STATISTICS OF TREES

by

John H. Halton

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DEDICATION

PATRICK D. HALTON celebrated his hundredth year on the 11th of August, 1978. His patience, understanding, support, encouragement, and love have been a constant inspiration to me; and his unquestioning faith and trust in me have allowed me to persevere and survive through times of doubt and discouragement. He has long been fascinated by trees of all kinds, and it is therefore particularly fitting that this paper is, hereby, humbly dedicated to him, with all my most heartfelt love and admiration.

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1. INTRODUCTION

We consider the *storage* of a sequence of unpredetermined length, of data which are susceptible to a *linear ordering*, the storage procedure being designed to facilitate the efficient *retrieval* of arbitrary, specified data. We may think of the data as *real numbers*, ordered by <; say, $c_1 < c_2 < \dots < c_m$. We shall limit ourselves to storage procedures which depend only on the ordering, not on the spacing, of the *keys* c_i. When considering the *statistics* of such data, it is then most reasonable to assume that every permutation of the m values is equally probable as an input to our storage algorithm. As a consequence, we note that, if $(m - 1)$ data have already been stored and we consider the statistics of the position of an m-th datum, this will have equal probability of being in any of the m intervals into which the previous data divide the real line.

The most straightforward algorithm simply stores the data sequentially as they are received; so that the work of storing the data will be proportional to their number, m; while the work of retrieving a specified datum (e. g., to find a key c or the datum whose key is closest to a given value d) will also be proportional to m, since every datum must be examined, in general (depending on the search-criterion, it may be sufficient to find one suitable datum: then the average length of search will be proportional to m/r, where r is the number of suitable data stored, randomly distributed, by our hypothesis.)

At the other extreme, we may fully order the data. This requires work of the order of $m \log m$ (As Knuth[1] points out, if m becomes really large, the keys used must be specified by more digits --- multiple precision --- and the work of storage is really of order $m (\log m)^2$.) The work of retrieval is of order $\log m$ (or of order $(\log m)^2$, if key-length is taken into account.)

We shall be concerned with an intermediate and frequently-occurring situation, when the data are stored in a structure taking the form of a *binary tree*. Such a tree is a directed graph, in which every *node* but one (the *root* of the tree) has exactly one entering line(from its *predecessor* node): the root has no predecessor; and each node has 0, 1, or 2 leaving lines (to its *successor* node(s)); there being no closed circuits. If a node has less than two successors, we may call the potential or latent successors *open nodes* (as opposed to actual or occupied nodes.) It is clear that each node is connected to the root by an unique path (traced backwards); so that it may be assigned an unique *level* (a positive integer) equal to the number of nodes in this path. Only the root has level 1, and there are at most 2^{k-1} nodes in level k.

Ordered data may be formed into a tree as follows. The first datum is placed at the root of the tree. The two open nodes attached to the root are labelled the *left* and *right* successors of the root. If the second datum is less than the first (more precisely, if the second key is less than the first), it is placed at the left successor of the root; if the second datum is greater than the first, it is placed at the right successor of the root. Thereafter, all nodes are identified as left or right successors, and when $(m - 1)$ data have already been inserted into the tree structure, the m-th is placed at an open node as follows: beginning at the root, the new datum is compared with that occupying the node being considered, and consideration moves to the left successor if the new datum is less, and to the right successor if the new datum is greater. As soon as an open node is encountered, it is filled by the new datum.

It is clear that the number of comparisons required to store a given datum is equal to the level of the open node at which the datum is stored. This is also the number of comparisons required to retrieve a datum. Thus the work required to construct an m-node tree is proportional to the *internal sum* of the tree --- the sum of the levels of all occupied nodes --- and the work required to retrieve a given datum, chosen at random in the tree, will be the *average level* of the nodes occupied --- the internal sum divided by m. Finally, the work required to store a new datum in a given m-node tree will be proportional to the *external average* --- $1/n$ times the *external sum*, defined as the sum of the levels of all open nodes; when n is the number of external nodes.

More complicated storage schemes sometimes involve the concept of a higher-order tree structure. We may define a *tree of order* s, or more briefly, an *s-ary tree*, as one in which each node has up to s successors. The level k may then have up to s^k occupied or open nodes. In what follows, since the theory generalizes readily, we shall deal with s-ary trees, pointing out along the way the specific results for binary trees. As explained above, all estimates of work required for operations on m-node trees (of any order) should be multiplied by log m if the trees are large enough to require multiple-precision keys to be used and compared.

Finally, we note that, trees being trees, questions of interest to quantitative dendrologists may be answered by the results obtained here.

2. THE STRUCTURE OF AN s-ARY TREE

At level k, let $m_{k0}, m_{k1}, \ldots, m_{ks}$ respectively denote the number of nodes having 0, 1, ..., s successors occupied. Then the number of nodes occupied at level k is

$$\mu_k = \sum_{r=0}^{s} m_{kr}; \tag{1}$$

and in the tree there are

$$m = \sum_{k=1}^{\infty} \mu_k . \tag{2}$$

nodes in all. We note that, since an m-node tree cannot reach beyond level m,

$$\mu_1 = 1 \quad \text{if} \quad m > 0, \qquad \text{and} \qquad \mu_k = 0 \quad \text{if} \quad k > m. \tag{3}$$

Counting successors, if $k > 1$, we see that

$$\mu_k = \sum_{r=0}^{s} r\, m_{(k-1)r} = \sum_{r=1}^{s} r\, m_{(k-1)r}. \tag{4}$$

Thus, if $k > 1$, (1) and (4) yield

$$m_{k0} = \sum_{r=1}^{s} [r\, m_{(k-1)r} - m_{kr}]. \tag{5}$$

If $k > 1$, the number of open nodes at level k is

$$\nu_k = \sum_{r=0}^{s} (s - r)\, m_{(k-1)r} = \sum_{r=0}^{s-1} (s - r)\, m_{(k-1)r}; \tag{6}$$

so that (1) and (4) give us that

$$\nu_k = s\, \mu_{k-1} - \mu_k . \tag{7}$$

We note that, by (3), $\quad \nu_1 = 0 \quad \text{if} \quad m > 0, \quad \text{and} \quad \nu_{k+1} = 0 \quad \text{if} \quad k > m.$ (8)

The total number of open nodes is thus, for $m > 0$, by (2), (3), and (7),

$$n = \sum_{k=2}^{\infty} \nu_k = \sum_{k=2}^{\infty} (s\, \mu_{k-1} - \mu_k) = s \sum_{k=1}^{\infty} \mu_k - \sum_{k=2}^{\infty} \mu_k = s\, m - (m - \mu_1);$$

that is,

$$n = (s - 1)m + 1. \tag{9}$$

The result (9) may also be obtained by counting lines: the total number of successors (occupied or open) of the nodes of an m-node s-ary tree is clearly $s\ m$; while every occupied node, except the root, has an (occupied) predecessor; $(m - 1)$ in all: (9) follows. We note the rather curious fact, that the number of open nodes depends only on the number of occupied nodes, not on the configuration of the tree.

When, in the course of our discussion, the value of m, the number of nodes in the tree, can vary; we shall write $m_{kr}^{(m)}$ for m_{kr}, $\mu_k^{(m)}$ for μ_k, $\nu_k^{(m)}$ for ν_k, and $n^{(m)}$ for n.

The internal sum of a given tree is defined as

$$F_m = \sum_{k=1}^{\infty} \mu_k^{(m)} k, \tag{10}$$

following Knuth[1], and we may generalize this to the *internal sum of degree* p:

$$F_m^{(p)} = \sum_{k=1}^{\infty} \mu_k^{(m)}\ k^p \quad [\ = 0 \text{ if } m = 0]. \tag{11}$$

The corresponding average p-th power of the levels of *internal* (i. e., occupied) nodes is then

$$Y_m^{(p)} = F_m^{(p)}\ /\ m. \tag{12}$$

Similarly, the external sum,

$$E_m = \sum_{k=1}^{\infty} \nu_k^{(m)} k \quad [\ = \sum_{k=2}^{\infty} \nu_k^{(m)} k \text{ if } m > 0], \tag{13}$$

generalizes to the *external sum of degree p*

$$E_m^{(p)} = \sum_{k=1}^{\infty} \nu_k^{(m)}\ k^p \quad [\ = \sum_{k=2}^{\infty} \nu_k^{(m)}\ k^p \text{ if } m > 0], \tag{14}$$

with the corresponding average,

$$X_m^{(p)} = E_m^{(p)}\ /\ n^{(m)}. \tag{15}$$

By (7) and (9), we see that

$$E_m^{(p)} = \sum_{k=2}^{\infty} [s\ \mu_{k-1}^{(m)} - \mu_k^{(m)}]\ k^p = s \sum_{k=1}^{\infty} \mu_k^{(m)}\ (k+1)^p - \sum_{k=2}^{\infty} \mu_k^{(m)}\ k^p;$$

whence

$$E_m^{(p)} = (s-1)\ F_m^{(p)} + s \sum_{q=1}^{p-1} \binom{p}{q} F_m^{(q)} + s\ m + 1, \tag{16}$$

since $F_m^{(0)} = m$, $\mu_1 = 1$, by (2) and (3), respectively, and by (11). Note, too, that $E_m^{(0)} = n^{(m)}$, by (9) and (14), consistently with (16); and similarly, $E_0^{(p)} = 1$.

3. STORING DATA IN BINARY TREES

We begin by formalizing the situation described in §1. Let $u_1, u_2, \ldots, u_{m-1}, u_m$ denote the real keys of m data, in the order in which they are received and allocated storage in a binary tree. Let the linear ordering of the keys be

$$u_{\rho(1)} < u_{\rho(2)} < \cdots < u_{\rho(m-1)} < u_{\rho(m)}, \tag{17}$$

where ρ denotes a permutation of the indices $1, 2, \ldots, m-1, m$.

<u>Assumption</u> 1. *Given m, every permutation ρ of the m indices is equally probable, in the ordering* (17) *of m data received for storage.*

<u>Lemma</u> 1. *If the ordering of the first $(m-1)$ data is fixed:*

$$u_{i_1} < u_{i_2} < \ldots < u_{i_{m-1}}, \tag{18}$$

where $[i_1, i_2, \ldots, i_{m-1}]$ is a permutation of $[1, 2, \ldots, m-1]$; then the only possible permutations ρ compatible with (18) *are:*

$$[m, i_1, \ldots, i_{m-1}],\ [i_1, m, i_2, \ldots, i_{m-1}],\ [i_1, i_2, m, i_3, \ldots, i_{m-1}],$$
$$\ldots,\qquad [i_1, \ldots, i_{m-2}, m, i_{m-1}],\ [i_1, \ldots, i_{m-1}, m]. \tag{19}$$

Proof. The permutations listed in (19) are obtained by merging the index m with the permutation (18) in every possible way. Since the permutations (19) are all distinct, Assumption 1 and Lemma 1 yield:

Corollary 1. *Given the ordering* (18) *of the first* $(m - 1)$ *data, the m-th has equal probability of falling into any of the m intervals into which the previous data divide the real line.*

Let us now define a *canonical representation* of binary trees: the root is represented by the origin $[0, 0]$ of the Euclidean real plane. For $k > 0$, the 2^k possible nodes at level $(k + 1)$ are represented by the points with ordinate $y = k$ and abscissae

$$x = -(2^k - 1) / 2^k,\ -(2^k - 3) / 2^k,\ \ldots,\ -1 / 2^k,\ +1 / 2^k,$$
$$+3 / 2^k,\ \ldots,\ (2^k - 3) / 2^k,\ (2^k - 1) / 2^k, \tag{20}$$

all odd multiples of 2^{-k}. The points $[(4j + 1) / 2^k,\ k]$ and $[(4j + 3) / 2^k,\ k]$ represent, respectively, the left and right successors of the point $[(2j + 1) / 2^{k-1},\ k - 1]$; that is to say, the successors of the node represented by $[x, k - 1]$ are $[x \pm 2^{-k},\ k]$. *See Appendix B for illustrations of this and the proofs below.*

When left and right successors are identified for all nodes, the algorithm described in §1 uniquely generates a binary tree from any given sequence of data. The configuration of this tree will depend only on the ordering of the data keys; that is, on the permutation ρ defined by (17).

Lemma 2. *The successors at all levels of the node represented canonically by the point* $[(2j + 1) / 2^k,\ k]$ *lie in the open interval defined by*

$$j / 2^{k-1} < x < (j + 1) / 2^{k-1}. \tag{21}$$

Proof. By the canonical representation above, the level $(l + 1)$ successors of the given point take the form

$$[(2j + 1) / 2^k \pm 2^{-k-1} \pm 2^{-k-2} \pm \ldots \pm 2^{-l},\ l]; \tag{22}$$

and, for all l, the '±' increments all lie between $\pm\ (2^{-k} - 2^{-l})$: the Lemma follows. By (20), we obtain:

Corollary 2. *There is no overlap between the abscissae of the sets of successors at all levels of any two points at one given level.* In other words: *All the successors of the node represented by* $[x, k]$ *lie strictly between* $x - 2^{-k}$ *and* $[x - 2^{1-k}, k]$ *and all its successors, on the left, and* $x + 2^{-k}$ *and* $[x + 2^{1-k}, k]$ *and all its successors, on the right.*

Lemma 3. *Given a set of m data, the linear ordering of their keys is the same as the order of the abscissae of the corresponding points in the canonical representation of the binary tree generated by the data.*

Proof. Let u and v be the keys of two data, and let $u < v$. Suppose that u is inserted into the tree before v. In the path, in the tree generated by the given data, from the root to each of the nodes corresponding to the keys u and v; there will be a branch-point, with key w, say, (w could be u or the key of the root of the tree), and $u \leq w < v$, from the way branches are chosen in the algorithm. Since u and v are, by definition, successors of w (possibly $u = w$), Corollary 2 tells us that u, which is w or the immediate left successor of w or a successor of the immediate left successor of w, corresponds to a point of the canonical tree whose abscissa is less than the abscissa of the point of the tree corresponding to v, which has to be the immediate right successor of w or a successor of the immediate right successor of w. If, instead, v is inserted into the tree before u, the only change in the above argument is that, now, the branch-point w could be v, rather than u: this has no appreciable effect on the logic. Therefore the Lemma is proved.

Lemma 4. *There is a one-to-one correspondence between the intervals into which the keys of given data divide the real line and the open nodes of the corresponding canonical tree representation.*

Proof. By Lemma 3, there is a one-to-one correspondence between the intervals into which the keys of given data divide the real line and the intervals into which the abscissae of corresponding nodes in a canonical tree representing these data divide the x-axis: in fact, the order of the keys and the corresponding abscissae is the same. Thus, to prove the Lemma, it suffices to show a one-to-one correspondence between open nodes of a canonical tree and the intervals into which the abscissae of occupied nodes divide the x-axis. To begin with, if the tree has m occupied nodes, then the number of intervals generated is clearly $(m + 1)$; and, by (9) with $s = 2$, this equals the number of open nodes of this tree.

