

Analysis of Combining Musical Scale Vectors



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BACKGROUND

The motivation for this project stems from a paper by Sethares [1]. One of the topics this paper explores is how scale vectors that represent musically significant collections of notes (MSCN's), such as musical scales, chords, and intervals –essentially any collection of notes with a periodic structure– can be expressed as linear combinations of other types of MSCN's. These scale vectors are 12 by 1 (for the 12 notes of the chromatic scale) and consist of 1's and 0's, where 1's symbolize the inclusion of a note, while the 0's indicate the exclusion of a note. For example, the C major scale can be written as the scale vector

$$[1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1]'$$

where the numbers in the scale vector coincide with the ordering of the note names given by

$$[C\ C\#/D\flat\ D\ D\#/E\flat\ E\ F\ F\#/G\flat\ G\ G\#/A\flat\ A\ A\#/B\flat\ B]'$$

These scale vectors are actually column vectors, but we have written them as single rows here for readability (the apostrophe indicates that these are column vectors).

Sethares devised a method for creating combinations using these scale vectors, and provides a couple examples of this method in action. Our goal was to use his method to generate more combinations and improve upon his method based on our results.

MATLAB CODE

```
% f = basis vector, m = vector from a
different basis, n = size of the
vectors
function changebasis(f, m, n)
% obtain the circulant matrix based on
f:
S = f;
for z = 1:n-1;
    fz = circshift(f, z);
    S = [S fz];
end
% iterative method:
format rat
T = rref([S m]);
A = sparse(T(:,1:12));
b = T(:,13);
t.fqmr(A,b)
% Sethares' method with pseudoinverse
if needed:
    if inv(S) == pinv(S)
        [N,D] = rat((S\m)');
    else
        [N,D] = rat((pinv(S)*m)');
    end
end
[N]
[D]
```

METHODS

An aspect common to all of the following methods of creating combinations is the use of a circulant matrix. This is a 12 by 12 matrix built from the scale vector for the MSCN which we intend to represent a different MSCN as a combination of. In order to construct this, we consider every possible rotation of the scale vector (i.e. the MSCN starting on each note value, beginning with C, then C#, and so on). We set each rotation as a column of the circulant matrix. The following is an example of the circulant matrix for the interval of a major third:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Where the scale vector for this MSCN is $[1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]'$.

1. INVERTIBLE CIRCULANT MATRIX

When the circulant matrix is invertible, Sethares' method can be used. This involves solving the equation

$$S * x = m$$

for x , where S is the circulant matrix and m is the scale vector for a different MSCN. By multiplying both sides of the equation by the inverse of S , denoted S^{-1} , on the left, we see that

$$x = S^{-1} * m.$$

The x produced is a new 12 by 1 matrix containing the coefficients needed to produce a linear combination of the columns of the circulant matrix. In other words, this yields the recipe for the combination that we desired.

2. SINGULAR CIRCULANT MATRIX-PSEUDOINVERSE

When the circulant matrix is singular (non-invertible), a slightly different approach to Sethares' method is used. Everything from the method above remains the same, except the pseudoinverse of the circulant matrix is taken instead. The downside of this method is that sometimes it produces very complex combinations when a much simpler combination does exist.

3. SINGULAR CIRCULANT MATRIX-ITERATIVE METHOD

Overall, the pseudoinverse doesn't yield the simplest combinations possible, which we define to contain the most 0, 1, and -1 coefficients and least amount of non-integer coefficients. The use of iterative methods shows promise in simplifying results. Instead of solving the equation $S * x = m$ by multiplying both sides of the equation by S^{-1} , it is equivalent to concatenate the circulant matrix and scale vector into the 12 by 13 augmented matrix

$$[S | m]$$

and reduce this augmented matrix into reduced-row echelon form. Then we deconcatenate this reduced matrix back into two matrices A and b , whose sizes are 12 by 12 and 12 by 1 respectively. Lastly we feed A and b into the MatLab function "tfqmr," which stands for Transpose-free quasi-minimal residual method [2]. This is an iterative method for solving systems of equations in the same manner as the first method, though it makes an initial guess at the solution and uses more sophisticated mathematics to improve upon the guess. Iterative methods work best with large, sparse matrices, which is why we reduced the augmented matrix earlier.

RESULTS

We produced many combinations using all three methods. In particular, we focused on the octatonic scale (alternating whole and half steps between notes) due to Dr. Gary Don's recommendation [3]. Below are a few examples from our results:

1. Major triad chord in basis of intervals of major thirds

Using method 1, a major triad chord can be written as a combination of intervals of major thirds using the following coefficients:

$$\frac{1}{2}[2\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ -1]'$$

This translates to a major triad chord being $\frac{1}{2}$ the combination of 4 major thirds are minor third apart from one another. The first major third weighted by 2 and the last one is negative, which indicates that this major third is subtracted instead of added to the others.

2. Octatonic scale in basis of diminished sevenths

Using method 2, the octatonic scale can be written as a combination of diminished seventh chords using the following coefficients:

$$\frac{1}{4}[1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1]'$$

Using method 3, the following combination is possible:

$$[1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]'$$

Clearly method 3 yields a simpler result, which translates to an octatonic scale being equivalent to one diminished seventh chord on the tonic added to one diminished seventh chord on the note a minor third above the tonic.

FUTURE WORK

In the coming summer, we will collaborate with Dr. Gary Don of the UWEC Music Department in trying to connect our results with real musical literature. We intend to find musical examples which imply the existence of one MSCN using a sequence or combination of a different MSCN (i.e. implied existence of a C octatonic scale through the composer's use of both a C and an E \flat diminished seventh chord). In later semesters, we will explore the mechanisms behind iterative methods further. In addition, we will craft a simple method of representing one MSCN as a combination of two or more different types of MSCN's (i.e. The diatonic scale written as combination of both major and minor thirds simultaneously), which is briefly discussed in Sethares' paper.

REFERENCES

- [1] E. Amoit and W.A. Sethares. "An algebra for periodic rhythms and scales." *Journal of Mathematics and Music*, 5(3), 2011, pp. 149-169
- [2] MathWorks. (2015). tfqmr: Transpose-free quasi-minimal residual method. *MATLAB*.
- [3] Dr. Gary Don, Professor of Music Theory at UWEC, Consultation

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