BETWEEN
ROTATING COAXIAL DISKS; A NUMERICAL EXPERIMENT

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ABSTRACT

A pair of nonlinear ordinary differential equations that arise in the study of fluid flow between rotating coaxial disks is studied numerically. Collocation at Gaussian points is used to find approximate solutions to these equations for different values of the ratio of the angular velocity of one disk to the other. Recently J. B. McLeod and S. V. Parter have proved rigorous analytic results for the case when the disks rotate with equal and opposite angular velocity. A comparison between their analytic results and the numerical results is presented for this case.

1. Introduction

The study of fluid flow between rotating coaxial disks has received attention from many authors for over fifty years. Von Kármán [9] considered the steady flow above an infinitely large rotating disk under the assumption that axial velocity was both angle and radius independent. Batchelor [1] extended von Kármán's ideas to the case of two coaxial disks and gave a qualitative picture of the flow in the case of infinite Reynolds number. Stewartson [19] also considered the case of two coaxial disks and reached conclusions different from Batchelor's for high Reynolds number flow. Mathematical and numerical studies using various techniques have been performed by Rodgers and Lance [10,16], Pearson [13], Mellor et al. [12], Greenspan [6], and Schultz and Greenspan [18]. Rasmussen [15] has used the method of matched asymptotic expansions to obtain analytic approximations to flows at high Reynolds number under the assumption that boundary layers are developed only on the disks and the effects of viscosity appear only in these layers.

Our purpose in this report is to describe a numerical study of the steady motion of a viscous incompressible fluid between two infinite rotating coaxial disks at high Reynolds number. As in Greenspan [6], we let the first disk be centered at $(0,0,0)$ in (x,y,z) space rotating in the plane $x=0$ with with angular velocity Ω_1. The second disk is centered at $(1,0,0)$ and rotates in the plane $x=1$ with angular velocity Ω_2. The dimensionless, steady state Navier-Stokes equations become (see e.g. H. P. Greenspan [7], Pearson [13]

(1.1) $\quad G'' + R(GH'-G'H) = 0 \qquad 0 \le x \le 1$

(1.2) $\quad H'''' - R(HH'''+4GG') = 0 \qquad 0 \le x \le 1$

with boundary conditions

(1.3) $\quad G(0) = \Omega_1 \qquad G(1) = \Omega_2$

(1.4) $\quad H(0) = H'(0) = 0 \qquad H(1) = H'(1) = 0$

where differentiation is with respect to x. If the cylindrical coordinates of (x,y,z) are (x,r,θ) and (V_x,V_r,V_θ) is the fluid velocity vector then

(1.5) $\quad V_x = H(x) \qquad V_r = -(1/2)rH'(x) \qquad V_\theta = rG(x)$.

Collocation at Gaussian points with piecewise polynomial functions was used to find approximate solutions to the system (1.1)-(1.4). In [4], de Boor and Swartz give theoretical justification for this method to approximate an isolated solution of a single nonlinear ordinary differential equation. Cerutti [2] gives an extension to systems of ordinary differential equations. Theoretical results for the nonlinear system (1.1) - (1.4) are limited, hence [2] and [4] provide only partial justification for using the collocation method for this problem. De Boor has considered the problem of effectively computing with piecewise polynomial functions and gives (see [3]) a package of FORTRAN subprograms for calculating with B-splines. In particular, example 5 of [2] is the basis for the FORTRAN program he wrote (and we used and modified) to find approximate solutions of (1.1) - (1.4).

ACKNOWLEDGEMENT

We are indebted to Professor Carl de Boor. His suggestions concerning these computations were very helpful.

2. Numerical Method

We seek two functions $G(x)$, $H(x)$ which are piecewise polynomials of order six (degree less than or equal to five) with breakpoints .05, .1, .2, .3, .4, .5, .6, .7, .8, .9, .95, satisfy the boundary conditions (1.3), (1.4), and satisfy the differential equations (1.1), (1.2) at the "local Gauss points". A discussion of this collocation method is found in de Boor and Swartz [4]. In order to solve the nonlinear problem we proceed as follows: we apply Newton's method to the nonlinear system (1.1)-(1.4) and then use collocation at the local Gauss points to obtain approximations to the solutions of these linear equations. That is, given H_0, G_0 (which do not necessarily satisfy the boundary conditions (1.3), (1.4)) we generate H_1, G_1 by solving the linear system obtained by requiring that the following equations

$$H''''_{n+1} - (RH_n)H'''_{n+1} - (RH'''_n)H_{n+1} = R(4G_nG'_n - H_nH'''_n) + 4R\{(G_{n+1}-G_n)G'_n + (G_{n+1}-G_n)'G_n\} \tag{2.1}$$

$$G''_{n+1} - (RH_n)G'_{n+1} + (RH'_n)G_{n+1} = R\{(H_{n+1}-H_n)G'_n - (H'_{n+1}-H'_n)G_n\} \tag{2.2}$$

$$H_{n+1}(0) = H'_{n+1}(0) = 0 \quad H_{n+1}(1) = H'_{n+1}(1) = 0 \tag{2.3}$$

$$G_{n+1}(0) = \Omega_1 \quad G_{n+1}(1) = \Omega_2 , \tag{2.4}$$

be satisfied at the local Gauss points. Note that the boundary conditions (2.3), (2.4) are satisfied for iterates H_n, G_n with $n \geq 1$.