By following the unique path from the root to a given node $[x, k]$, we see that we can always find ± signs such that

$$x = \pm 2^{-1} \pm 2^{-2} \pm \ldots \pm 2^{-k} \quad \text{if } k > 0; \quad x = 0 \quad \text{if } k = 0. \tag{23}$$

Now, either all the ± signs in (23) are the same, or we can find an unique $l < k$, such that

$$x = \pm 2^{-1} \pm \ldots \pm 2^{1-l} \pm (2^{-l} - 2^{-l-1} - \ldots - 2^{-k}) = a \pm 2^{-k}. \tag{24}$$

Without loss of generality, we may assume that either $x = 2^{-1} + 2^{-2} + \ldots + 2^{-k}$, or $x = a + 2^{-k}$: if the signs are all '-' instead, the logic of the argument is virtually identical.

In the first case, if $[x, k]$ is an open node of the canonical tree, it is the right successor of the rightmost node, $[2^{-1} + \ldots + 2^{1-k}, k - 1]$ of the tree. (It is easy to verify that no occupied node can lie to the right of this one.)

The negative of this case yields the open node which is the left successor of the leftmost node of the tree. Thus, each of the extreme, semi-infinite intervals generated by the abscissae contains exactly one open node.

In the second case, when (24) holds,

$$a < x < b = a + 2^{1-k}; \tag{25}$$

and, by the form of the tree-storage algorithm and the definition of an open node, if $[x, k]$ is an open node, then the points $[a, l - 1]$ and $[b, k - 1]$ are two occupied nodes of the tree (the latter node being a successor of the former, and the immediate predecessor of the open node $[x, k]$.)

By Lemma 2, $[b, k - 1]$ is succeeded by points having abscissae strictly between a and $a + 2^{2-k}$, and any node between a and b must be a successor of $[b, k - 1]$. Now, the immediate successors of this node are $[x, k]$ itself (which, being open, has no successors, occupied or open) and $[b + 2^{-k}, k]$ (which, together with all its successors, by Lemma 2, lies to the right of $[b, k - 1]$.) This proves that the only node of the tree with abscissa between those of the nodes $[a, l - 1]$ and $[b, k - 1]$ is the single open node $[x, k]$. If we have the negative of the present case, the argument is identical. Thus, to each open node of our tree, there corresponds a distinct one of the intervals into which the abscissae of the nodes divide the x-axis (namely, that interval which contains that open node): since the number of open nodes equals the number of such intervals, the Lemma holds.

From Corollary 1 and Lemma 4, we obtain:

Corollary 3. *Given a binary tree generated by a sequence of data in the manner described in §1, the next datum has equal probability of being placed at any of the open nodes of the tree.*

Generalizing to the s-ary tree, we may now adopt:

Assumption 2. *Given an s-ary tree, generated by a sequence of data, we assume that the generating algorithm is such that the next datum has equal probability of being placed at any of the open nodes of the tree.*

4. STATISTICAL RELATIONSHIPS

By Assumption 2, in an m-node s-ary tree, each of the open nodes has probability

$$\omega_m = 1 / n^{(m)} = 1 / [(s - 1) m + 1], \tag{26}$$

by (9), of being the next node filled. We note that the probability associated with level k is then

$$p_k^{(m)} = \nu_k^{(m)} \omega_m = [s \mu_{k-1}^{(m)} - \mu_k^{(m)}] / n^{(m)}, \tag{27}$$

by (7); so that, by (2) and (9), as well as directly from the definition of $n^{(m)}$,

$$\sum_{k=2}^{\infty} p_k^{(m)} = \omega_m \sum_{k=2}^{\infty} \nu_k^{(m)} = 1. \tag{28}$$

It is of incidental interest (though not directly relevant to our present discussion) to observe that, if we associate with each open node at level k a probability

$$\kappa^{(k)} = s^{1-k}, \tag{29}$$

independent of m; so that the total probability associated with level k becomes

$$r_k^{(m)} = \nu_k^{(m)} \kappa^{(k)} = s^{2-k} \mu_{k-1}^{(m)} - s^{1-k} \mu_k^{(m)}; \tag{30}$$

then we get, as required, that

$$\sum_{k=2}^{\infty} r_k^{(m)} = \sum_{k=2}^{\infty} s^{2-k}\, \mu_{k-1}^{(m)} - \sum_{k=2}^{\infty} s^{1-k}\, \mu_k^{(m)} = \mu_1^{(m)} = 1, \tag{31}$$

by (3). We shall not pursue this possibility here.

Returning to (26) and (27), we shall denote the mathematical expectations of the various parameters defined in §2 as follows:

$$\mathbf{M}_k^{(m)} = \mathrm{E}[\mu_k^{(m)}],\quad \mathbf{N}_k^{(m)} = \mathrm{E}[\nu_k^{(m)}],\quad \mathbf{F}_m^{(p)} = \mathrm{E}[F_m^{(p)}],\quad \mathbf{Y}_m^{(p)} = \mathrm{E}[Y_m^{(p)}],$$
$$\mathbf{E}_m^{(p)} = \mathrm{E}[E_m^{(p)}],\quad \mathbf{X}_m^{(p)} = \mathrm{E}[X_m^{(p)}]; \tag{32}$$

where, in each case, E[...] denotes the expected value over all trees generated by random data inserted in accordance with Assumption 2. From (3), (7), (8), (11), (12), (14), and (15), we obtain that, for $k > 1$ and $m > 0$,

$$\mathbf{M}_1^{(m)} = 1,\quad \mathbf{N}_1^{(m)} = 0,\quad \text{and}\quad \mathbf{N}_k^{(m)} = s\, \mathbf{M}_{k-1}^{(m)} - \mathbf{M}_k^{(m)}; \tag{33}$$

$$\mathbf{F}_m^{(p)} = \sum_{k=1}^{\infty} \mathbf{M}_k^{(m)}\, k^p \quad \text{and} \quad \mathbf{Y}_m^{(p)} = \mathbf{F}_m^{(p)} / m; \tag{34}$$

$$\mathbf{E}_m^{(p)} = \sum_{k=2}^{\infty} \mathbf{N}_k^{(m)}\, k^p = (s - 1)\, \mathbf{F}_m^{(p)} + s \sum_{q=1}^{p-1} \binom{p}{q} \mathbf{F}_m^{(q)} + s\, m + 1$$
$$\text{and} \quad \mathbf{X}_m^{(p)} = \mathbf{E}_m^{(p)} / [(s - 1)\, m + 1]; \tag{35}$$

where the last results use (9) and (16) also. In addition, we note, from (3) and (8), that

$$\mathbf{M}_k^{(m)} = 0,\quad \mathbf{N}_{k+1}^{(m)} = 0,\quad \text{if} \quad k > m. \tag{36}$$

Consider now an $(m - 1)$-node tree to which an m-th node is added at level l. Then, clearly, if $m > 0$,

$$\mu_k^{(m)} = \mu_k^{(m-1)} + \delta_{kl}, \tag{37}$$

where δ_{kl} is the Kronecker symbol (= 1, if $k = l$; = 0, otherwise); and so, if $k > 0$,

$$\mathbf{M}_k^{(m)} = \mathbf{M}_k^{(m-1)} + E[\sum_{l=1}^{\infty} \delta_{kl}\, p_l^{(m-1)}] = \mathbf{M}_k^{(m-1)} + E[p_k^{(m-1)}],$$

that is, by (26) and (27),

$$\mathbf{M}_k^{(m)} = \alpha_m\, \mathbf{M}_k^{(m-1)} + \beta_m\, \mathbf{M}_{k-1}^{(m-1)}, \tag{38}$$

where

$$\alpha_m = \frac{(s-1)\,(m-1)}{(s-1)\,(m-1)+1}, \quad \beta_m = \frac{s}{(s-1)\,(m-1)+1}. \tag{39}$$

Also, inserting the m-th node in level l reduces the number of open nodes at this level by one, but adds s new open nodes at level $(l+1)$, so that, if $m > 0$,

$$\nu_k^{(m)} = \nu_k^{(m-1)} - \delta_{kl} + s\,\delta_{k(l+1)}; \tag{40}$$

whence, by (26) and (27), if $k > 1$,

$$\mathbf{N}_k^{(m)} = \mathbf{N}_k^{(m-1)} - E[p_k^{(m-1)}] + s\,E[p_{k-1}^{(m-1)}] = \alpha_m\, \mathbf{N}_k^{(m-1)} + \beta_m\, \mathbf{N}_{k-1}^{(m-1)}. \tag{41}$$

We note, too, by (2) and (9), that

$$\sum_{k=1}^{\infty} \mathbf{M}_k^{(m)} = m \quad \text{and} \quad \sum_{k=2}^{\infty} \mathbf{N}_k^{(m)} = (s-1)\,m + 1. \tag{42}$$

Applying (38) to (34), with (33) and (42), we see that

$$\begin{aligned}
\mathbf{F}_m^{(p)} &= 1 + \sum_{k=2}^{\infty} \mathbf{M}_k^{(m)}\, k^p = 1 + \alpha_m \sum_{k=2}^{\infty} \mathbf{M}_k^{(m-1)}\, k^p + \beta_m \sum_{k=2}^{\infty} \mathbf{M}_{k-1}^{(m-1)}\, k^p \\
&= 1 + \alpha_m\, [\mathbf{F}_{m-1}^{(p)} - 1] + \beta_m \sum_{k=1}^{\infty} \mathbf{M}_k^{(m-1)}\, (k+1)^p \\
&= \sigma_m + \tau_m\, \mathbf{F}_{m-1}^{(p)} + \beta_m \sum_{q=1}^{p-1} \binom{p}{q}\, \mathbf{F}_{m-1}^{(q)},
\end{aligned} \tag{43}$$

where

$$\sigma_m = 1 - \alpha_m + \beta_m\,(m-1), \quad \tau_m = \alpha_m + \beta_m. \tag{44}$$

Similarly, applying (41) to (35), with (33) and (42), we get that

$$\underset{\sim}{E}_m^{(p)} = \alpha_m \sum_{k=2}^{\infty} \underset{\sim}{N}_k^{(m-1)} k^p + \beta_m \sum_{k=2}^{\infty} \underset{\sim}{N}_k^{(m-1)} (k+1)^p$$

$$= s + \tau_m \underset{\sim}{E}_{m-1}^{(p)} + \beta_m \sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{E}_{m-1}^{(q)}. \tag{45}$$

Finally, by (34) and (35), we see that

$$\underset{\sim}{Y}_m^{(p)} = \frac{\sigma_m}{m} + \frac{m-1}{m} \tau_m \underset{\sim}{Y}_{m-1}^{(p)} + \frac{m-1}{m} \beta_m \sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{Y}_{m-1}^{(q)} \tag{46}$$

and

$$\underset{\sim}{X}_m^{(p)} = \beta_{m+1} + \underset{\sim}{X}_{m-1}^{(p)} + \beta_{m+1} \sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{X}_{m-1}^{(q)}. \tag{47}$$

(The last result follows from (39) and (44), when we observe that $\tau_m = \alpha_m + \beta_m = n^{(m)} / n^{(m-1)}$ and $\beta_m = s / n^{(m-1)}$, whence $\beta_m / \tau_m = \beta_{m+1}$.)

5. SOLVING THE RECURRENCES

We observe that the recurrence-relations (38) and (41) are the same; and that (43), (45), (46), and (47) are essentially of the same form: the former, homogeneous linear recurrences, and the latter, inhomogeneous linear recurrences.

The recurrence (38), with (36), yields

$$\underset{\sim}{M}_k^{(m)} = \sum_{h=k}^{m} \alpha_m \alpha_{m-1} \cdots \alpha_{h+1} \beta_h \underset{\sim}{M}_{k-1}^{(h-1)}, \tag{48}$$

for any $m + 1 \geq k > 1$. Similarly, the recurrence (41), with (36), yields

$$\underset{\sim}{N}_k^{(m)} = \sum_{h=k-1}^{m} \alpha_m \alpha_{m-1} \cdots \alpha_{h+1} \beta_h \underset{\sim}{N}_{k-1}^{(h-1)}, \tag{49}$$

for any $m + 2 \geq k > 2$. If $m < k$ in (48) or $m < k - 1$ in (49), the sums on the right are empty and are interpreted as zero, consistently with (36). In particular, for $m = 1$, we get that

$$\mathbf{M}_1^{(1)} = 1, \quad \mathbf{M}_k^{(1)} = 0 \text{ if } k > 1, \quad \text{and} \quad \mathbf{N}_2^{(1)} = s, \quad \mathbf{N}_k^{(1)} = 0 \text{ if } k > 2, \tag{50}$$

by (36), and because a single node is placed at the root of the tree (level 1) and so has s (open) successors in level 2. When $m + 2 > k = 2$ (i. e., $m > 0$), (41) yields

$$\mathbf{N}_2^{(m)} = \alpha_m \, \alpha_{m-1} \, \cdots \, \alpha_2 \, \beta_1, \tag{51}$$

by (33) and (50), since $\beta_1 = s$. Actually, (51) may be interpreted as a case of (49), if we remember that, by (33), $\mathbf{N}_1^{(h-1)} = 0$ if $h > 1$, while $\mathbf{N}_1^{(0)} = 1$, being the number of open nodes at level 1 in a 0-node (i. e., empty) tree.

