

CONCEPTUAL UNDERSTANDING OF LINEAR RELATIONSHIPS ACROSS VARIOUS
MATHEMATICS COURSES

by

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ABSTRACT

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This cross-sectional study investigated the conceptual understanding of linear relationships for 195 students enrolled in a single school in a large, urban district across five mathematics courses: Grade 7 Math ($n = 24$), Grade 8 Math ($n = 52$), Geometry ($n = 43$), Algebra 1 ($n = 31$), and Algebra 2 ($n = 45$). The following questions guided this study: (1) What differences exist in students' conceptual understanding of linear relationships across mathematics courses? (2) What are common strengths and weaknesses in students' conceptual understanding of linear relationships?

An assessment was created to assess three constructs of conceptual understanding of linear relationships: (1) Identifying unit rates in proportional relationships, (2) Moving fluidly between representations of the same linear relationships, and (3) Interpreting different representations of linear relationships. The assessment contained eight multiple-choice items and seven free-response items, with five items assessing each construct. Each student completed the assessment in their regular mathematics course in a single class period. The assessment was scored and analyzed by mathematics course.

The overall test score and the score on each construct were analyzed using an ANOVA with Tukey and Games-Howell post hoc analyses. A significant difference was found in overall understanding of linear relationships across courses. The difference existed between students

enrolled in the follow mathematics courses: Grade 7 Math and Algebra 2, Grade 7 Math and Geometry, and Grade 8 Math and Algebra 2. Results also highlighted a significant difference between Grade 7 Math and Algebra 2 students in all three linear relationship constructs, with Algebra 2 students scoring significantly higher than Grade 7 Math students. Overall, students did not demonstrate a strong understanding of linear relationships, although Algebra 2 students were the most successful. The area in which students in all mathematics courses showed the greatest understanding was calculating unit rates in familiar contexts (e.g. speed, units per hour). Areas of weakness included interpreting linear relationships from any representation (e.g. table, graph, equation, verbal description) and moving fluidly between representations of linear relationships.

The results of this study suggested that students need to be given more opportunities to engage in learning experiences where they are interpreting multiple representations of the same linear relationship across all mathematics courses. Students should be asked to translate between tables, graphs, equations, and verbal representations of linear relationships in all directions (e.g., table to equation, verbal to equation, equation to graph). Curricular materials and learning experiences that only ask students to translate in select familiar directions between representations (e.g., table to graph, equation to graph) may be contributing to inequitable mathematics learning experiences and outcomes. Engaging in learning experiences in all directions and with all representations of linear relationships should help increase students' conceptual understanding of linear relationships. Effectively implementing conceptual-based curricular materials along with research-based best teaching practices will help provide a more equitable mathematics experience for all students.

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Dedication

To all of my students: past, present, and future. You inspire me to be the best teacher I can be and I promise to always advocate for your access to high-quality mathematics instruction.

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CHAPTER ONE

INTRODUCTION

Algebra is widely recognized as a gatekeeper to higher level mathematics courses, college acceptance and success, and other enhanced opportunities after high school (Boaler, 2006; Choike, 2000). While students across the United States take a formal algebra course at different times during their educational careers, they are introduced to algebraic concepts as early as first grade according to the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010).

In algebra, concrete numbers and ideas from earlier mathematics courses start transforming into abstract ideas through the use of variables. According to Witzel et al. (2003), "Algebra is considered a gateway to abstract thought" (p. 121). Courses such as calculus require students to be able to think abstractly, but they must understand algebraic foundations to be able to do this effectively. One of these foundational topics is the idea of a linear relationship between two quantities. These relationships can be represented verbally, graphically, in a table of values, or as an equation with two variables. Linear relationships are a major component of the mathematics curriculum in middle and high school, which is why they were chosen as the focus of this study.

Background

Students learn about linear relations starting in Grade 6 with proportional relationships and continue working with linear relations well into high school (NGA & CCSSO, 2010). Linear relations are a foundational idea in algebra, and without the understanding and ability to fluidly move between multiple representations of linear relations, students struggle with future

mathematical topics (Dreyfus & Eisenberg, 1982; Keller & Hirsch, 1998, Ozgun-Koca, 1998, 2001).

Linear Relationships

By definition, a relationship is linear if the value of an expression changes at a constant rate as the variable changes. This requires the power of the variable to be one, because any other power would cause changes in the value of the expression at a nonconstant rate. Students learn about the idea of a constant rate of change in a few different ways beginning in grade six (NGA Center & CCSSO, 2010). First, students learn about *ratio relationships* between two quantities, where as the first quantity increases by one, the other quantity increases or decreases by a constant rate, called the *unit rate*. In grade seven, students learn that ratio relationships are also called *proportional relationships* and can be represented using equations of the form $y = kx$, where k is a constant representing the unit rate, which is now called the *constant of proportionality*. Since the quantities in a proportional relationship are related by a constant rate of change, graphing these relationships gives us a straight line, which provides a visual representation showing that proportional relationships are linear. In grade eight, students graph proportional relationships and learn that the *unit rate* is called the *slope* of the graph. They then derive the equation $y = mx + b$, where m represents the slope, which is the constant rate of change and b represents the value at which the line intersects with the vertical axis. This equation models a linear relationship and therefore all equations in the form $y = mx + b$ are represented graphically by straight lines. In high school, students work to solve and graph single linear equations and systems of linear equations, as well as explore linear and nonlinear functions in more depth.