In order to facilitate the comparison of solutions for different values of Ω_1, Ω_2 we fix

$$R = 1000$$

unless another value is specifically indicated. All computations were carried out in double precision (~18 decimal digits) on the Univac 1108 and Univac 1110 at the University of Wisconsin.

3. Computational Results

Because of an inconsistency in the results of Greenspan [6] (as shown in [11]) in the case of counterrotating disks ($\Omega_1=1$, $\Omega_2=-1$) the Fortran program of A. Schubert (see Appendix [17]) was converted to double precision. Using $R=1000$ an effort was made to clarify this discrepancy. The results of [6] were used as starting values for an extended run, but the computation did not converge. All other computations reported on hereafter were carried out with the de Boor program as described in Section 2.

In the case $\Omega_1=1$, $\Omega_2=-1$ we obtain figure 1. This result agrees with the computations done by Jones [8] and by Schultz and Greenspan [18]. Jones [8] also gives two computational solutions for the case $\Omega_1=1$, $\Omega_2=0$ and our results (see figures 2, 3) agree with his.

In the case $\Omega_1=\Omega_2=1$ we obtain (in addition to the trivial solution $H\equiv 0$, $G\equiv 1$) three other solutions (see figures 4, 5). With these boundary conditions if $H(x)$, $G(x)$ is a solution then so is $-H(1-x)$ and $G(1-x)$; hence the non-symmetrical solution (figure 4) represents two computational solutions. The theoretical results known for this case do not lead one to expect solutions of the form given in figures 4 and 5. In particular, the approximate solutions represented in figure 5 are surprising and certainly require further study. It is known, however, that the trivial (rigid motion) solution is not a bifurcation point (see appendix 1).

In order to gain more insight into the number and type of solutions for $R=1000$ and to see how the computational solutions above might be "connected", Ω_2 was varied while Ω_1 was fixed (see figures 6, 7). The computational solution obtained in figure 1 (counterrotating case) was used as a starting point and $G(1)=\Omega_2$ was incremented to $-.1$ by steps of $.1$. At each step the previous computational solution was

used as the initial value in the Newton iteration. At each step convergence was obtained in less than 5 iterations but the step from $G(1)=-.2$ to $G(1)=-.1$ did not converge after 31 iterations. These results are described in figure 6.

In figure 7 we describe the results obtained by starting with the case $\Omega_2=\Omega_1=1$ and $H\equiv 0$, $G\equiv 1$. $G(1)$ was decremented by steps of $.1$ to 0. In each case less than 5 Newton iterations were necessary to obtain convergence, except in the step from $G(1)=.1$ to $G(1)=0$ where the computation failed to converge after 25 iterations. When the computation failed to converge after ~30 iterations another parameter (e.g. $G'(0)$ rather than $G(1)=\Omega_2$) was selected and varied. Using this technique the graph of $G(1)$ versus $G'(0)$ was obtained (see figure 8 and a detail of figure 8 in figure 9). Figure 10 shows a plot of $G(1)$ versus $G'(1)$ for the same computational solutions.

4. Analytical Comparisons

Only recently (see Mc Leod and Parter [11]) have rigorous analytic results been proved for the system (1.1)-(1.4) with large Reynolds number. These results are confined to the case of counter-rotating disks ($\Omega_1=1$, $\Omega_2=-1$) and show the existence of a solution in which G is monotone. The computational result of figure 1 is in qualitative agreement with the analytical results of [11]. Mc Leod and Parter [11] also give the behavior of H and G as $R \uparrow \infty$. In order to compare our computations with some of those results we computed solutions for $R = 62.5, 125, 250, 500, 1000, 2000, 4000$ for the case $\Omega_1=1$, $\Omega_2=-1$. If x_1 denotes the zero of H in $(0,1/2)$ and $\varepsilon=1/R$ then $x_1 = O(\varepsilon^{1/2}\ln\varepsilon)$ (Theorem II of [11]). In particular, $x_1 \to 0$ slower than $\varepsilon^{1/2}$ and this behavior is seen in table 1.

TABLE 1

R	x_1	ratio $(x_1)_{R/2}/(x_1)_R$	$(x_1)_R/\varepsilon^{1/2}$ ($\varepsilon=1/R$)
62.5	.260		2.05
125.	.240	.92	2.69
250.	.212	.88	3.34
500	.176	.83	3.93
1000	.140	.80	4.43
2000	.106	.76	4.73
4000	.0764	.72	4.83

It is also shown in [11] that $-H''(0) = O(\varepsilon^{-1/2})$ and $\varepsilon^{-1/2} = O(-H''(0))$. The ratio $H''(0)$ at R divided by $H''(0)$ at $2R$ shown in Table 2 is in approximate agreement with this behavior.