Applying (48) and (49) repeatedly, with (33) and (51), we obtain

$$\mathbf{M}_k^{(m)} = \sum_{\{k,m\}}^{\dagger} \alpha_m \cdots \alpha_{h_1+1} \, \beta_{h_1} \, \alpha_{h_1-1} \cdots \alpha_{h_2+1} \, \beta_{h_2} \, \alpha_{h_2-1} \cdots$$
$$\cdots \alpha_{h_{k-1}+1} \, \beta_{h_{k-1}} , \tag{52}$$

where $\sum\limits_{\{k,m\}}^{\dagger}$ denotes the sum over all values of $h_1, h_2, \ldots, h_{k-1}$ such that

$$m + 1 > h_1 > h_2 > \ldots > h_{k-1} > 1; \tag{53}$$

and

$$\mathbf{N}_k^{(m)} = \sum_{\{k-1,m\}}^{\dagger} \alpha_m \cdots \alpha_{h_1+1} \, \beta_{h_1} \, \alpha_{h_1-1} \cdots \alpha_{h_2+1} \, \beta_{h_2} \, \alpha_{h_2-1} \cdots$$
$$\cdots \alpha_{h_{k-2}+1} \, \beta_{h_{k-2}} \, \alpha_{h_{k-2}-1} \cdots \alpha_2 \, \beta_1 . \tag{54}$$

If we write $\theta = 1 / (s - 1)$, then (39) and (44) become

$$\alpha_m = \frac{m - 1}{m - 1 + \theta}, \quad \beta_m = \frac{1 + \theta}{m - 1 + \theta}, \quad \sigma_m = \frac{m - 1 + m\theta}{m - 1 + \theta}, \quad \tau_m = \frac{m + \theta}{m - 1 + \theta}. \tag{55}$$

Hence (52) and (54) become

$$\mathbf{M}_k^{(m)} = \sum_{\{k,m\}}^{\dagger} \frac{(m - 1) \ldots (h_{k-1} - 1)}{(m - 1 + \theta) \ldots (h_{k-1} - 1 + \theta)} \prod_{r=1}^{k-1} \left(\frac{1 + \theta}{h_r - 1}\right) \tag{56}$$

and

$$\underset{\sim}{N}_k^{(m)} = \sum_{\{k-1,m\}}^{\dagger} \frac{(m-1)\ \ldots\ 2}{(m-1+\theta)\ \ldots\ (2+\theta)\ \theta} \prod_{r=1}^{k-2} \left(\frac{1+\theta}{h_r - 1}\right). \tag{57}$$

Now, if we define

$$P_m(z) = \prod_{h=1}^{m-1} (z+h), \tag{58}$$

it is clear that $\underset{\sim}{N}_k^{(m)}$ will equal the coefficient of z^{k-2} in

$$\frac{(1+\theta)^{k-1}}{(m-1+\theta)\ \ldots\ (1+\theta)\ \theta} P_m(z), \tag{59}$$

when $m + 2 > k > 2$; while if

$$Q_m^{(k)}(z) = \sum_{l=2}^{m-k+2} \frac{1}{(m-1+\theta)\ \ldots\ (l-1+\theta)} \prod_{h=l}^{m-1} (z+h), \tag{60}$$

then, when $m + 1 > k > 1$, $\underset{\sim}{M}_k^{(m)}$ will equal the coefficient of z^{k-2} in

$$(1+\theta)^{k-1}\ Q_m^{(k)}(z), \tag{61}$$

if, when $l = m$ (only possible when $k = 2$), the term in (60) is $1/(m-1+\theta)$. If, further,

$$R_{lm}(z) = \prod_{h=l}^{m-1} (z+h) \quad \text{and} \quad S_{lm}^{(q)}(z) = \sum_{h=l}^{m-1} (z+h)^{-q}, \tag{62}$$

with $R_{mm}(z) = 1$ and $S_{mm}^{(q)}(z) = 0$, then

$$(\partial/\partial z)\ R_{lm}(z) = R_{lm}(z)\ S_{lm}^{(1)}(z) \quad \text{and} \quad (\partial/\partial z)\ S_{lm}^{(q)}(z) = -\,q\ S_{lm}^{(q+1)}(z). \tag{63}$$

Since (58) and (60) are constant linear combinations of the polynomials $R_{lm}(z)$; we see from (59) and (61) that, when $m + 2 > k > 1$ (extended to $k = 2$ via (50) and (51)),

$$\underset{\sim}{N}_k^{(m)} = \frac{(1+\theta)^{k-1}\ /\ (k-2)!}{(m-1+\theta)\ \ldots\ (1+\theta)\ \theta} \left[(\partial/\partial z)^{k-2}\ R_{1m}(z)\right]_{z=0}; \tag{64}$$

and, when $m + 1 > k > 1$,

$$\underset{\sim}{M}_k^{(m)} = \frac{(1+\theta)^{k-1}}{(k-2)!} \sum_{l=2}^{m-k+2} \frac{1}{(m-1+\theta)\ \ldots\ (l-1+\theta)} \left[(\partial/\partial z)^{k-2}\ R_{lm}(z)\right]_{z=0}; \tag{65}$$

and that these expressions are finitely computable, in terms of

$$R_{lm}(0) = l\ (l+1)\ \ldots\ (m-1)$$

and the sums

$$S_{lm}^{(q)}(0) = \frac{1}{l^q} + \frac{1}{(l+1)^q} + \ldots + \frac{1}{(m-1)^q}\,. \tag{66}$$

For example,

$$[(\partial/\partial z)^{k-2} R_{lm}(z)]_{z=0} = \begin{array}{ll} \text{for } k=2, & R \\ \text{for } k=3, & R\, S_1 \\ \text{for } k=4, & R\, (S_1{}^2 - S_2) \\ \text{for } k=5, & R\, (S_1{}^3 - 3\, S_1\, S_2 + 2\, S_3) \\ \text{for } k=6, & R\, (S_1{}^4 - 6\, S_1{}^2\, S_2 + 8\, S_1\, S_3 + 3\, S_2{}^2 - 6\, S_4), \end{array} \tag{67}$$

and so on, where we write R for $R_{lm}(0)$ and S_q for $S^{(q)}_{lm}(0)$. The corresponding expressions for $\underset{\sim}{N}^{(m)}_k$ and $\underset{\sim}{M}^{(m)}_k$ follow immediately from (64) and (65).

To solve for $\underset{\sim}{E}^{(p)}_m$ and $\underset{\sim}{F}^{(p)}_m$ (from which $\underset{\sim}{X}^{(p)}_m$ and $\underset{\sim}{Y}^{(p)}_m$ follow immediately, by (34) and (35)), we may use (59) and (61), first noting that, by (34) and (35), with (33) and (36),

$$\underset{\sim}{E}^{(p)}_m = \sum_{k=2}^{m+1} \underset{\sim}{N}^{(m)}_k\, k^p \quad \text{and} \quad \underset{\sim}{F}^{(p)}_m = 1 + \sum_{k=2}^{m} \underset{\sim}{M}^{(m)}_k\, k^p. \tag{68}$$

We then obtain that

$$\underset{\sim}{E}^{(p)}_m = [(m - 1 + \theta)\ \dots\ (2 + \theta)\ \theta]^{-1}\ [z^{-2}\ (z\partial/\partial z)^p\ z^2\ P_m(z)]_{z=1+\theta} \tag{69}$$

and

$$\underset{\sim}{F}^{(p)}_m = 1 + \sum_{l=2}^{m} [(m - 1 + \theta)\ \dots\ (l - 1 + \theta)]^{-1} \times$$

$$\times\ [z^{-1}\ (z\partial/\partial z)^p\ z^2\ R_{lm}(z)]_{z=1+\theta}, \tag{70}$$

by matching powers of z. If, using (62), we define

$$T^{(q)}_{lm}(z) = z^q\ S^{(q)}_{lm}(z), \tag{71}$$

we see that, by (63),

$$(z\partial/\partial z)\ z^2\ R_{lm}(z) = z^2\ R_{lm}(z)\ \{T^{(1)}_{lm}(z) + 2\}$$

and (72)

$$(z\partial/\partial z)\ T^{(q)}_{lm}(z) = q\ \{T^{(q)}_{lm}(z) - T^{(q+1)}_{lm}(z)\}.$$

Writing R^* for $(1 + \theta)^2\ R_{lm}(1 + \theta)$ and T^*_q for $T^{(q)}_{lm}(1 + \theta)$, as in (67), we obtain

$$[(z\partial/\partial z)^p\, z^2\, R_{lm}(z)]_{z=1+\theta} = \begin{cases} \text{for } p=0, & R^* \\ \text{for } p=1, & R^*\,(T_1^* + 2) \\ \text{for } p=2, & R^*\,(T_1^{*2} + 5\,T_1^* - T_2^* + 4) \\ \text{for } p=3, & R^*\,(T_1^{*3} + 9\,T_1^{*2} - 3\,T_1^*\,T_2^* + 19\,T_1^* \\ & \quad - 9\,T_2^* + 2\,T_3^* + 8) \\ \text{for } p=4, & R^*\,(T_1^{*4} + 14\,T_1^{*3} - 6\,T_1^{*2}\,T_2^* + 55\,T_1^{*2} \\ & \quad - 42\,T_1^*\,T_2^* + 8\,T_1^*\,T_3^* + 3\,T_2^{*2} \\ & \quad + 65\,T_1^* - 55\,T_2^* + 28\,T_3^* \\ & \quad - 6\,T_4^* + 16), \end{cases} \tag{73}$$

and so on. Again, the corresponding expressions for $\underset{\sim}{E}_m^{(p)}$, $\underset{\sim}{F}_m^{(p)}$, $\underset{\sim}{X}_m^{(p)}$, and $\underset{\sim}{Y}_m^{(p)}$ follow immediately from (69), (70), (34), and (35).

Although these expressions (derived from (67) and (73)) are finitely computable, they are not in a form which we shall find useful here. Thus, we take another approach, to solve the inhomogeneous recurrences for successive values of p.

Consider first (43), for $p = 1$:

$$\underset{\sim}{F}_m^{(1)} = \sigma_m + \tau_m\,\underset{\sim}{F}_{m-1}^{(1)}. \tag{74}$$

By (50), $\underset{\sim}{F}_1^{(1)} = 1$; so it is readily seen that, since $\sigma_1 = 1$ (by (55)),

$$\underset{\sim}{F}_m^{(1)} = \sum_{l=1}^{m} \tau_m\,\tau_{m-1} \cdots \tau_{l+1}\,\sigma_l; \tag{75}$$

and by (55), this becomes

$$\underset{\sim}{F}_m^{(1)} = \sum_{l=1}^{m} \frac{(m+\theta)\,(l-1+l\theta)}{(l-1+\theta)\,(l+\theta)} = (m+\theta)\,\Big\{\frac{\theta}{\theta} - \frac{\theta}{1+\theta} + \frac{1+2\theta}{1+\theta} - \frac{1+2\theta}{2+\theta} + $$

$$+ \frac{2+3\theta}{2+\theta} - \cdots - \frac{m-2+(m-1)\,\theta}{m-1+\theta} + \frac{m-1+m\,\theta}{m-1+\theta} - \frac{m-1+m\,\theta}{m+\theta}\Big\}.$$

The sum telescopes, yielding

$$\mathbf{F}_m^{(1)} = (m + \theta)\ (1 + \theta)\ [\frac{1}{1 + \theta} + \ldots + \frac{1}{m + \theta}] - m\ \theta$$

$$= (m + \theta)\ T_{0m}^{(1)}(1 + \theta) - m\ \theta, \tag{76}$$

by the notation of (62) and (71). By a somewhat laborious calculation, it is possible to verify that (70), with (73), agrees with (76).

Taking (43) with $p = 2$, we get an equation similar to (74), except that σ_m is replaced by $\sigma_m + 2\ \beta_m\ \mathbf{F}_{m-1}^{(1)} = \sigma_m^{(2)}$, say. Therefore, since $\sigma_1^{(2)} = \sigma_1 = 1$ and (actually, for all $p > 0$)

$$\mathbf{F}_1^{(p)} = 1, \tag{77}$$

we see that (75) holds for $p = 2$, when σ_l is replaced by $\sigma_l^{(2)}$. Again applying (55), we obtain, first, that

$$\sigma_l^{(2)} = \frac{l - 1 + l\ \theta}{l - 1 + \theta} + 2\ (1 + \theta)^2\ [\frac{1}{1 + \theta} + \ldots + \frac{1}{l - 1 + \theta}] - \frac{2\ (l - 1)\ (1 + \theta)\ \theta}{l - 1 + \theta},$$

and then, that

$$\mathbf{F}_m^{(2)} = \sum_{l=1}^{m} \frac{m + \theta}{l + \theta}\ \sigma_l^{(2)} = (m + \theta)\ (1 + \theta)\ [\frac{1}{\theta} - \frac{1}{1 + \theta} + \frac{2}{1 + \theta} - \frac{2}{2 + \theta} + \frac{3}{2 + \theta} -$$

$$\ldots - \frac{m - 1}{m - 1 + \theta} + \frac{m}{m - 1 + \theta} - \frac{m}{m + \theta}] - (m + \theta)\ [\frac{1}{\theta} - \frac{1}{1 + \theta} + \frac{1}{1 + \theta} - \ldots$$

$$- \frac{1}{m - 1 + \theta} + \frac{1}{m - 1 + \theta} - \frac{1}{m + \theta}] + 2\ (m + \theta)\ (1 + \theta)^2\ [\frac{1}{2 + \theta}\ (\frac{1}{1 + \theta}) +$$

$$\frac{1}{3 + \theta}\ (\frac{1}{1 + \theta} + \frac{1}{2 + \theta}) + \ldots + \frac{1}{m + \theta}\ (\frac{1}{1 + \theta} + \ldots + \frac{1}{m - 1 + \theta})] - 2\ (m + \theta)\ (1 +$$

$$\theta)\ \theta\ [\frac{1}{1 + \theta} - \frac{1}{2 + \theta} + \frac{2}{2 + \theta} - \ldots - \frac{m - 2}{m - 1 + \theta} + \frac{m - 1}{m - 1 + \theta} - \frac{m - 1}{m + \theta}]$$

$$= (m + \theta)\ (1 + \theta)\ [\frac{1}{\theta} + \frac{1}{1 + \theta} + \ldots + \frac{1}{m + \theta}] - (m + 1)\ (1 + \theta) - (m + \theta)\ [\frac{1}{\theta} -$$

$$\frac{1}{m + \theta}] + (m + \theta)\ (1 + \theta)^2\ \{[\frac{1}{1 + \theta} + \ldots + \frac{1}{m + \theta}]^2 - [\frac{1}{(1 + \theta)^2} + \ldots + \frac{1}{(m + \theta)^2}]\}$$

$$- 2\ (m + \theta)\ (1 + \theta)\ \theta\ [\frac{1}{1 + \theta} + \ldots + \frac{1}{m + \theta}] + 2\ m\ (1 + \theta)\ \theta,$$

which simplifies to

$$\underset{\sim}{F}_m^{(2)} = (m + \theta)\ (1 + \theta)^2\ \{[\frac{1}{1 + \theta} + \dots + \frac{1}{m + \theta}]^2 - [\frac{1}{(1 + \theta)^2} + \dots + \frac{1}{(m + \theta)^2}]\}$$
$$+ (m + \theta)\ (1 - 2\ \theta)\ (1 + \theta)\ [\frac{1}{1 + \theta} + \dots + \frac{1}{m + \theta}] + m\ \theta + 2\ m\ \theta^2$$
$$= (m + \theta)\ \{[T_{0m}^{(1)}(1 + \theta)]^2 - T_{0m}^{(2)}(1 + \theta)\} + (m + \theta)\ (1 - 2\ \theta)\ T_{0m}^{(1)}(1 + \theta)$$
$$+ m\ \theta + 2\ m\ \theta^2. \quad (78)$$

By a very laborious calculation, it is again possible to verify that (70), with (73), agrees with (78). From this point on, the computations take on increasingly heroic proportions; but it is clear that successive formulae for $\underset{\sim}{F}_m^{(p)}$ may be obtained for $p = 3, 4, \dots$, from (43), and that these computations are simpler than those we would need to obtain these formulae from (70) with the aid of (72). Fortunately, for our purposes, (76) and (78) will suffice!