Up to 30 percent of the items on any given ACT test will cover algebra and functions, in addition to 40-43 percent of the test items assessing a students' ability to "integrate essential skills" (ACT, 2022). This being said, the understanding of functions and equations is essential to student success on college admissions exams, which will then impact their college experience. On the National Assessment of Educational Progress (NAEP), a student in Grade 12 who can "apply proportional reasoning, when necessary, to solve problems in nonroutine settings" (National Assessment Governing Board, 2017, p. 74), which calls for the student to have a conceptual understanding of a linear relation, would earn a *Proficient* score. In 2019, 40 percent of twelfth-graders who took the NAEP scored "below *NAEP Basic*" (National Center for Educational Statistics, 2019), which means that they could not successfully demonstrate this understanding.

Conceptual Understanding

Hiebert and Lefevre (1986) stated that conceptual knowledge is "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3). The relationship aspect of conceptual knowledge is something that has been echoed by other scholars. The National Research Council (2001) described conceptual understanding:

Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (p. 118)

Some scholars do not deem connections and relationships necessary to the definition of conceptual understanding or knowledge. Star (2005) argued that conceptual and procedural understanding are not black and white and that educators and scholars entangle the terms based on knowledge type and knowledge quality. As Star (2005) explained:

The term *conceptual knowledge* has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g., deeply and with rich connections). Similarly, the term *procedural knowledge* indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known (e.g., superficially and without rich connections). (p. 408)

Star argues that both procedural and conceptual knowledge contain a hierarchy of superficial knowledge and deep knowledge. Based on this idea, he created a 2x2 matrix to classify understanding in which he described procedural knowledge as typically superficial and conceptual knowledge as typically deep, but questioned whether deep procedural knowledge or superficial conceptual knowledge existed, and how they could be demonstrated if they exist. This leveled approach somewhat supports the argument that Hiebert and LeFevre (1986) also made about the descriptions of knowledge:

Not all knowledge can be usefully described as either conceptual or procedural. Some knowledge seems to be a little of both, and some knowledge seems to be neither.

Nevertheless, we believe that it is possible to distinguish between the two types of knowledge and that such a distinction provides a way of interpreting the learning process that helps us better understand students' failures and successes. (p. 3)

For the purposes of this study, *conceptual understanding* will refer to the ability to fluidly and precisely use and understand multiple representations to solve problems, based on a connected web of knowledge.

Research Problem

Sufficient research exists on students' understanding of the equal sign (Knuth et al., 2006; Matthews et al., 2012), but not much research has focused on the understanding of linear relationships. Rakes et al. (2010) noted that “students have difficulty assigning meaning to variables and fail to recognize the systemic consistency in the multiple uses of variables” (p. 373). This is an area of weakness for students with linear relationships. Students often view each representation of linear relations, such as equations, graphs, and tables as separate and independent instead of being able to see the connections and benefits of moving between them to solve problems (Birgin, 2012; Knuth, 2000; Wilkie & Ayalon, 2018). From my own teaching experience, I have found that students struggle to see connections between representations of linear relationships. For example, students may be able to graph a line when given a linear equation in slope-intercept form following a procedure they were taught, and are able to substitute a value into that same equation for x to solve for y , but the connection between their calculated y -values and ordered pairs on their graphed line is not immediately clear to them. When students are not able to make connections between representations, they are working at a procedural level and are not necessarily understanding the meaning behind what they are doing, which shows they do not have a strong conceptual understanding of linear relationships. The lack of conceptual understanding and ability to move between representations of linear relationships contributes to a performance gap between American students and students in other countries

(Brenner et al., 1999) and an achievement gap on mathematics standardized tests in general (Beyranevand, 2010).

In 2008, the National Mathematics Advisory Panel (NMAP) suggested that the federal government should be prioritizing research in four different areas. Specifically, NMAP called for research in four areas: “(a) effective instructional practices and materials, (b) mechanisms of learning, (c) ways to enhance teachers’ effectiveness, including teacher education that focus on learning processes and outcomes, and (d) item and test features that improve the assessment of mathematical knowledge” (NMAP, 2008, p. 63). My research will help fill two areas of need presented by the Panel, by examining the conceptual understanding of linear relationships for students enrolled in mathematics courses from Grade 7 Math through Algebra 2. The mechanisms of learning is addressed by bringing attention to gaps in students’ conceptual understanding of linear relationships, which can help teachers adjust their teaching strategies and how they engage students in conceptual thinking. The results of this study will also address item and test features that improve the assessment of mathematical knowledge, by helping curriculum and standards authors think about what types of items should be in curriculum materials and on assessments to help students practice and demonstrate their conceptual understanding of linear relationships. Hopefully, this study will provide valuable information to add to the collective pool of studies that are being done in the mathematics education community.