TABLE 2

R	$-H''(0)$	ratio
62.5	71.1	
125	113.	1.59
250	164.	1.45
500	231.	1.41
1000	326.	1.41
2000	470.	1.44
4000	709.	1.51

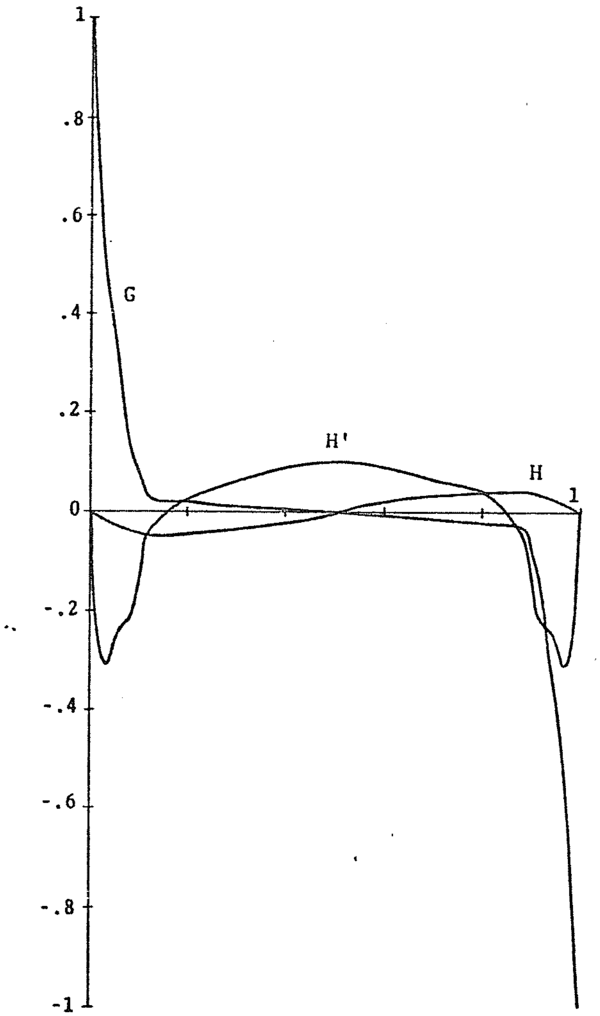


Figure 1

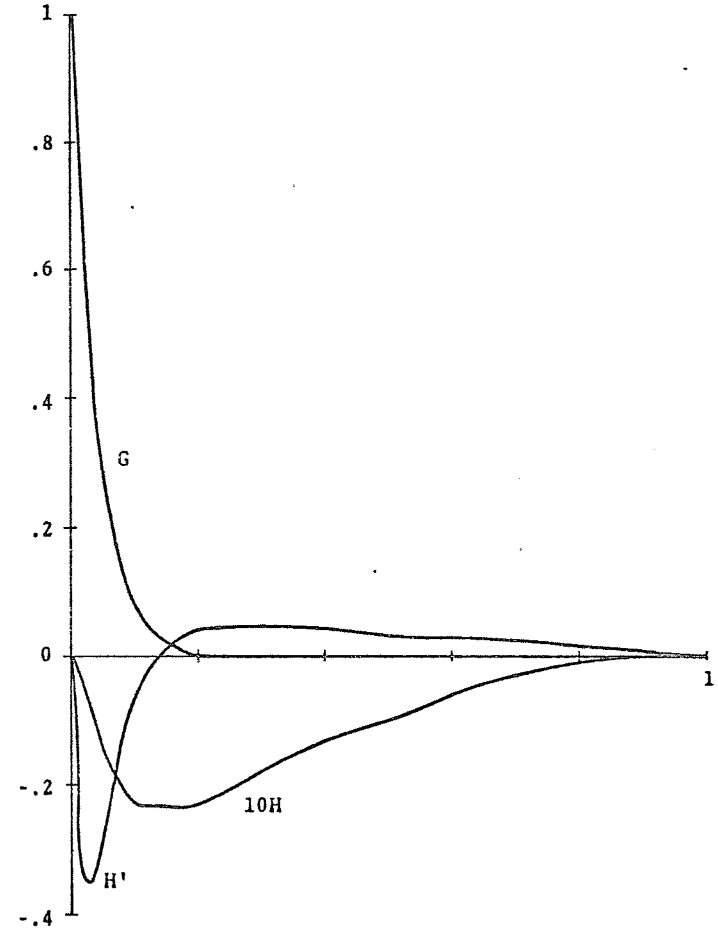
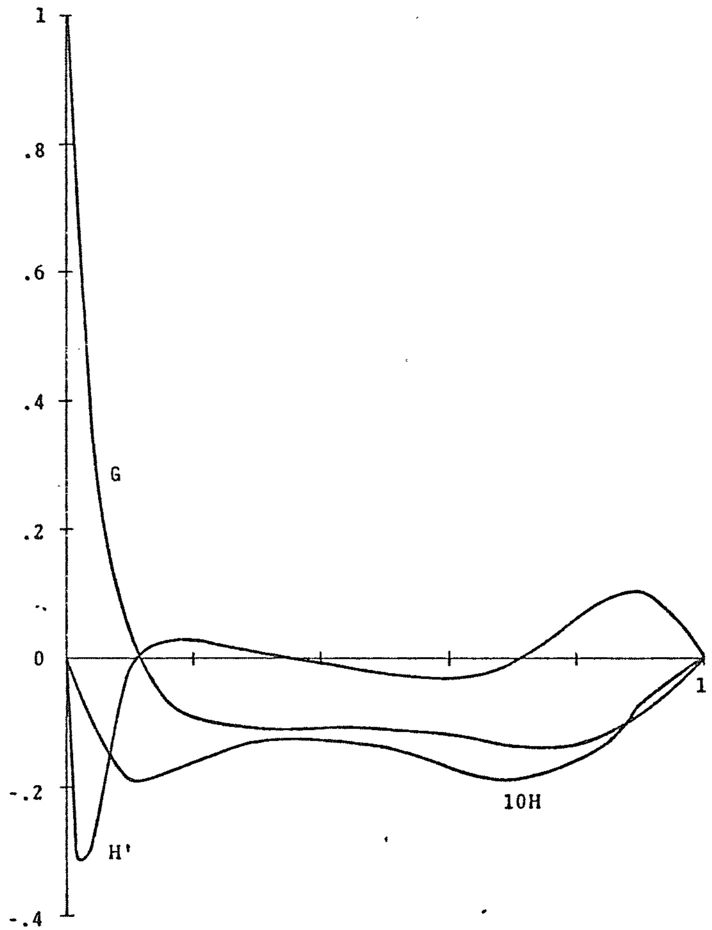