To get $\underset{\sim}{Y}_m^{(p)}$, we simply use (34). Thus the *expected internal sum* is $\underset{\sim}{F}_m^{(1)}$, given in (76); and the *expected average level of internal (occupied) nodes* is given by

$$\underset{\sim}{Y}_m^{(1)} = \frac{m + \theta}{m}\ T_{0m}^{(1)}(1 + \theta) - \theta. \quad (79)$$

Similarly, the *expected internal sum of degree 2*, that is, the expected sum of squares of levels of internal nodes, is $F_m^{(2)}$, given in (78); so that the *expected average square of levels of internal nodes* is given by

$$\underset{\sim}{Y}_m^{(2)} = \frac{m + \theta}{m}\ \{[T_{0m}^{(1)}(1 + \theta)]^2 - T_{0m}^{(2)}(1 + \theta) + (1 - 2\ \theta)\ T_{0m}^{(1)}(1 + \theta)\}$$
$$+ \theta + 2\ \theta^2. \quad (80)$$

To obtain $\underset{\sim}{E}_m^{(p)}$ and $\underset{\sim}{X}_m^{(p)}$, it is best to use (35), with (76) and (78); thus, when $p = 1$, we have

$$\underset{\sim}{E}_m^{(1)} = \frac{1}{\theta}\,\underset{\sim}{F}_m^{(1)} + \frac{1+\theta}{\theta}\,m + 1 = \frac{m+\theta}{\theta}\,[T_{0m}^{(1)}(1+\theta) + 1]; \tag{81}$$

and, when $p = 2$, we similarly get

$$\begin{aligned}\underset{\sim}{E}_m^{(2)} &= \frac{1}{\theta}\,\underset{\sim}{F}_m^{(2)} + 2\,\frac{1+\theta}{\theta}\,\underset{\sim}{F}_m^{(1)} + \frac{1+\theta}{\theta}\,m + 1\\ &= \frac{m+\theta}{\theta}\,\{[T_{0m}^{(1)}(1+\theta)]^2 - T_{0m}^{(2)}(1+\theta) + 3\,T_{0m}^{(1)}(1+\theta) + 1\}.\end{aligned} \tag{82}$$

Thus, we have explicit forms for the *expected external sums of degrees* 1 *and* 2. Finally, the explicit forms for the *expected averages of the first and second powers of levels of external (open) nodes* are, respectively,

$$\underset{\sim}{X}_m^{(1)} = T_{0m}^{(1)}(1+\theta) + 1 \tag{83}$$

and

$$\underset{\sim}{X}_m^{(2)} = [T_{0m}^{(1)}(1+\theta)]^2 - T_{0m}^{(2)}(1+\theta) + 3\,T_{0m}^{(1)}(1+\theta) + 1. \tag{84}$$

6. THE VARIANCES

We now turn to the second moments of the parameters, defined by

$$\begin{aligned}&\underset{\sim}{A}_{kk'}^{(m)} = E[\nu_k^{(m)}\ \nu_{k'}^{(m)}], \quad \underset{\sim}{B}_{kk'}^{(m)} = E[\mu_k^{(m)}\ \mu_{k'}^{(m)}],\\ &\underset{\sim}{C}_{pp'}^{(m)} = E[E_m^{(p)}\ E_m^{(p')}], \quad \underset{\sim}{D}_{pp'}^{(m)} = E[F_m^{(p)}\ F_m^{(p')}],\\ &\underset{\sim}{G}_{pp'}^{(m)} = E[X_m^{(p)}\ X_m^{(p')}], \quad \underset{\sim}{H}_{pp'}^{(m)} = E[Y_m^{(p)}\ Y_m^{(p')}].\end{aligned} \tag{85}$$

First, by (40), we obtain that, if $m > 0$,

$$\underset{\sim}{A}_{kk'}^{(m)} = E[\{\nu_k^{(m-1)} - \delta_{kl} + s\ \delta_{k(l+1)}\}\ \{\nu_{k'}^{(m-1)} - \delta_{k'l} + s\ \delta_{k'(l+1)}\}];$$

and, by first taking expectations for the last inserted node, and then for all preceding ones, as in obtaining (38) and (41), we get that

$$\underset{\sim}{A}_{kk'}^{(m)} = E[\nu_k^{(m-1)}\ \nu_{k'}^{(m-1)} - \nu_k^{(m-1)}\ p_{k'}^{(m-1)} + s\ \nu_k^{(m-1)}\ p_{k'-1}^{(m-1)} - p_k^{(m-1)}\ \nu_{k'}^{(m-1)}$$

$$+ p_k^{(m-1)}\ \delta_{kk'} - s\ p_k^{(m-1)}\ \delta_{k(k'-1)} + s\ p_{k-1}^{(m-1)}\ \nu_{k'}^{(m-1)}$$

$$- s\ p_{k-1}^{(m-1)}\ \delta_{(k-1)k'} + s^2\ p_{k-1}^{(m-1)}\ \delta_{kk'}],$$

and so, by (27),

$$\underset{\sim}{A}_{kk'}^{(m)} = \frac{m-1-\theta}{m-1+\theta}\underset{\sim}{A}_{kk'}^{(m-1)} + \frac{1+\theta}{m-1+\theta}[\underset{\sim}{A}_{k(k'-1)}^{(m-1)} + \underset{\sim}{A}_{(k-1)k'}^{(m-1)}]$$

$$+ \frac{1+\theta}{m-1+\theta}\delta_{kk'}\ [\frac{\theta}{1+\theta}\underset{\sim}{N}_k^{(m-1)} + \frac{1+\theta}{\theta}\underset{\sim}{N}_{k-1}^{(m-1)}]$$

$$- \frac{1+\theta}{m-1+\theta}[\underset{\sim}{N}_k^{(m-1)}\ \delta_{k(k'-1)} + \underset{\sim}{N}_{k'}^{(m-1)}\ \delta_{(k-1)k'}]. \tag{86}$$

Similarly, by (3), (27), and (37), if $m > 0$ and $k,\ k' > 1$,

$$\underset{\sim}{B}_{11}^{(m)} = 1, \qquad \underset{\sim}{B}_{k1}^{(m)} = \underset{\sim}{B}_{1k}^{(m)} = \underset{\sim}{M}_k^{(m)}, \tag{87}$$

and

$$\underset{\sim}{B}_{kk'}^{(m)} = E[\{\mu_k^{(m-1)} + \delta_{kl}\}\ \{\mu_{k'}^{(m-1)} + \delta_{k'l}\}]$$

$$= \frac{m-1-\theta}{m-1+\theta}\underset{\sim}{B}_{kk'}^{(m-1)} + \frac{1+\theta}{m-1+\theta}[\underset{\sim}{B}_{k(k'-1)}^{(m-1)} + \underset{\sim}{B}_{(k-1)k'}^{(m-1)}]$$

$$+ \frac{\theta}{m-1+\theta}\delta_{kk'}\ \underset{\sim}{N}_k^{(m-1)}. \tag{88}$$

Using the initial values given in (3), (8), (50), (51), and Appendix A, as well as known values of $\underset{\sim}{M}_k^{(m)}$ and $\underset{\sim}{N}_k^{(m)}$ from (52), (54), (64), and (65); we can proceed to solve the recurrences (86) and (88), much as we solved (38) and (41) in §5, successively, for increasing values of k and k', and of m. Of course, the labor required grows very fast. In the simplest cases, we get

$$\underset{\sim}{A}_{11}^{(m)} = \underset{\sim}{A}_{1k}^{(m)} = \underset{\sim}{A}_{k1}^{(m)} = 0 \quad \text{if} \quad m > 0; \tag{89}$$

and

$$\underset{\sim}{A}_{22}^{(m)} = \frac{m-1-\theta}{m-1+\theta} \underset{\sim}{A}_{22}^{(m-1)} + \frac{\theta}{m-1+\theta} \underset{\sim}{N}_{2}^{(m-1)}$$

$$= \frac{m-1-\theta}{m-1+\theta} \underset{\sim}{A}_{22}^{(m-1)} + \frac{(m-2)!}{(m-1+\theta)\ \dots\ (2+\theta)} \quad \text{if} \quad m > 1, \tag{90}$$

by (64); which is solved for $m > 1$ by

$$\underset{\sim}{A}_{22}^{(m)} = \frac{(m-1-\theta)\ \dots\ (2-\theta)}{(m-1+\theta)\ \dots\ (2+\theta)} \left(\frac{1}{\theta}\right)^2$$

$$+ \sum_{l=2}^{m-1} \frac{(m-1-\theta)\ \dots\ (l+1-\theta)\ (l-1)!}{(m-1+\theta)\ \dots\ (2+\theta)}; \tag{91}$$

and

$$\underset{\sim}{B}_{12}^{(m)} = \underset{\sim}{B}_{21}^{(m)} = \underset{\sim}{M}_{2}^{(m)} = (1+\theta) \sum_{l=2}^{m} \frac{(m-1)\ \dots\ l}{(m-1+\theta)\ \dots\ (l-1+\theta)} \tag{92}$$

if $m > 0$, by (65); and, by (64) and (92), if $m > 1$,

$$\underset{\sim}{B}_{22}^{(m)} = \frac{m-1-\theta}{m-1+\theta} \underset{\sim}{B}_{22}^{(m-1)} + 2\,\frac{1+\theta}{m-1+\theta} \underset{\sim}{B}_{12}^{(m-1)} + \frac{\theta}{m-1+\theta} \underset{\sim}{N}_{2}^{(m-1)}$$

$$= \frac{m-1-\theta}{m-1+\theta} \underset{\sim}{B}_{22}^{(m-1)} + \frac{(m-2)!}{(m-1+\theta)\ \dots\ (2+\theta)}$$

$$+ 2\,(1+\theta)^2 \sum_{l=2}^{m-1} \frac{(m-2)\ \dots\ l}{(m-1+\theta)\ \dots\ (l-1+\theta)}; \tag{93}$$

which is solved for $m > 1$ by

$$\underset{\sim}{B}_{22}^{(m)} = \frac{(m-1-\theta)\ \dots\ (2-\theta)}{(m-1+\theta)\ \dots\ (2+\theta)} + \sum_{l=2}^{m-1} \left\{\frac{(m-1-\theta)\ \dots\ (l+1-\theta)\ (l-1)!}{(m-1+\theta)\ \dots\ (2+\theta)}\right.$$

$$\left. + 2\,(1+\theta)^2 \sum_{h=l}^{m-1} \frac{(m-1-\theta)\ \dots\ (h+1-\theta)\ (h-1)\ \dots\ l}{(m-1+\theta)\ \dots\ (l-1+\theta)}\right\}. \tag{94}$$

Now, by (9), (11), (12), (14), (15), and (85), we see that

$$\begin{aligned} \underset{\sim}{C}^{(m)}_{pp'} &= \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \underset{\sim}{A}^{(m)}_{kk'}\, k^{p}\, k'^{p'}, \qquad \underset{\sim}{G}^{(m)}_{pp'} = \left(\frac{\theta}{m+\theta}\right)^2 \underset{\sim}{C}^{(m)}_{pp'}, \\ \underset{\sim}{D}^{(m)}_{pp'} &= \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \underset{\sim}{B}^{(m)}_{kk'}\, k^{p}\, k'^{p'}, \qquad \underset{\sim}{H}^{(m)}_{pp'} = \left(\frac{1}{m}\right)^2 \underset{\sim}{D}^{(m)}_{pp'}. \end{aligned} \tag{95}$$

Also, as was mentioned at the end of §2,

$$E^{(0)}_{m} = \sum_{k=1}^{\infty} \nu^{(m)}_{k} = n^{(m)}, \qquad X^{(0)}_{m} = 1, \qquad F^{(0)}_{m} = \sum_{k=1}^{\infty} \mu^{(m)}_{k} = m, \qquad Y^{(0)}_{m} = 1; \tag{96}$$

whence

$$\begin{aligned} &\underset{\sim}{C}^{(m)}_{00} = \left(\frac{m+\theta}{\theta}\right)^2, \quad \underset{\sim}{C}^{(m)}_{0p} = \underset{\sim}{C}^{(m)}_{p0} = \left(\frac{m+\theta}{\theta}\right) \underset{\sim}{E}^{(p)}_{m}, \quad \underset{\sim}{G}^{(m)}_{00} = 1, \quad \underset{\sim}{G}^{(m)}_{0p} = \underset{\sim}{G}^{(m)}_{p0} = \underset{\sim}{X}^{(p)}_{m}, \\ &\underset{\sim}{D}^{(m)}_{00} = m^2, \qquad \underset{\sim}{D}^{(m)}_{0p} = \underset{\sim}{D}^{(m)}_{p0} = m\, \underset{\sim}{F}^{(p)}_{m}, \qquad \underset{\sim}{H}^{(m)}_{00} = 1, \quad \underset{\sim}{H}^{(m)}_{0p} = \underset{\sim}{H}^{(m)}_{p0} = \underset{\sim}{Y}^{(p)}_{m}. \end{aligned} \tag{97}$$

Thus, by (86) and (88), with (34), (35), (87), and (89), we obtain that, if $m > 1$,

$$\begin{aligned} \underset{\sim}{G}^{(m)}_{pp'} &= \left(\frac{\theta}{m+\theta}\right)^2 \sum_{k=2}^{\infty} \sum_{k'=2}^{\infty} \underset{\sim}{A}^{(m)}_{kk'}\, k^{p}\, k'^{p'} \\ &= \left[1 - \left(\frac{1}{m+\theta}\right)^2\right] \underset{\sim}{G}^{(m-1)}_{pp'} + \left(\frac{1}{m+\theta}\right)^2 \Big\{(1+\theta)\,(m-1+\theta) \Big[\sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{G}^{(m-1)}_{qp'} \\ &\quad + \sum_{q'=1}^{p'-1} \binom{p'}{q'} \underset{\sim}{G}^{(m-1)}_{pq'}\Big] + \underset{\sim}{X}^{(p+p')}_{m-1} + (1+\theta) \Big[\sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{X}^{(q+p')}_{m-1} \\ &\quad + \sum_{q'=1}^{p'-1} \binom{p'}{q'} \underset{\sim}{X}^{(p+q')}_{m-1}\Big] + (1+\theta)^2 \sum_{q=1}^{p-1} \sum_{q'=1}^{p'-1} \binom{p}{q} \binom{p'}{q'} \underset{\sim}{X}^{(q+q')}_{m-1} \\ &\quad + (1+\theta)\,(m+\theta) \Big[\underset{\sim}{X}^{(p)}_{m-1} + \underset{\sim}{X}^{(p')}_{m-1}\Big] + (1+\theta)^2 \Big[\sum_{q=1}^{p-1} \binom{p}{q} \underset{\sim}{X}^{(q)}_{m-1} \\ &\quad + \sum_{q'=1}^{p'-1} \binom{p'}{q'} \underset{\sim}{X}^{(q')}_{m-1}\Big] + (1+\theta)^2 \Big\} \end{aligned} \tag{98}$$

and

$$\underset{\sim}{\mathrm{H}}_{pp'}^{(m)} = m^{-2} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \underset{\sim}{\mathrm{B}}_{kk'}^{(m)} k^{p} k'^{p'} = m^{-2} \{ \sum_{k=2}^{\infty} \sum_{k'=2}^{\infty} \underset{\sim}{\mathrm{B}}_{kk'}^{(m)} k^{p} k'^{p'}$$

$$+ \sum_{k=2}^{\infty} \underset{\sim}{\mathrm{B}}_{k1}^{(m)} k^{p} + \sum_{k'=2}^{\infty} \underset{\sim}{\mathrm{B}}_{1k'}^{(m)} k'^{p'} + \underset{\sim}{\mathrm{B}}_{11}^{(m)} \}$$

$$= m^{-2} \{ \underset{\sim}{\mathrm{Z}}_{pp'}^{(m)} + m \underset{\sim}{\mathrm{Y}}_{m}^{(p)} + m \underset{\sim}{\mathrm{Y}}_{m}^{(p')} - 1 \}, \tag{99}$$

where

$$\underset{\sim}{\mathrm{Z}}_{pp'}^{(m)} = \sum_{k=2}^{\infty} \sum_{k'=2}^{\infty} \underset{\sim}{\mathrm{B}}_{kk'}^{(m)} k^{p} k'^{p'} = m^{2} \underset{\sim}{\mathrm{H}}_{pp'}^{(m)} - m \underset{\sim}{\mathrm{Y}}_{m}^{(p)} - m \underset{\sim}{\mathrm{Y}}_{m}^{(p')} + 1, \tag{100}$$

whence
$$\underset{\sim}{\mathrm{Z}}_{00}^{(m)} = (m-1)^{2}, \quad \underset{\sim}{\mathrm{Z}}_{0p}^{(m)} = \underset{\sim}{\mathrm{Z}}_{p0}^{(m)} = (m-1)[m \underset{\sim}{\mathrm{Y}}_{m}^{(p)} - 1]; \tag{101}$$

and if we define the operators

$$\overset{p}{\underset{q}{\mathrm{T}}} z_q = \sum_{q=0}^{p} \binom{p}{q} z_q \quad \text{and} \quad \overset{p}{\underset{q}{\Xi}} z_q = \sum_{q=1}^{p-1} \binom{p}{q} z_q, \tag{102}$$

so that

$$\overset{p}{\underset{q}{\mathrm{T}}} 1 = 2^{p}, \quad \overset{p}{\underset{q}{\mathrm{T}}} h^{q} = (h+1)^{p}, \quad \overset{p}{\underset{q}{\Xi}} 1 = 2^{p} - 2, \quad \overset{p}{\underset{q}{\Xi}} h^{q} = (h+1)^{p} - h^{p} - 1, \tag{103}$$

we get, using also (33) and (87), that

$$\underset{\sim}{\mathrm{Z}}_{pp'}^{(m)} = \frac{m-1-\theta}{m-1+\theta} \sum_{k=2}^{\infty} \sum_{k'=2}^{\infty} \underset{\sim}{\mathrm{B}}_{kk'}^{(m-1)} k^{p} k'^{p'} + \frac{1+\theta}{m-1+\theta} \sum_{k=2}^{\infty} \sum_{l'=1}^{\infty} \underset{\sim}{\mathrm{B}}_{kl'}^{(m-1)} k^{p} (l'+1)^{p'}$$

$$+ \frac{1+\theta}{m-1+\theta} \sum_{l=1}^{\infty} \sum_{k'=2}^{\infty} \underset{\sim}{\mathrm{B}}_{lk'}^{(m-1)} (l+1)^{p} k'^{p'} + \frac{\theta}{m-1+\theta} \sum_{k=2}^{\infty} \underset{\sim}{\mathrm{N}}_{k}^{(m-1)} k^{p+p'}$$

$$= \frac{m-1-\theta}{m-1+\theta} \underset{\sim}{\mathrm{Z}}_{pp'}^{(m-1)} + \frac{1+\theta}{m-1+\theta} \{ \overset{p'}{\underset{q'}{\mathrm{T}}} \underset{\sim}{\mathrm{Z}}_{pq'}^{(m-1)} + 2^{p'} [(m-1) \underset{\sim}{\mathrm{Y}}_{m-1}^{(p)} - 1]$$

$$+ \overset{p}{\underset{q}{\mathrm{T}}} \underset{\sim}{\mathrm{Z}}_{qp'}^{(m-1)} + 2^{p} [(m-1) \underset{\sim}{\mathrm{Y}}_{m-1}^{(p')} - 1] \} + \frac{\theta}{m-1+\theta} \{ \frac{1+\theta}{\theta} \overset{p}{\underset{q}{\mathrm{T}}} \overset{p'}{\underset{q'}{\mathrm{T}}} (m-1) \underset{\sim}{\mathrm{Y}}_{m-1}^{(q+q')}$$

$$- (m-1) \underset{\sim}{\mathrm{Y}}_{m-1}^{(p+p')} + 1 \}$$

or

$$\underset{\sim}{Z}_{pp'}^{(m)} = \frac{m+1+\theta}{m-1+\theta}\,\underset{\sim}{Z}_{pp'}^{(m-1)} + \frac{1+\theta}{m-1+\theta}\left[\overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{Z}_{qp'}^{(m-1)} + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Z}_{pq'}^{(m-1)}\right]$$

$$+ \frac{(m-1)\,(1+\theta)}{m-1+\theta}\left[\frac{1}{1+\theta}\,\underset{\sim}{Y}_{m-1}^{(p+p')} + \overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q+p')} + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(p+q')}\right.$$

$$+ \overset{p}{\underset{q}{\Xi}}\,\overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q+q')} + (2^{p'}+m-1)\,\underset{\sim}{Y}_{m-1}^{(p)} + (2^{p}+m-1)\,\underset{\sim}{Y}_{m-1}^{(p')}$$

$$\left. + \overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q)} + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q')}\right] + \frac{\theta-(1+\theta)\,(m-3+2^{p}+2^{p'})}{m-1+\theta}; \qquad (104)$$

whence, finally, using also (46) and (55), we obtain that

$$\underset{\sim}{H}_{pp'}^{(m)} = \left(\frac{m-1}{m}\right)^2 \left\{\frac{m+1+\theta}{m-1+\theta}\,\underset{\sim}{H}_{pp'}^{(m-1)} + \frac{1+\theta}{m-1+\theta}\left[\overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{H}_{qp'}^{(m-1)} + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{H}_{pq'}^{(m-1)}\right]\right\}$$

$$+ \frac{(1+\theta)\,(m-1)}{m^2\,(m-1+\theta)}\left\{\frac{1}{1+\theta}\,\underset{\sim}{Y}_{m-1}^{(p+p')} + \overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q+p')} + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(p+q')}\right.$$

$$+ \overset{p}{\underset{q}{\Xi}}\,\overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q+q')} + \left(m+1-\frac{1}{1+\theta}\right)\left[\underset{\sim}{Y}_{m-1}^{(p)} + \underset{\sim}{Y}_{m-1}^{(p')}\right] + \overset{p}{\underset{q}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q)}$$

$$\left. + \overset{p'}{\underset{q'}{\Xi}}\,\underset{\sim}{Y}_{m-1}^{(q')}\right\} + \frac{\theta+(1+\theta)\,(m-1)}{m^2\,(m-1+\theta)}\,. \qquad (105)$$

We have now finally arrived at recurrences, (98) and (105), whose solutions will yield the *covariances* and *variances* which are the goals of this section:

$$\begin{aligned} \mathrm{cov}[X_m^{(p)}, X_m^{(p')}] &= \underset{\sim}{G}_{pp'}^{(m)} - \underset{\sim}{X}_m^{(p)}\,\underset{\sim}{X}_m^{(p')}, \quad \mathrm{var}[X_m^{(p)}] = \underset{\sim}{G}_{pp}^{(m)} - [\underset{\sim}{X}_m^{(p)}]^2, \\ \mathrm{cov}[Y_m^{(p)}, Y_m^{(p')}] &= \underset{\sim}{H}_{pp'}^{(m)} - \underset{\sim}{Y}_m^{(p)}\,\underset{\sim}{Y}_m^{(p')}, \quad \mathrm{var}[Y_m^{(p)}] = \underset{\sim}{H}_{pp}^{(m)} - [\underset{\sim}{Y}_m^{(p)}]^2. \end{aligned} \qquad (106)$$

While the recurrences above are, in principle, soluble for all values of the indices, the herculean task this presents dwarfs even the laborious and highly accident-prone computations of the previous section. We shall therefore limit ourselves to the case (most interesting in practice) of $p = p' = 1$.

For brevity, following (62), let us write

$$S_q^\dagger = S_{mq}^* = S_{0(m-1)}^{(q)}(1+\theta) = \frac{1}{(1+\theta)^q} + \frac{1}{(2+\theta)^q} + \ldots + \frac{1}{(m-1+\theta)^q}. \tag{107}$$

Then, by (71), (79), (80), (83), and (84),

$$\begin{aligned} \underset{\sim}{X}_{m-1}^{(1)} &= (1+\theta)\, S_1^\dagger + 1, \quad \underset{\sim}{X}_m^{(1)} = \underset{\sim}{X}_{m-1}^{(1)} + \frac{1+\theta}{m+\theta}, \\ \underset{\sim}{X}_{m-1}^{(2)} &= (1+\theta)^2\, (S_1^{\dagger 2} - S_2^\dagger) + 3\,(1+\theta)\, S_1^\dagger + 1, \end{aligned} \tag{108}$$

and

$$\begin{aligned} \underset{\sim}{Y}_{m-1}^{(1)} &= \frac{m-1+\theta}{m-1}\,(1+\theta)\, S_1^\dagger - \theta, \\ \underset{\sim}{Y}_m^{(1)} &= \frac{m+\theta}{m}\,(1+\theta)\, S_1^\dagger + \frac{1+\theta}{m} - \theta, \\ \underset{\sim}{Y}_{m-1}^{(2)} &= \frac{m-1+\theta}{m-1}\,[(1+\theta)^2 (S_1^{\dagger 2} - S_2^\dagger) + (1-2\theta)\,(1+\theta)\, S_1^\dagger] + \theta\,(1+2\theta). \end{aligned} \tag{109}$$

Now, by (98),

$$\underset{\sim}{G}_{11}^{(m)} = [1 - (\frac{1}{m+\theta})^2]\, \underset{\sim}{G}_{11}^{(m-1)} + (\frac{1}{m+\theta})^2\, [\underset{\sim}{X}_{m-1}^{(2)} + 2\,(1+\theta)\,(m+\theta)\, \underset{\sim}{X}_{m-1}^{(1)} + (1+\theta)^2]; \tag{110}$$

so that, by (106), with (108),

$$\begin{aligned} \mathrm{var}[X_m^{(1)}] = {} & [1 - (\frac{1}{m+\theta})^2]\, \mathrm{var}[X_{m-1}^{(1)}] + [1 - (\frac{1}{m+\theta})^2]\, \underset{\sim}{X}_{m-1}^{(1)2} - \underset{\sim}{X}_m^{(1)2} \\ & + (\frac{1}{m+\theta})^2\, [\underset{\sim}{X}_{m-1}^{(2)} + 2\,(1+\theta)\,(m+\theta)\, \underset{\sim}{X}_{m-1}^{(1)} + (1+\theta)^2], \end{aligned}$$

or, upon simplification,

$$\mathrm{var}[X_m^{(1)}] = [1 - (\frac{1}{m+\theta})^2]\, \mathrm{var}[X_{m-1}^{(1)}] + \frac{1+\theta}{(m+\theta)^2}\, [S_1^\dagger - (1+\theta)\, S_2^\dagger]. \tag{111}$$

Since $\mathrm{var}[X_1^{(1)}] = \underset{\sim}{G}_{11}^{(1)} - \underset{\sim}{X}_1^{(1)2} = 0$, we can solve (111), observing that

$$\prod_{h=m}^{l+1} [1 - (\frac{1}{h+\theta})^2] = \frac{(m+1+\theta)(m-1+\theta)(m+\theta)(m-2+\theta)\ \ldots\ (l+2+\theta)(l+\theta)}{(m+\theta)^2 \qquad (m-1+\theta)^2 \qquad\qquad (l+1+\theta)^2}$$

telescopes to

$$\prod_{h=m}^{l+1} [1 - (\frac{1}{h + \theta})^2] = \frac{(m + 1 + \theta)\ (l + \theta)}{(m + \theta)\ (l + 1 + \theta)}, \tag{112}$$

yielding

$$\mathrm{var}[X_m^{(1)}] = \sum_{l=2}^{m} \frac{(m + 1 + \theta)\ (1 + \theta)}{(m + \theta)\ (l + \theta)\ (l + 1 + \theta)} [\ S^*_{l1} - (1 + \theta)\ S^*_{l2}]. \tag{113}$$

Now, we observe that, by telescoping the sums below, we get that

$$\sum_{l=2}^{m} \frac{S^*_{l1}}{(l + \theta)\ (l + 1 + \theta)} = \sum_{l=2}^{m} (\frac{1}{l + \theta} - \frac{1}{l + 1 + \theta})\ (\frac{1}{1 + \theta} + \ldots + \frac{1}{l - 1 + \theta})$$

$$= \sum_{l=2}^{m} \frac{S^*_{l1}}{l + \theta} - \sum_{l=3}^{m+1} \frac{S^*_{(l-1)1}}{l + \theta} = \sum_{l=2}^{m} \frac{1}{(l + \theta)\ (l - 1 + \theta)} - \frac{S^*_{m1}}{m + 1 + \theta}$$

$$= \sum_{l=2}^{m} (\frac{1}{l - 1 + \theta} - \frac{1}{l + \theta}) - \frac{S^*_{m1}}{m + 1 + \theta}$$

$$= \frac{1}{1 + \theta} - \frac{1}{m + \theta} - \frac{1}{m + 1 + \theta} S^\dagger_1, \tag{114}$$

and

$$\sum_{l=2}^{m} \frac{S^*_{l2}}{(l + \theta)\ (l + 1 + \theta)} = \sum_{l=2}^{m} \frac{1}{(l + \theta)\ (l - 1 + \theta)^2} - \frac{S^*_{m2}}{m + 1 + \theta}$$

$$= \sum_{l=2}^{m} [\frac{1}{l + \theta} - \frac{1}{l - 1 + \theta} + \frac{1}{(l - 1 + \theta)^2}] - \frac{S^*_{m2}}{m + 1 + \theta}$$

$$= \frac{1}{m + \theta} - \frac{1}{1 + \theta} + [1 - \frac{1}{m + 1 + \theta}]\ S^\dagger_2. \tag{115}$$

Thus, finally, (113) yields that

$$\mathrm{var}[X_m^{(1)}] = (2 + \theta)\ [1 - \frac{\theta}{m + \theta} - \frac{1 + \theta}{(m + \theta)^2}] - (1 + \theta)^2\ S^\dagger_2 - \frac{1 + \theta}{m + \theta} S^\dagger_1. \tag{116}$$

From the last result, we obtain at once that

$$\underset{\sim}{G}_{11}^{(m)} = (1 + \theta)^2 S_1^{\dagger 2} + (1 + \theta)\ (2 + \frac{1 + 2\theta}{m + \theta})\ S_1^{\dagger} - (1 + \theta)^2 S_2^{\dagger}$$
$$+ [3 + \theta + \frac{2 - \theta^2}{m + \theta} - \frac{1 + \theta}{(m + \theta)^2}]. \tag{117}$$

To complete our computations, we could proceed to solve (105), much as we solved (98), from the recurrence

$$\underset{\sim}{H}_{11}^{(m)} = (\frac{m - 1}{m})^2\ (\frac{m + 1 + \theta}{m - 1 + \theta})\ \underset{\sim}{H}_{11}^{(m-1)} + \frac{m - 1}{m^2\ (m - 1 + \theta)}\ \{\underset{\sim}{Y}_{m-1}^{(2)} + 2[m\ (1 + \theta)$$
$$+ \theta]\ \underset{\sim}{Y}_{m-1}^{(1)} + 1 + \theta\} + \frac{\theta}{m^2\ (m - 1 + \theta)}. \tag{118}$$

However, very fortunately, a considerable shortcut presents itself in equation (16), which yields

$$E_m^{(1)} = \frac{1}{\theta} F_m^{(1)} + \frac{1 + \theta}{\theta} m + 1; \tag{119}$$

whence, by (9), (12), and (15),

$$Y_m^{(1)} = \frac{m + \theta}{m}\ [X_m^{(1)} - 1] - \theta; \tag{120}$$

and this, in turn, yields, by (79) and (83), that

$$Y_m^{(1)} - \underset{\sim}{Y}_m^{(1)} = \frac{m + \theta}{m}\ [X_m^{(1)} - \underset{\sim}{X}_m^{(1)}]. \tag{121}$$

Thus we see mmediately that

$$\mathrm{var}[Y_m^{(1)}] = (\frac{m + \theta}{m})^2\ \mathrm{var}[X_m^{(1)}] \tag{122}$$

or, finally, by (116),

$$\mathrm{var}[Y_m^{(1)}] = m^{-2}\ \{(2 + \theta)\ [(m + \theta)^2 - \theta\ (m + \theta) - (1 + \theta)]$$
$$- (1 + \theta)^2\ (m + \theta)^2\ S_2^{\dagger} - (1 + \theta)\ (m + \theta)\ S_1^{\dagger}\}. \tag{123}$$

From this, by (106), we can readily derive the expression for $\underset{\sim}{H}_{11}^{(m)}$.