Figure 2



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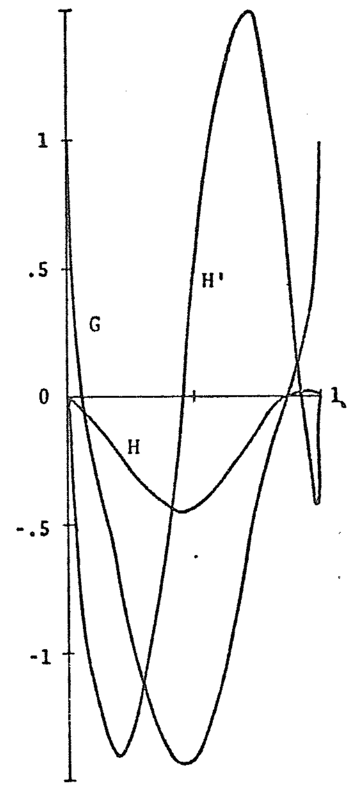

Figure 4

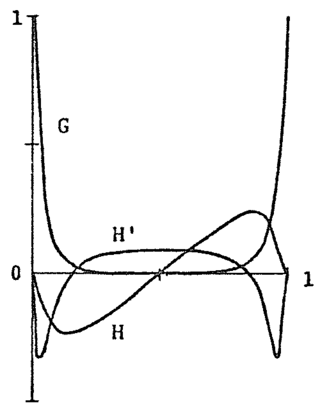

Figure 5

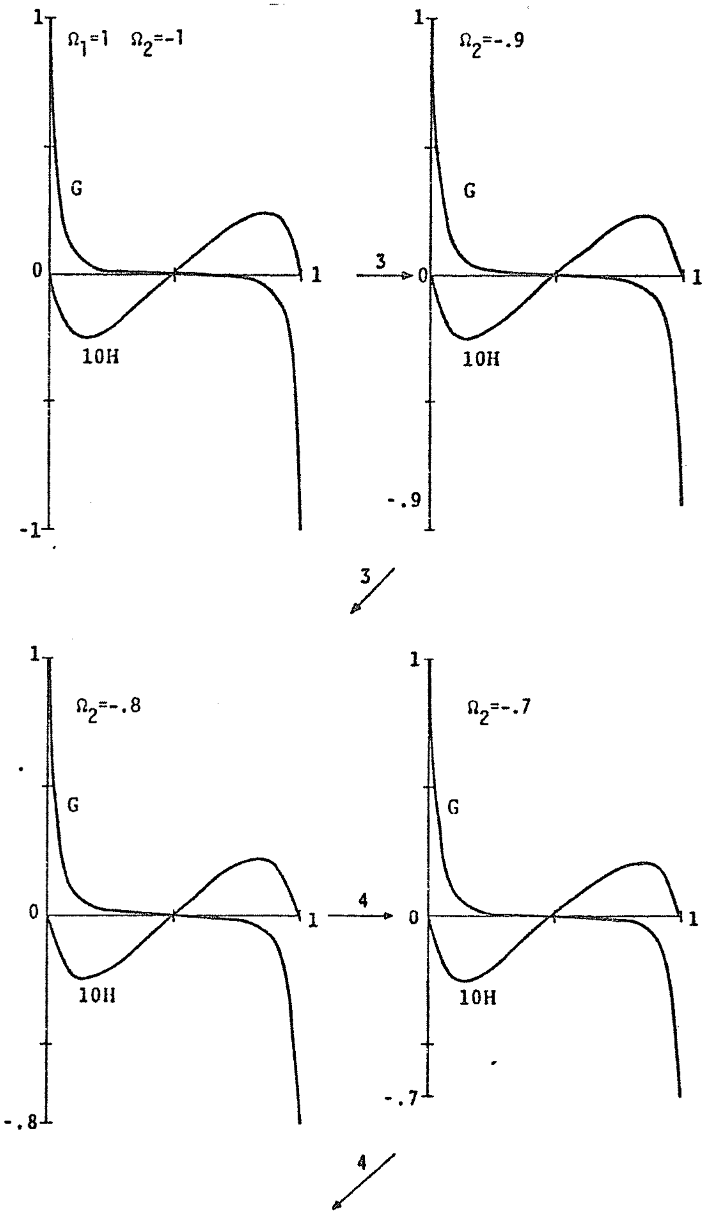

Figure 6a

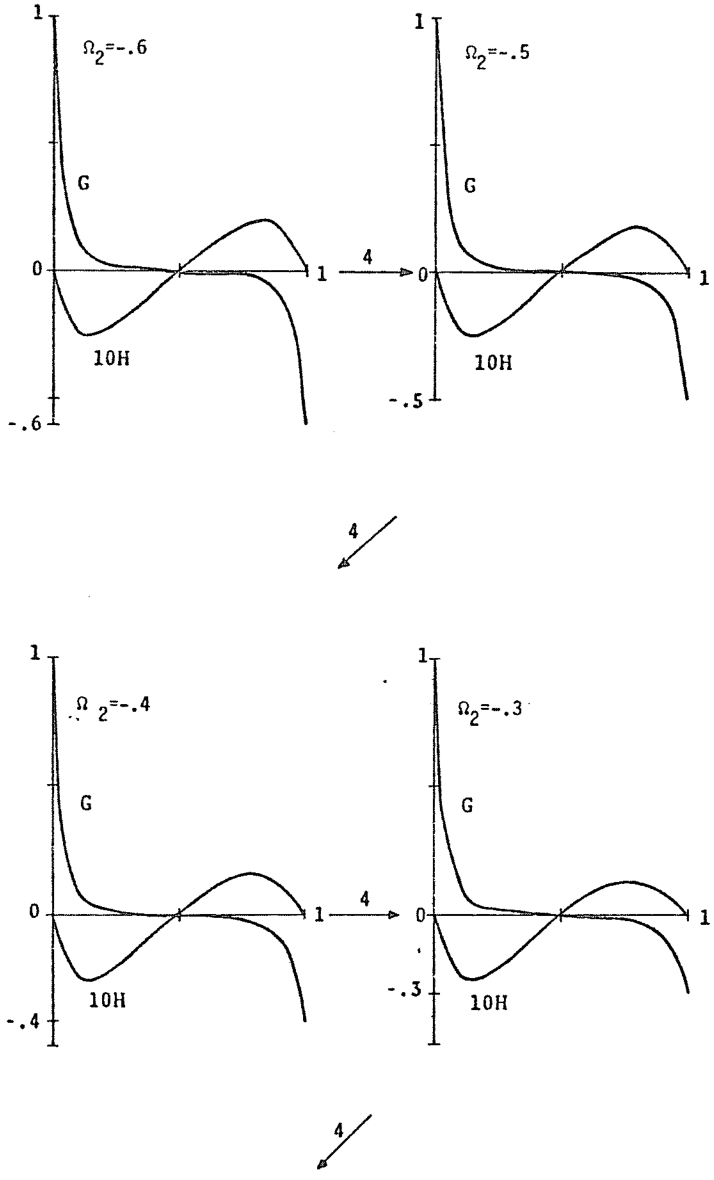

Figure 6b

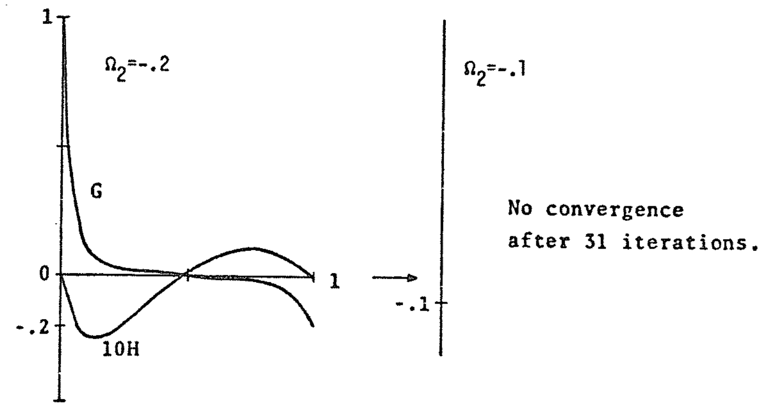

Figure 6c

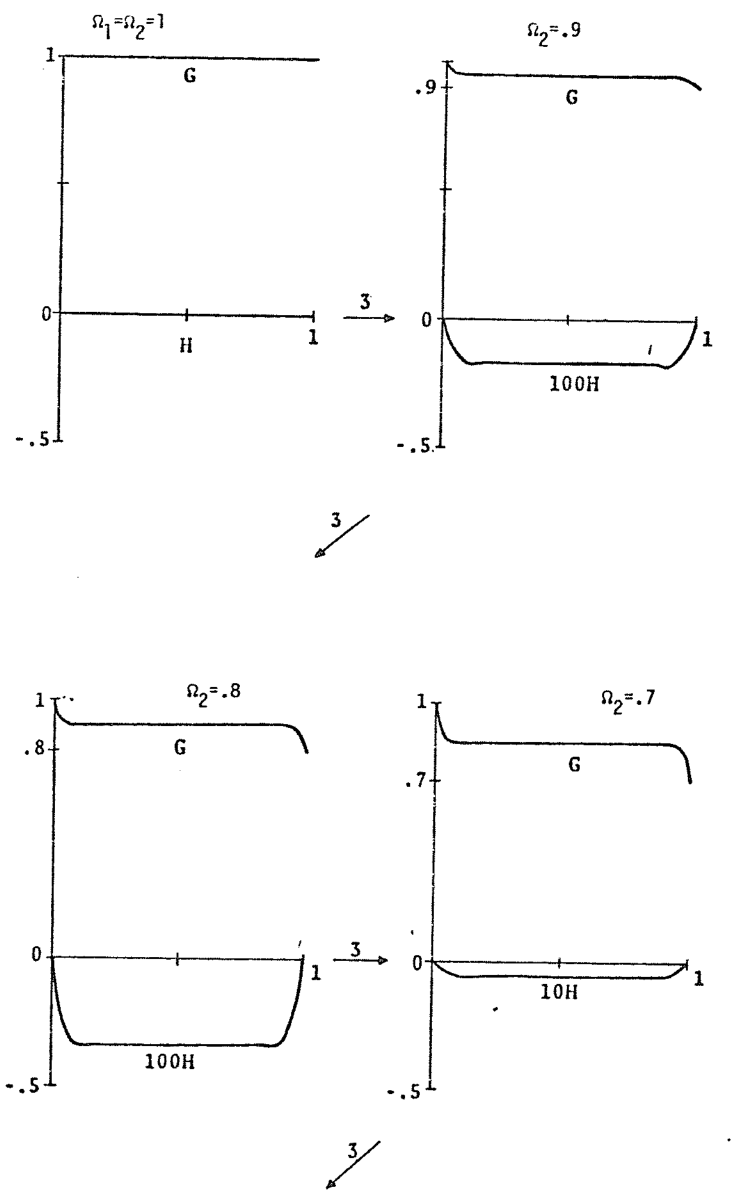

Figure 7a

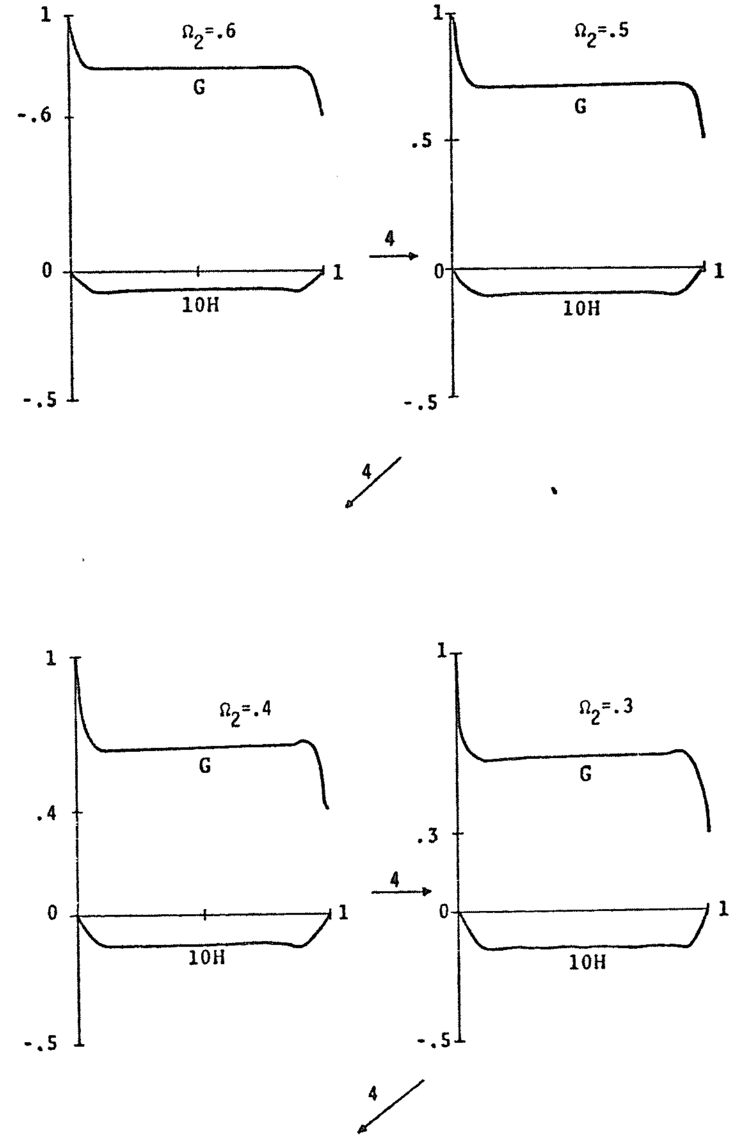

Figure 7b