In Sections 2, 4, 5, and 6, we have obtained exact formulae for some of the fundamental statistical parameters of s-ary [one might say $(\frac{1+\theta}{\theta})$-ary!] trees. It is perhaps useful to formalize the more important results at this point.

Lemma 5. *Under Assumption 2, the mathematical expectations of the number of internal (occupied) and external (open) nodes in level k of an s-ary tree with m (occupied) nodes are, respectively,* $\underset{\sim}{M}_k^{(m)}$ *and* $\underset{\sim}{N}_k^{(m)}$*, as given in equations* (65) *and* (64)*, with* (62), (33)*, and* (36). *Alternative formulae are given by* (52) *and* (54)*, with* (53) *and* (55).

Lemma 6. *The* internal sum of degree p *is defined in* (11)*: its expectation is given in* (70). *The* external sum of degree p *is defined in* (14)*: its expectation is given in* (69). *From these, the expectations of the corresponding* average p-th powers of levels, *as defined in* (12) *and* (15), *respectively, are trivial to derive.*

Lemma 7. *The second moments of the six quantities whose first moments (mathematical expectations) are given in Lemmas* 5 *and* 6 *satisfy recurrences given by* (85), (86), (88), (95), (98), *and* (105), *from which (in principle) they can be obtained.*

Theorem 1. *The mathematical expectations of the average levels of internal (occupied) and external (open) nodes of an m-node s-ary tree, under Assumption* 2, *are given, respectively by equations* (79) *and* (83), *with* (71) *and* (62). *Corresponding sums are given by* (76) *and* (81). *The corresponding statistics for degree* 2 *are given by* (80), (84), (78), *and* (82), *respectively.*

Theorem 2. *The variances of the average levels of internal and external nodes of an m-node s-ary tree are given respectively by equations* (123) *and* (116). *The variances of the corresponding sums are immediately derivable, and (in principle) the corresponding variances for degree* 2 *(or more) can similarly be explicitly obtained.*

Turning to the particular case of *binary* trees, when $s = 2$ and $\theta = 1$, let us adopt the abbreviated notations

$$T_q^\dagger = 1 + [S_q^\dagger]_{\theta=1} = 1 + \frac{1}{2^q} + \frac{1}{3^q} + \ldots + \frac{1}{m^q} ,$$

and (124)

$$X_m^\dagger = [X_m^{(1)}]_{\theta=1} \qquad \text{and} \qquad Y_m^\dagger = [Y_m^{(1)}]_{\theta=1} .$$

Then Theorems 1 and 2 readily yield:

Corollary 4. *The mathematical expectations of the average levels of internal and external nodes of an m-node binary tree are, respectively,*

$$\mathrm{E}[Y_m^\dagger] = 2 \, (1 + \frac{1}{m}) \, T_1^\dagger - 3$$

and (125)

$$\mathrm{E}[X_m^\dagger] = 2 \, T_1^\dagger - \frac{m - 1}{m + 1} .$$

Corollary 5. *The variances of the average levels of internal and external nodes of an m-node binary tree are, respectively,*

$$\mathrm{var}[Y_m^\dagger] = \frac{1}{m^2} \, \{m \, (7m + 13) - 4 \, (m + 1)^2 \, T_2^\dagger - 2 \, (m + 1) \, T_1^\dagger\}$$

and (126)

$$\mathrm{var}[X_m^\dagger] = \frac{1}{(m + 1)^2} \, \{m \, (7m + 13) - 4 \, (m + 1)^2 \, T_2^\dagger - 2 \, (m + 1) \, T_1^\dagger\}.$$

7. ANALYTIC AND ASYMPTOTIC RESULTS

We proceed to use the results of Theorems 1 and 2 to estimate the behavior of $X_m^{(1)}$ and $Y_m^{(1)}$ when m grows in size. To this end, we first examine the properties of the sums (107).

We note that $s \geqslant 2$, so that

$$0 < \theta \leqslant 1. \tag{127}$$

By (107) and (124), we see that

$$T_q^\dagger - 1 = [S_q^\dagger]_{\theta=1} \leqslant S_q^\dagger < [S_q^\dagger]_{\theta=0} = T_q^\dagger - \frac{1}{m^q} . \tag{128}$$

Now we observe (see, e. g., Copson[2] or Whittaker and Watson[3]) that these sums are related to the Riemann zeta function ---

$$T_q^\dagger \uparrow \zeta(q) \quad \text{as} \quad m \to \infty \tag{129}$$

--- and to the Hurwitz generalization thereof ---

$$S_q^\dagger \uparrow \zeta(q, 1 + \theta) \quad \text{as} \quad m \to \infty. \tag{130}$$

We also obtain at once that (see, e. g., Abramowitz and Stegun[4] or Mitrinović[5])

$$T_1^\dagger - \log m = \gamma_m \to \gamma \quad \text{as} \quad m \to \infty, \tag{131}$$

where γ is Euler's (or Mascheroni's) constant, 0.5772156649...; and

$$T_2^\dagger \uparrow \zeta(2) = \pi^2/6 \quad \text{as} \quad m \to \infty. \tag{132}$$

When $s = 3$, $\theta = \frac{1}{2}$ and it is known that

$$[S_2^\dagger]_{\theta=\frac{1}{2}} \uparrow \pi^2/2 - 4 \quad \text{as} \quad m \to \infty. \tag{133}$$

Higher values of s give more difficulty. However, we see that

$$0 < \frac{1}{2}\left(\frac{1}{1+\theta} - \frac{1}{m+\theta}\right) = \frac{1}{2}\sum_{l=1}^{m-1}\left(\frac{1}{l+\theta} - \frac{1}{l+\theta+1}\right) = \sum_{l=1}^{m-1}\int_0^1 \frac{z \; dz}{(l+\theta)(l+\theta+1)}$$

$$\leqslant \lambda_m(\theta) = \sum_{l=1}^{m-1}\int_0^1 \frac{z \; dz}{(l+\theta)(l+\theta+z)} = \sum_{l=1}^{m-1}\int_0^1 \left(\frac{1}{l+\theta} - \frac{1}{l+\theta+z}\right) dz$$

$$= \sum_{l=1}^{m-1}\frac{1}{l+\theta} - \sum_{l=1}^{m-1}\log\left(\frac{l+\theta+1}{l+\theta}\right) = S_1^\dagger - \log\left(\frac{m+\theta}{1+\theta}\right). \tag{134}$$

Similarly, by (128) and (132),

$$\lambda_m(\theta) \leqslant \sum_{l=1}^{m-1} \int_0^1 \frac{z\ dz}{(l+\theta)^2} = \frac{1}{2} \sum_{l=1}^{m-1} \frac{1}{(l+\theta)^2} = \frac{1}{2} S_2^\dagger < \frac{1}{2} T_2^\dagger < \pi^2/6. \tag{135}$$

Another result which is of interest, though it will not be too useful here, is

$$\begin{aligned} S_q^\dagger &= \sum_{l=1}^{m-1} \frac{1}{(l+\theta)^q} = \sum_{l=1}^{m-1} \frac{1}{(l+1)^q} \left(1 - \frac{1-\theta}{l+1}\right)^{-q} \\ &= \sum_{l=1}^{m-1} \frac{1}{(l+1)^q} \sum_{h=0}^{\infty} \binom{q+h-1}{h} \left(\frac{1-\theta}{l+1}\right)^h \\ &= \sum_{h=0}^{\infty} \binom{q+h-1}{h} (1-\theta)^h \left(T_{q+h}^\dagger - 1\right) \\ &\uparrow \sum_{h=0}^{\infty} \binom{q+h-1}{h} (1-\theta)^h \left[\zeta(q+h) - 1\right] \quad \text{as} \quad m \to \infty . \end{aligned} \tag{136}$$

Applying these results to (108), (109), (116), and (123), we obtain that

$$\underset{\sim}{X}_m^{(1)} = (1+\theta)\left[\log\left(\frac{m+\theta}{1+\theta}\right) + \lambda_m(\theta)\right] + \frac{1+\theta}{m+\theta} + 1, \tag{137}$$

$$\underset{\sim}{Y}_m^{(1)} = \frac{m+\theta}{m} (1+\theta)\left[\log\left(\frac{m+\theta}{1+\theta}\right) + \lambda_m(\theta)\right] + \frac{1+\theta}{m} - \theta, \tag{138}$$

$$\begin{aligned} \operatorname{var}[X_m^{(1)}] = (2+\theta)\left[1 - \frac{\theta}{m+\theta} - \frac{1+\theta}{(m+\theta)^2}\right] - (1+\theta)^2 S_2^\dagger \\ - \frac{1+\theta}{m+\theta}\left[\log\left(\frac{m+\theta}{1+\theta}\right) + \lambda_m(\theta)\right], \end{aligned} \tag{139}$$

$$\begin{aligned} \operatorname{var}[Y_m^{(1)}] = m^{-2} \{(2+\theta)\left[(m+\theta)^2 - \theta(m+\theta) - (1+\theta)\right] \\ - (1+\theta)^2 (m+\theta)^2 S_2^\dagger - (1+\theta)(m+\theta)\left[\log\left(\frac{m+\theta}{1+\theta}\right) + \lambda_m(\theta)\right]\}. \end{aligned} \tag{140}$$

Now $\lambda_m(\theta)$ is an increasing function of m (a sum of $(m-1)$ positive terms, by (134)), bounded above, by (135); so it converges to a limit

$$\lambda_m(\theta) \uparrow \lambda(\theta) \quad \text{as} \quad m \to \infty; \tag{141}$$

with
$$\lambda_m(0) = \gamma_m - \frac{1}{m} \uparrow \gamma = \lambda(0) \quad \text{as} \quad m \to \infty. \tag{142}$$

Going to asymptotic forms, as $m \to \infty$, we see that

$$\underset{\sim}{X}_m^{(1)} \sim (1 + \theta) \log m + (1 + \theta) [\lambda(\theta) - \log (1 + \theta)] + 1, \tag{143}$$

$$\underset{\sim}{Y}_m^{(1)} \sim (1 + \theta) \log m + (1 + \theta) [\lambda(\theta) - \log (1 + \theta)] - \theta, \tag{144}$$

$$\text{var}[X_m^{(1)}] \sim (2 + \theta) - (1 + \theta)^2 S_2^{\dagger}, \tag{145}$$

$$\text{var}[Y_m^{(1)}] \sim (2 + \theta) - (1 + \theta)^2 S_2^{\dagger}. \tag{146}$$

We note in passing, that

$$\underset{\sim}{X}_m^{(1)} - \underset{\sim}{Y}_m^{(1)} \to 1 + \theta \quad \text{as} \quad m \to \infty. \tag{147}$$

By (128) and (132), with (130), we observe that

$$\text{var}[X_m^{(1)}] \sim \text{var}[Y_m^{(1)}] \sim (2 + \theta) - (1 + \theta)^2 \zeta(2, 1 + \theta) = \Omega_\theta, \tag{148}$$

where $\frac{\pi^2}{6} - 1 \leqslant \zeta(2, 1 + \theta) \leqslant \frac{\pi^2}{6}$. Thus the variances are asymptotically constant, for large m; that is, they are bounded for all m. Thus, we have

<u>Theorem</u> 3. *The expectations of the average levels of both internal and external nodes of an s-ary tree are asymptotic to* $(1 + \theta) \log m$*, as m, the number of (occupied) nodes, tends to infinity.*

<u>Theorem</u> 4. *The variances of the average levels of both internal and external nodes of an s-ary tree tend to the limit* Ω_θ *defined in* (148)*, as m, the number of nodes, tends to infinity; thus, these variances are bounded over all values of m.*

<u>Corollary</u> 6. *The expectation of average levels of both internal and external nodes in a binary tree is asymptotic to* 2 log *m, as the number of nodes, m, tends to infinity.*

This Corollary is simply the case $\theta = 1$, for which also

$$\Omega_1 = 3 - 2^2 \zeta(2, 2) = 3 - 4 [\zeta(2) - 1] = 7 - 4 \pi^2/6 = 0.42026373... \qquad (149)$$

For $s = 3$ and $\theta = \frac{1}{2}$, we similarly get, by (133), that

$$\Omega_{\frac{1}{2}} = \frac{5}{2} - \frac{9}{4} (\frac{\pi^2}{2} - 4) = 0.396695... \qquad (150)$$

Now, by the Chebyshev inequality (see, e. g., Feller[6] or Tucker[7]), for any $\varepsilon > 0$,

$$\text{Prob}[|X_m^{(1)} - \underset{\sim}{X}_m^{(1)}| \geqslant \varepsilon \underset{\sim}{X}_m^{(1)}] \leqslant \frac{\text{var}[X_m^{(1)}]}{\varepsilon^2 \underset{\sim}{X}_m^{(1)2}} \sim \frac{\Omega_\theta}{\varepsilon^2 (1 + \theta)^2 (\log m)^2}$$

$$\to 0 \quad \text{as} \quad m \to \infty. \qquad (151)$$

Similarly,

$$\text{Prob}[|Y_m^{(1)} - \underset{\sim}{Y}_m^{(1)}| \geqslant \varepsilon \underset{\sim}{Y}_m^{(1)}] \to 0 \quad \text{as} \quad m \to \infty. \qquad (152)$$

Theorem 5. *The average levels of both internal (occupied) and external (open) nodes in s-ary trees distributed statistically according to Assumption 2, are asymptotic* in probability *to* $(1 + \theta) \log m$, *as* m, *the number of occupied nodes, tends to infinity.*

Note, once again, that the averages referred-to in all our results are averages over the nodes (internal or external, as the case may be) of a single tree: the statistical averages over all trees are always referred-to as (mathematical) expectations.