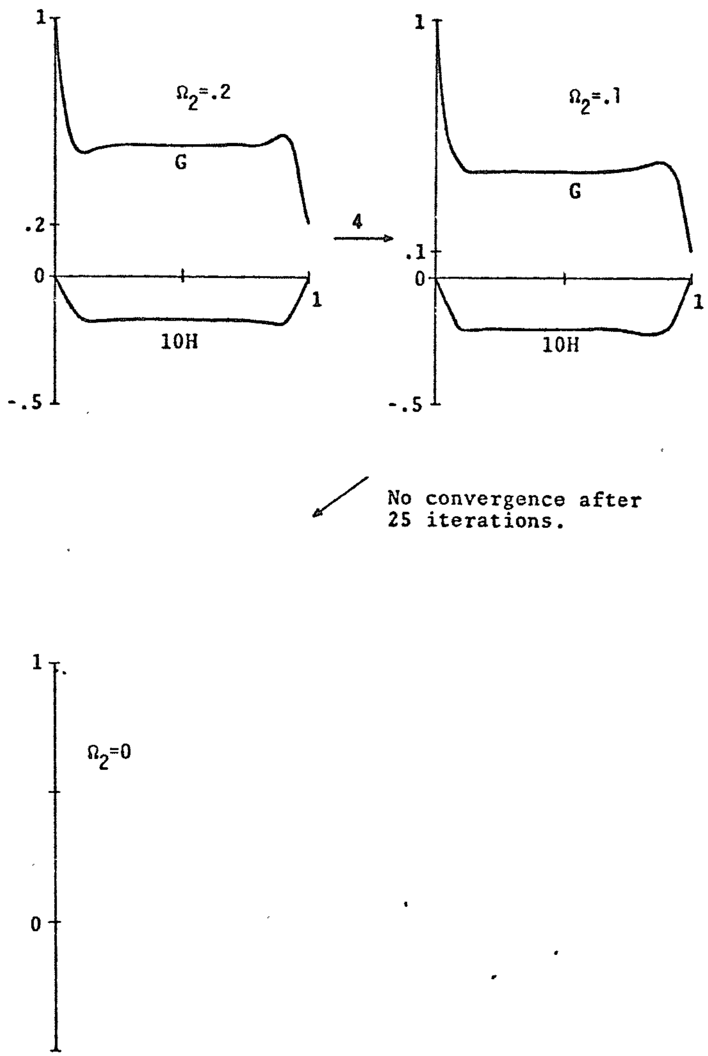

Figure 7c

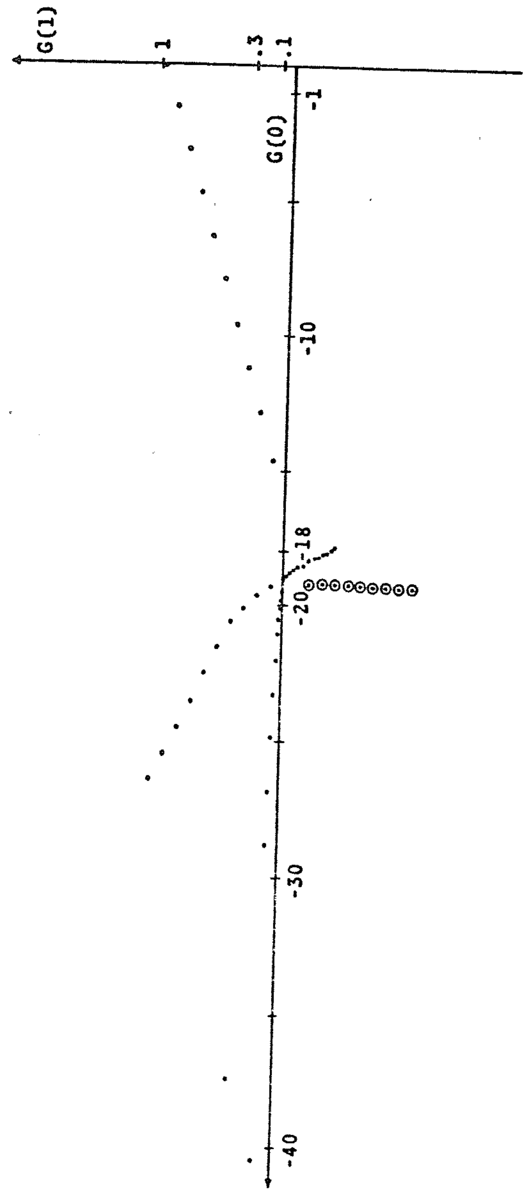

Figure 8

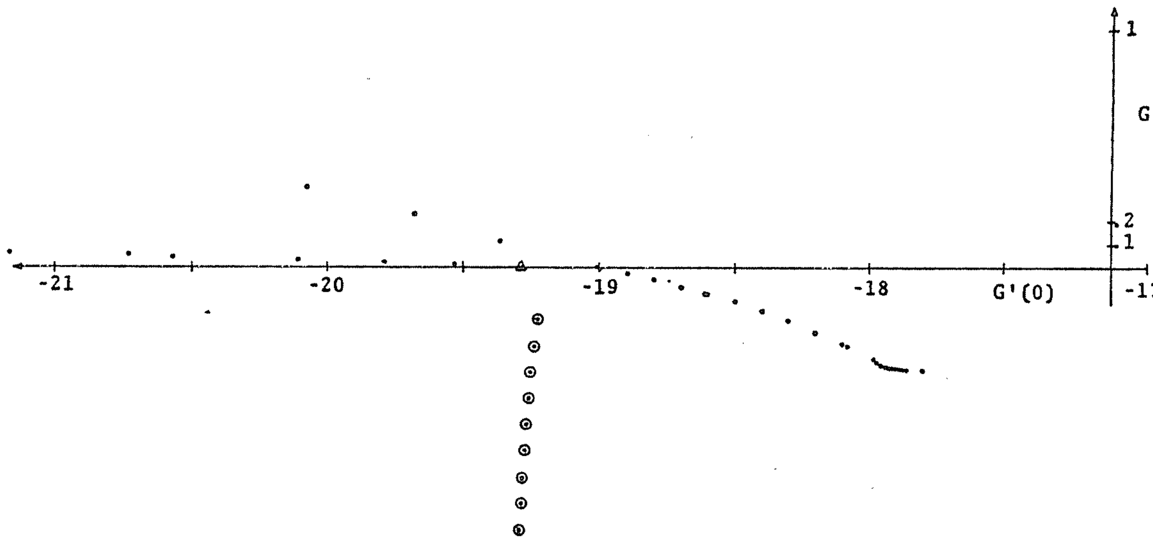

Figure 9

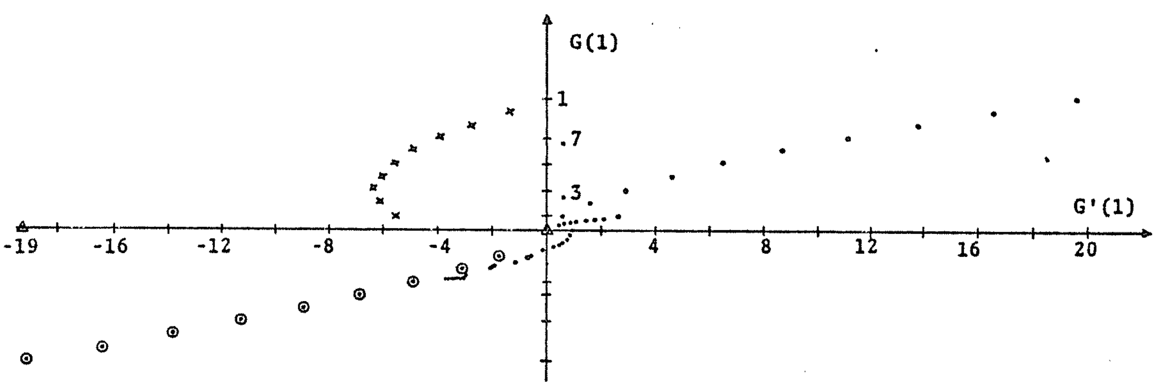

Figure 10

APPENDIX 1

To see that the trivial solution $H \equiv 0$, $G \equiv 1$ of (1.1)-(1.4) in the case $\Omega_1 = \Omega_2 = 1$ is locally unique we show that the derivative is nonsingular at this point. The derivative at this point is

$$(1)\ \phi'''' - 4R\chi' = 0 \qquad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0\ ,$$

$$(2)\ \chi'' + R\phi' = 0 \qquad \chi(0) = \chi(1) = 1\ .$$

Replacing χ by $\Psi+1$ gives rise to (2') with zero boundary conditions and we have

$$(1')\quad \phi'''' - 4R\Psi' = 0 \qquad \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0,$$

$$(2')\quad \Psi'' + R\phi' = 0 \qquad \Psi(0) = \Psi(1) = 0.$$

Multiplying (1') by ϕ and (2') by 4Ψ and subtracting shows

$$\int_0^1 \phi\phi'''' - 4R\phi\Psi' - 4\Psi\Psi'' - 4R\Psi\phi' = 0\ .$$

Integrating by parts, using the zero boundary conditions and noting

$$\int_0^1 - 4R[\phi\Psi'+\Psi\phi'] = \int_0^1 -4R(\phi\Psi)' = -4R\phi\Psi\Big|_0^1 = 0$$

we see

$$\int_0^1 \phi''^2 + 4\Psi'^2 = 0\ .$$

Hence $\phi \equiv \Psi \equiv \chi + 1 \equiv 0$ and the trivial solution is not a bifurcation point.

Elcrat [5] has shown that if Ω_1 and Ω_2 are sufficiently close then solutions exist for large Reynold's number. His results show that if Ω_1 and Ω_2 are of the same sign and $|\Omega_1{}^2 - \Omega_2^2| \le c^2/4R$ (where c is a constant, $c \sim 1.5$) then (1.1) - (1.4) has a solution. This result is a perturbation about the trivial (rigid motion) solution.

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