Corollary 7. *The internal and external sums of an s-ary tree are asymptotic in probability to* $(1 + \theta) m \log m$ *and* $\frac{(1 + \theta)}{\theta} m \log m$, *respectively, as* m *tends to infinity.*

We may write these results as

$$X_m^{(1)} \sim Y_m^{(1)} \sim (1 + \theta) \log m, \quad E_m^{(1)} \sim \frac{(1 + \theta)}{\theta} m \log m, \quad F_m^{(1)} \sim (1 + \theta) m \log m$$

$$\text{in probability}, \quad \text{as} \quad m \to \infty. \qquad (153)$$

The asymptotic inequality (151) (and the similar result for $Y_m^{(1)}$) is, in fact, stronger than the assertion of Theorem 5. Indeed, the zero limits found in (151) and (152) will prevail even if ε depends on m, so long as

$$\varepsilon \, \underset{\sim}{X}_m^{(1)} \to \infty, \quad \text{or} \quad \varepsilon \log m \to \infty, \quad \text{as} \quad m \to \infty. \tag{154}$$

Thus,
$$\text{Prob}[\,|X_m^{(1)} - \underset{\sim}{X}_m^{(1)}| \geqslant \eta_m] \to 0$$

and
$$\text{Prob}[\,|Y_m^{(1)} - \underset{\sim}{Y}_m^{(1)}| \geqslant \eta_m] \to 0, \quad \text{as} \quad m \to \infty, \tag{155}$$

so long as
$$\eta_m \to \infty \quad \text{as} \quad m \to \infty.$$

We may express this state of affairs by

Theorem 6. *The average levels of both internal and external nodes in an s-ary tree remain, in probability, within finite bounds of the asymptotic forms* (143) *and* (144), *as* $m \to \infty$.

Another way of putting this is to say that the *distribution* of internal and of external average levels approaches a constant finite-variance distribution about the corresponding expectations.

8. CONCLUSIONS

Our main conclusions are contained in Lemma 3, Lemma 4, Corollary 3 (based on Assumption 1, and leading us to the generalized Assumption 2), Lemma 5, Lemma 6, Lemma 7, Theorem 1, Theorem 2, Theorem 5, and Theorem 6. Essentially, we record the fact that, as the number, m, of internal (occupied) nodes in a tree increases, the average levels of both internal and external (open) nodes approaches and remains boundedly close, in probability, to the corresponding expectations, which are asymptotic to $(1 + \theta) \log m$, where θ is $1 / (s - 1)$ for s-ary trees.

These results will affect the decision, whether or not to bother to adopt the procedure to store data in *balanced tree* structures, rather than in simple trees. With the average levels hewing so closely to their expectations, which we have shown to be asymptotic to 2 log m, in a binary tree (Corollary 6); the justification of the additional work of balancing becomes harder to find, unless the assumptions of randomness are sufficiently in question to make abnormally-unbalanced trees a real (and costly) probability. Unfortunately, precise results for balanced trees are not yet available (to the best of my knowledge): for an illuminating discussion of this and other matters, the reader is referred to Knuth[1].

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APPENDIX A

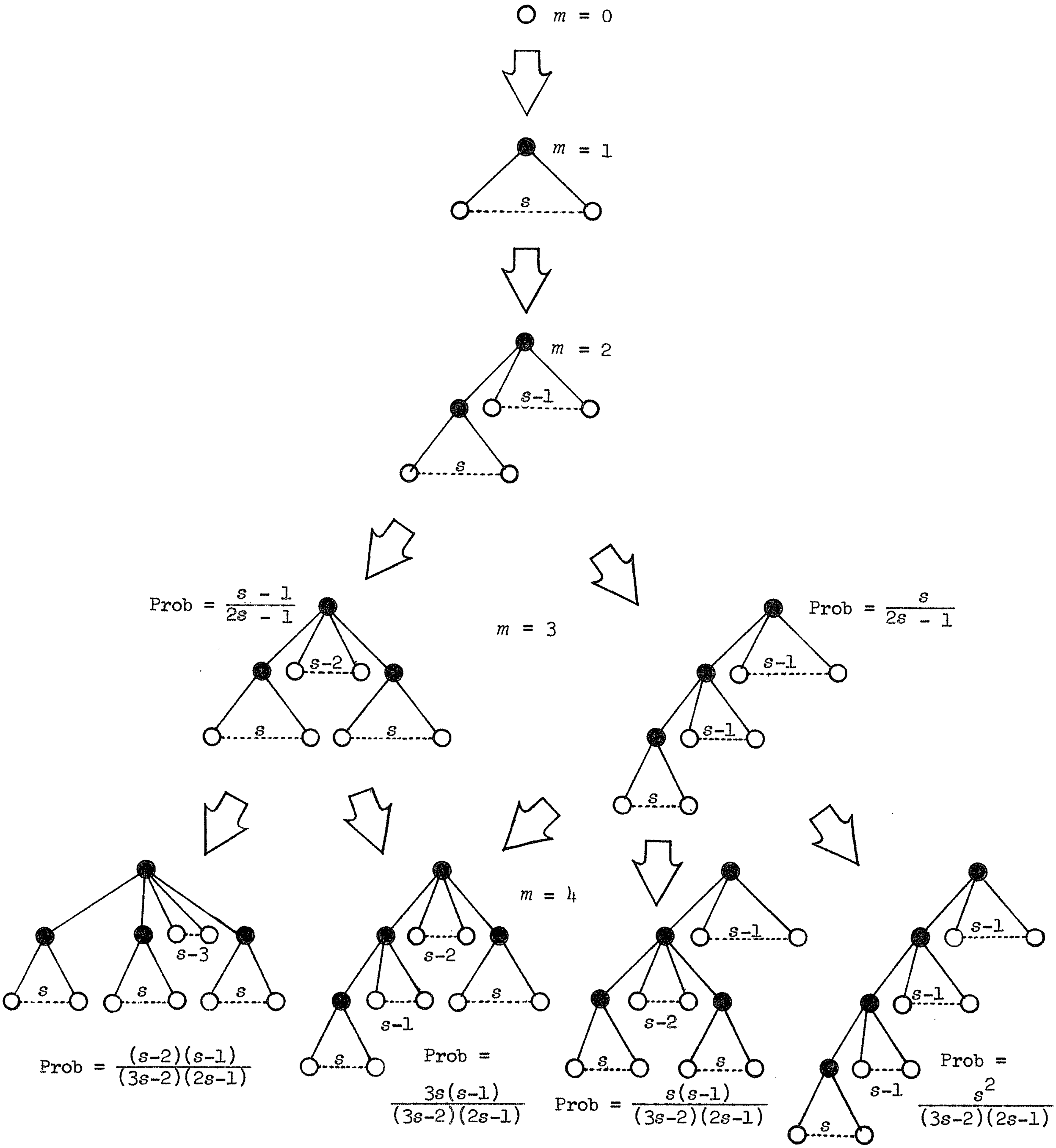
SPECIAL CASES

We list the values of the various statistics for an s-ary tree of m nodes, when m = 0, 1, 2, 3, and 4. We note that $Y_0^{(p)}$, $\mathbf{Y}_0^{(p)}$, and $\mathbf{H}_{pp'}^{(0)}$ are undefined: we take their values to be zero, by fiat. The figure shows the possible tree-configurations for the various cases being considered: black circles denote internal (occupied) nodes and white circles denote external (open) nodes. The arrows indicate which trees lead to which, as m increases; and "Prob" is the probability of the particular configuration, given m. These are the probabilities we use in computing the expectations $\mathbf{M}_k^{(m)}$, $\mathbf{N}_k^{(m)}$, $\mathbf{F}_m^{(p)}$, $\mathbf{E}_m^{(p)}$, $\mathbf{Y}_m^{(p)}$, $\mathbf{X}_m^{(p)}$, $\mathbf{B}_{kk'}^{(m)}$, $\mathbf{A}_{kk'}^{(m)}$, $\mathbf{D}_{pp'}^{(m)}$, $\mathbf{C}_{pp'}^{(m)}$, $\mathbf{H}_{pp'}^{(m)}$, and $\mathbf{G}_{pp'}^{(m)}$; from the values of $\mu_k^{(m)}$, $\nu_k^{(m)}$, $F_m^{(p)}$, $E_m^{(p)}$, $Y_m^{(p)}$, $X_m^{(p)}$, $\mu_k^{(m)}\ \mu_{k'}^{(m)}$, $\nu_k^{(m)}\ \nu_{k'}^{(m)}$, and so on. In these listings, we write

$$s = (1 + \theta) / \theta, \quad \mathbf{A}_{ii} = \delta_{ki}\ \delta_{k'i}, \quad \mathbf{A}_{ij} = \delta_{ki}\ \delta_{k'j} + \delta_{kj}\ \delta_{k'i} \quad (i \neq j),$$

$$\theta = 1 / (s - 1), \quad \mathbf{L}_{ii} = i^{p+p'}, \quad \mathbf{L}_{ij} = i^p\ j^{p'} + j^p\ i^{p'} \quad (i \neq j),$$

for brevity.



$m = 0$
$m = 1$
s
$m = 2$
$s-1$
s
Prob = $\frac{s-1}{2s-1}$
$m = 3$
Prob = $\frac{s}{2s-1}$
$s-2$
s
s
$s-1$
$s-1$
s
$m = 4$
$s-3$
s
s
s
$s-2$
$s-1$
s
s
$s-1$
$s-2$
s
s
$s-1$
$s-1$
$s-1$
s
Prob = $\frac{(s-2)(s-1)}{(3s-2)(2s-1)}$
Prob = $\frac{3s(s-1)}{(3s-2)(2s-1)}$
Prob = $\frac{s(s-1)}{(3s-2)(2s-1)}$
Prob = $\frac{s^2}{(3s-2)(2s-1)}$

m	$\mu_k^{(m)}$	Prob	$n^{(m)}$	$\nu_k^{(m)}$
0	0	1	1	δ_{k1}
1	δ_{k1}	1	$s \quad = \frac{1+\theta}{\theta}$	$\frac{1+\theta}{\theta}\ \delta_{k2}$
2	$\delta_{k1} + \delta_{k2}$	1	$2s-1 = \frac{2+\theta}{\theta}$	$\frac{1}{\theta}\ \delta_{k2} + \frac{1+\theta}{\theta}\ \delta_{k3}$
3	$\delta_{k1}+2\delta_{k2}$ $\delta_{k1}+\delta_{k2}+\delta_{k3}$	$\frac{s-1}{2s-1} = \frac{1}{2+\theta}$ $\frac{s}{2s-1} = \frac{1+\theta}{2+\theta}$	$3s-2 = \frac{3+\theta}{\theta}$	$\frac{1-\theta}{\theta}\delta_{k2}+2\frac{1+\theta}{\theta}\delta_{k3}$ $\frac{1}{\theta}(\delta_{k2}+\delta_{k3})+\frac{1+\theta}{\theta}\delta_{k4}$
4	$\delta_{k1}+3\delta_{k2}$ $\delta_{k1}+2\delta_{k2}+\delta_{k3}$ $\delta_{k1}+\delta_{k2}+2\delta_{k3}$ $\delta_{k1}+\delta_{k2}+\delta_{k3}+\delta_{k4}$	$\frac{(s-2)(s-1)}{(3s-2)(2s-1)} = \frac{1-\theta}{(3+\theta)(2+\theta)}$ $\frac{3s(s-1)}{(3s-2)(2s-1)} = \frac{3(1+\theta)}{(3+\theta)(2+\theta)}$ $\frac{s(s-1)}{(3s-2)(2s-1)} = \frac{1+\theta}{(3+\theta)(2+\theta)}$ $\frac{s^2}{(3s-2)(2s-1)} = \frac{(1+\theta)^2}{(3+\theta)(2+\theta)}$	$4s-3 = \frac{4+\theta}{\theta}$	$\frac{1-2\theta}{\theta}\delta_{k2}+3\frac{1+\theta}{\theta}\delta_{k3}$ $\frac{1-\theta}{\theta}\delta_{k2}+\frac{2+\theta}{\theta}\delta_{k3}+\frac{1+\theta}{\theta}\delta_{k4}$ $\frac{1}{\theta}\delta_{k2}+\frac{1-\theta}{\theta}\delta_{k3}+2\frac{1+\theta}{\theta}\delta_{k4}$ $\frac{1}{\theta}(\delta_{k2}+\delta_{k3}+\delta_{k4})+\frac{1+\theta}{\theta}\delta_{k5}$

m	$\underset{\sim}{M}_k^{(m)}$	$\underset{\sim}{N}_k^{(m)}$
0	0	δ_{k1}
1	δ_{k1}	$\frac{1+\theta}{\theta}\ \delta_{k2}$
2	$\delta_{k1} + \delta_{k2}$	$\frac{1}{\theta}\ \delta_{k2} + \frac{1+\theta}{\theta}\ \delta_{k3}$
3	$\delta_{k1}+\frac{3+\theta}{2+\theta}\delta_{k2}+\frac{1+\theta}{2+\theta}\delta_{k3}$	$[2\delta_{k2}+3(1+\theta)\delta_{k3}+(1+\theta)^2\delta_{k4}]/\theta(2+\theta)$

m	$\underset{\sim}{M}_k^{(m)}$	$\underset{\sim}{N}_k^{(m)}$
4	$\delta_{k1}+\frac{11+6\theta+\theta^2}{(3+\theta)(2+\theta)}\delta_{k2}+\frac{(6+\theta)(1+\theta)}{(3+\theta)(2+\theta)}\delta_{k3}+\frac{(1+\theta)^2}{(3+\theta)(2+\theta)}\delta_{k4}$	$\frac{6\delta_{k2}+11(1+\theta)\delta_{k3}+6(1+\theta)^2\delta_{k4}+(1+\theta)^3\delta_{k5}}{\theta(3+\theta)(2+\theta)}$

m	$\underset{\sim}{F}_m^{(p)}$	$\underset{\sim}{E}_m^{(p)}$
0	0	1
1	1	$\frac{1+\theta}{\theta}\ 2^p$
2	$1 + 2^p$	$\frac{1}{\theta}\ 2^p + \frac{1+\theta}{\theta}\ 3^p$
3	$1+\frac{3+\theta}{2+\theta}2^p+\frac{1+\theta}{2+\theta}3^p$	$[2^{p+1}+(1+\theta)3^{p+1}+(1+\theta)^2 4^p]/\theta(2+\theta)$
4	$1+\frac{11+6\theta+\theta^2}{(3+\theta)(2+\theta)}2^p+\frac{(6+\theta)(1+\theta)}{(3+\theta)(2+\theta)}3^p+\frac{(1+\theta)^2}{(3+\theta)(2+\theta)}4^p$	$\frac{6(2)^p+11(1+\theta)3^p+6(1+\theta)^2 4^p+(1+\theta)^3 5^p}{(3+\theta)(2+\theta)\theta}$

m	$\underset{\sim}{Y}_m^{(p)}$	$\underset{\sim}{X}_m^{(p)}$
0	[0]	1
1	1	2^p

m	$\underset{\sim}{Y}_m^{(p)}$	$\underset{\sim}{X}_m^{(p)}$
2	$(1 + 2^p)/2$	$[2^p + (1+\theta)\, 3^p]/(2+\theta)$
3	$\frac{1}{3} + \frac{(3+\theta)2^p + (1+\theta)3^p}{3(2+\theta)}$	$\frac{2^{p+1} + (1+\theta)3^{p+1} + (1+\theta)^2 4^p}{(3+\theta)(2+\theta)}$
4	$\frac{1}{4} + [(11+6\theta+\theta^2)2^p + (6+\theta)(1+\theta)3^p$ $+ (1+\theta)^2 4^p]/4(3+\theta)(2+\theta)$	$\frac{6(2)^p + 11(1+\theta)3^p + 6(1+\theta)^2 4^p + (1+\theta)^3 5^p}{(4+\theta)(3+\theta)(2+\theta)}$

m	$n^{(m)}$	$\nu_k^{(m)}$	Prob
0	1	δ_{k1}	1
	$\underset{\sim}{A}_{kk'}^{(0)} = \underset{\sim}{A}_{11}$		
1	$\frac{1+\theta}{\theta}$	$\frac{1+\theta}{\theta}\,\delta_{k2}$	1
	$\underset{\sim}{A}_{kk'}^{(1)} = \left(\frac{1+\theta}{\theta}\right)^2 \underset{\sim}{A}_{22}$		
2	$\frac{2+\theta}{\theta}$	$\frac{1}{\theta}\,\delta_{k2} + \frac{1+\theta}{\theta}\,\delta_{k3}$	1
	$\underset{\sim}{A}_{kk'}^{(2)} = \frac{1}{\theta^2}\underset{\sim}{A}_{22} + \frac{1+\theta}{\theta^2}\underset{\sim}{A}_{23} + \left(\frac{1+\theta}{\theta}\right)^2 \underset{\sim}{A}_{33}$		
3	$\frac{3+\theta}{\theta}$	$\frac{1-\theta}{\theta}\,\delta_{k2} + 2\,\frac{1+\theta}{\theta}\,\delta_{k3}$	$\frac{1}{2+\theta}$
		$\frac{1}{\theta}(\delta_{k2} + \delta_{k3}) + \frac{1+\theta}{\theta}\,\delta_{k4}$	$\frac{1+\theta}{2+\theta}$
	$\underset{\sim}{A}_{kk'}^{(3)} = [(2-\theta+\theta^2)\,\underset{\sim}{A}_{22} + (3-2\theta)(1+\theta)\underset{\sim}{A}_{23} + (1+\theta)^2\underset{\sim}{A}_{24} + (5+4\theta)(1+\theta)\underset{\sim}{A}_{33}$ $+ (1+\theta)^2\underset{\sim}{A}_{34} + (1+\theta)^3\underset{\sim}{A}_{44}]/\theta^2(2+\theta)$		

m	$n^{(m)}$	$\nu_k^{(m)}$	Prob
4	$\frac{4+\theta}{\theta}$	$\frac{1-2\theta}{\theta}\delta_{k2}+3\frac{1+\theta}{\theta}\delta_{k3}$	$\frac{1-\theta}{(3+\theta)(2+\theta)}$
		$\frac{1-\theta}{\theta}\delta_{k2}+\frac{2+\theta}{\theta}\delta_{k3}+\frac{1+\theta}{\theta}\delta_{k4}$	$\frac{3(1+\theta)}{(3+\theta)(2+\theta)}$
		$\frac{1}{\theta}\delta_{k2}+\frac{1-\theta}{\theta}\delta_{k3}+2\frac{1+\theta}{\theta}\delta_{k4}$	$\frac{1+\theta}{(3+\theta)(2+\theta)}$
		$\frac{1}{\theta}(\delta_{k2}+\delta_{k3}+\delta_{k4})+\frac{1+\theta}{\theta}\delta_{k5}$	$\frac{(1+\theta)^2}{(3+\theta)(2+\theta)}$

$$\underset{\sim}{A}^{(4)}_{kk'} = [(6-5\theta+6\theta^2-\theta^3)\underset{\sim}{A}_{22}+(11-12\theta+3\theta^2)(1+\theta)\underset{\sim}{A}_{23}+3(2-\theta)(1+\theta)^2\underset{\sim}{A}_{24}+(1+\theta)^3\underset{\sim}{A}_{25}$$
$$+(23+11\theta-5\theta^2)(1+\theta)\underset{\sim}{A}_{33}+(9+\theta)(1+\theta)^2\underset{\sim}{A}_{34}+(1+\theta)^3\underset{\sim}{A}_{35}+(8+7\theta)(1+\theta)^2\underset{\sim}{A}_{44}$$
$$+(1+\theta)^3\underset{\sim}{A}_{45}+(1+\theta)^4\underset{\sim}{A}_{55}]/\theta^2(3+\theta)(2+\theta)$$

m	$\mu_k^{(m)}$	Prob	$\underset{\sim}{B}^{(m)}_{kk'}$
0	0	1	0
1	δ_{k1}	1	$\underset{\sim}{A}_{11}$
2	$\delta_{k1}+\delta_{k2}$	1	$\underset{\sim}{A}_{11}+\underset{\sim}{A}_{12}+\underset{\sim}{A}_{22}$
3	$\delta_{k1}+2\,\delta_{k2}$	$\frac{1}{2+\theta}$	$\underset{\sim}{A}_{11}+\frac{(3+\theta)\underset{\sim}{A}_{12}+(1+\theta)\underset{\sim}{A}_{13}+(5+\theta)\underset{\sim}{A}_{22}+(1+\theta)\underset{\sim}{A}_{23}+(1+\theta)\underset{\sim}{A}_{33}}{2+\theta}$
	$\delta_{k1}+\delta_{k2}+\delta_{k3}$	$\frac{1+\theta}{2+\theta}$	
4	$\delta_{k1}+3\delta_{k2}$	$\frac{1-\theta}{(3+\theta)(2+\theta)}$	$\underset{\sim}{A}_{11}+[(11+6\theta+\theta^2)\underset{\sim}{A}_{12}+(6+\theta)(1+\theta)\underset{\sim}{A}_{13}+(1+\theta)^2\underset{\sim}{A}_{14}$
	$\delta_{k1}+2\delta_{k2}+\delta_{k3}$	$\frac{3(1+\theta)}{(3+\theta)(2+\theta)}$	$+(23+6\theta+\theta^2)\underset{\sim}{A}_{22}+(9+\theta)(1+\theta)\underset{\sim}{A}_{23}+(1+\theta)^2\underset{\sim}{A}_{24}$
	$\delta_{k1}+\delta_{k2}+2\delta_{k3}$	$\frac{1+\theta}{(3+\theta)(2+\theta)}$	$+(8+\theta)(1+\theta)\underset{\sim}{A}_{33}+(1+\theta)^2\underset{\sim}{A}_{34}$
	$\delta_{k1}+\delta_{k2}+\delta_{k3}+\delta_{k4}$	$\frac{(1+\theta)^2}{(3+\theta)(2+\theta)}$	$+(1+\theta)^2\underset{\sim}{A}_{44}]/(3+\theta)(2+\theta)$

<table>
<tr><th>m</th><th>$\mathbf{C}_{pp'}^{(m)}$</th><th>$\mathbf{G}_{pp'}^{(m)}$</th></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>$(\frac{1+\theta}{\theta})^2\mathbf{L}_{22}$</td><td>$\mathbf{L}_{22}$</td></tr>
<tr><td>2</td><td>$\frac{\mathbf{L}_{22}+(1+\theta)\mathbf{L}_{23}+(1+\theta)^2\mathbf{L}_{33}}{\theta^2}$</td><td>$\frac{\mathbf{L}_{22}+(1+\theta)\mathbf{L}_{23}+(1+\theta)^2\mathbf{L}_{33}}{(2+\theta)^2}$</td></tr>
<tr><td rowspan="2">3</td><td colspan="2">$(2-\theta+\theta^2)\mathbf{L}_{22} + (3-2\theta)(1+\theta)\mathbf{L}_{23} + (1+\theta)^2\mathbf{L}_{24} + (5+4\theta)(1+\theta)\mathbf{L}_{33} + (1+\theta)^2\mathbf{L}_{34} + (1+\theta)^3\mathbf{L}_{44}$</td></tr>
<tr><td>$\div\ \theta^2\ (2+\theta)$, for $\mathbf{C}_{pp'}^{(3)}$</td><td>$\div\ (3+\theta)^2\ (2+\theta)$, for $\mathbf{G}_{pp'}^{(3)}$</td></tr>
<tr><td rowspan="2">4</td><td colspan="2">$(6-5\theta+6\theta^2-\theta^3)\mathbf{L}_{22} + (11-12\theta+3\theta^2)(1+\theta)\mathbf{L}_{23} + 3(2-\theta)(1+\theta)^2\mathbf{L}_{24} + (1+\theta)^3\mathbf{L}_{25}$
$+ (23+11\theta-5\theta^2)(1+\theta)\mathbf{L}_{33} + (9+\theta)(1+\theta)^2\mathbf{L}_{34} + (1+\theta)^3\mathbf{L}_{35} + (8+7\theta)(1+\theta)^2\mathbf{L}_{44}$
$+ (1+\theta)^3\mathbf{L}_{45} + (1+\theta)^4\mathbf{L}_{55}$</td></tr>
<tr><td>$\div\ \theta^2\ (3+\theta)\ (2+\theta)$, for $\mathbf{C}_{pp'}^{(4)}$</td><td>$\div\ (4+\theta)^2\ (3+\theta)\ (2+\theta)$, for $\mathbf{G}_{pp'}^{(4)}$</td></tr>
</table>

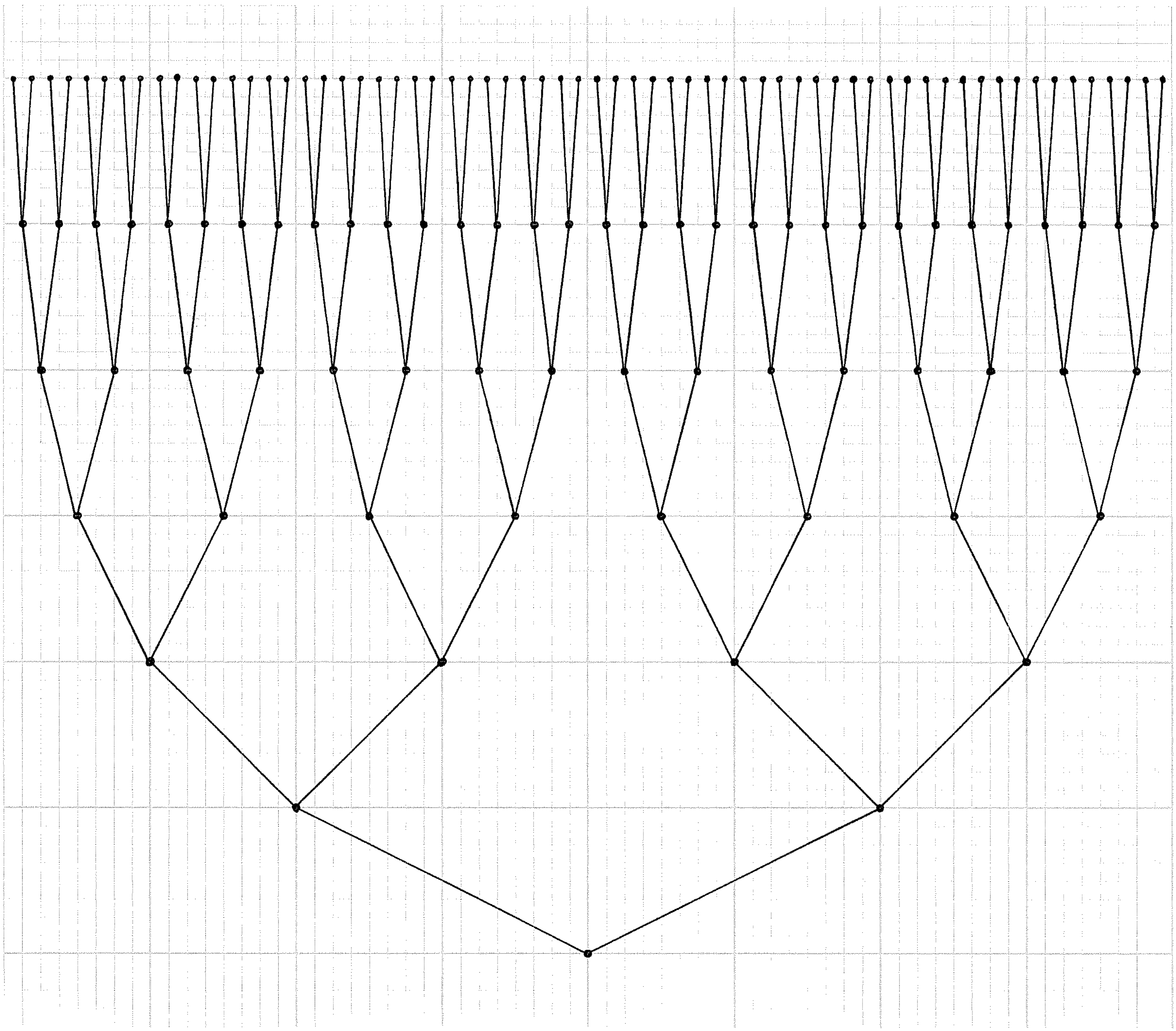
m	$\mathbf{D}_{pp'}^{(m)}$	$\mathbf{H}_{pp'}^{(m)}$
0	0	[0]
1	1	1
2	$1 + \mathbf{L}_{12} + \mathbf{L}_{22}$	$(1 + \mathbf{L}_{12} + \mathbf{L}_{22})/4$

m	$\underset{\sim}{D}^{(m)}_{pp'}$	$\underset{\sim}{H}^{(m)}_{pp'}$
3	$1 + \dfrac{(3+\theta)\underset{\sim}{L}_{12}+(1+\theta)\underset{\sim}{L}_{13}+(5+\theta)\underset{\sim}{L}_{22}+(1+\theta)\underset{\sim}{L}_{23}+(1+\theta)\underset{\sim}{L}_{33}}{2+\theta}$	
	thus for $\underset{\sim}{D}^{(3)}_{pp'}$	$\div 9$, for $\underset{\sim}{H}^{(3)}_{pp'}$
4	$(3+\theta)(2+\theta)+(11+6\theta+\theta^2)\underset{\sim}{L}_{12}+(6+\theta)(1+\theta)\underset{\sim}{L}_{13}+(1+\theta)^2\underset{\sim}{L}_{14}+(23+6\theta+\theta^2)\underset{\sim}{L}_{22}+(9+\theta)(1+\theta)\underset{\sim}{L}_{23}$ $+(1+\theta)^2\underset{\sim}{L}_{24}+(8+\theta)(1+\theta)\underset{\sim}{L}_{33}+(1+\theta)^2\underset{\sim}{L}_{34}+(1+\theta)^2\underset{\sim}{L}_{44}$	
	$\div (3+\theta)\ (2+\theta)$, for $\underset{\sim}{D}^{(4)}_{pp'}$	$\div 16\ (3+\theta)\ (2+\theta)$, for $\underset{\sim}{H}^{(4)}_{pp'}$

These particular values could clearly be indefinitely extended, with fast-increasing labor and complexity, as m increases. The values given here were used to check the general formulae derived in this paper.

APPENDIX B

CANONICAL BINARY TREE REPRESENTATION



Canonical representation of typical binary tree, illustrating the arguments of various Lemmas in §1.