Purpose Statement

The purpose of this cross-sectional study is to determine the level of conceptual understanding that students have of linear relationships and their applications through the use of multiple representations at different stages in school for students enrolled in Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, and Algebra 2 at a small, urban school in the Midwest.

Students enrolled in Grade 7 Math through Algebra 2 will take the same assessment on linear relationships and will be scored using the same rubric.

I will be comparing scores and levels of student understanding by mathematics course, in order to see how conceptual understanding is impacted by repeated exposure to similar topics with linear relations in multiple mathematics courses. Students should, in theory, develop stronger conceptual understanding of linear relationships as they have repeated exposure and increased learning opportunities with this content area across the mathematics courses.

To learn more about students' conceptual understanding of linear relationships throughout middle and high school, the following questions were investigated:

1. What differences exist in students' conceptual understanding of linear relationships across mathematics courses?
 - a. Is there a difference in overall conceptual understanding of linear relationships between mathematics courses for students enrolled in Grade 7 Math through Algebra 2?
 - b. Are there differences in students' conceptual understanding of constructs for linear relationships in Grade 7 Math through Algebra 2?
2. What are common strengths and weaknesses in students' conceptual understanding of linear relationships?
 - a. What are overall strengths and weaknesses in the conceptual understanding of linear relationships for middle and high school students?
 - b. What are course-specific strengths and weaknesses in the conceptual understanding of linear relationships for students in Grade 7 Math through Algebra 2?

Conceptual Framework

Three constructs essential to the conceptual understanding of linear relationships were identified for analysis in this study based on the standards and progression in the CCSSM domains of Expressions and Equations (EE) and Functions (F) and the test frameworks for the NAEP, Partnership for Assessment of Readiness for College and Careers (PARCC), and American College Test (ACT) assessments. The three constructs are:

Construct 1: Identifying unit rates in proportional relationships

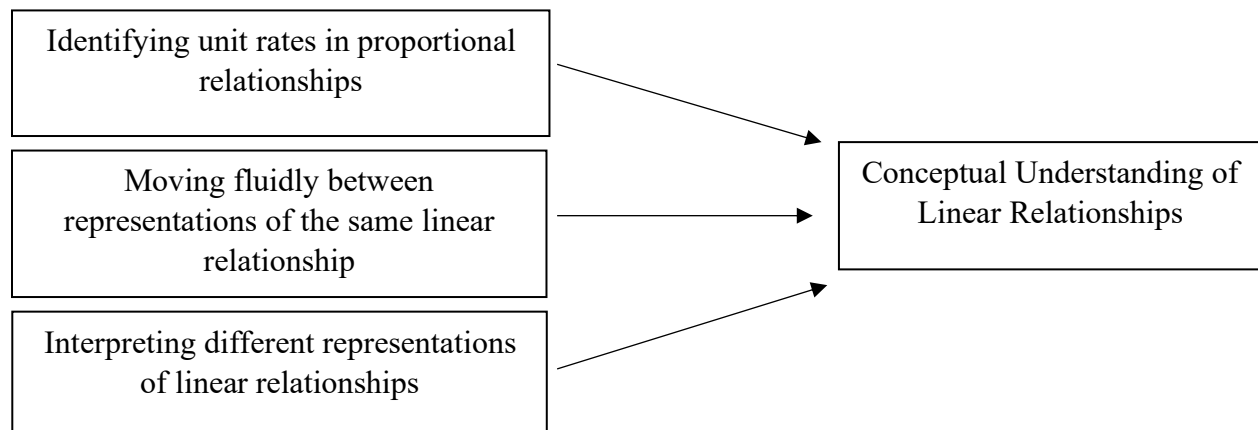
Construct 2: Moving fluidly between representations of the same linear relationship

Construct 3: Interpreting different representations of linear relationships

Figure 1.1 shows the conceptual framework that guided this study. Each construct on the left contributes to an aspect of conceptual understanding. When proficiency is demonstrated in all three constructs, it can be concluded that a student has a robust conceptual understanding of linear relationships.

Figure 1.1

Conceptual Framework Guiding this Study



Overview of Methodology

An assessment of linear relationships was created for use as the instrument in this study using items from other assessments (e.g. NAEP and curricular materials). Each of the three constructs in the conceptual framework for this study are assessed by five test items, resulting in an assessment with a total of 15 items. Students enrolled in Grade 7 Math through Algebra 2 were introduced to the study and asked to bring back a signed student assent and parent consent form if they wanted to participate in the study. The students who elected to participate took the assessment, which were then scored and analyzed.

To address the first research question, a one-way analysis of variance (ANOVA) with post-hoc tests was conducted to determine whether differences in conceptual understanding of linear relationships existed between mathematics courses. To answer research question 1a, an ANOVA was conducted using mathematics courses Grade 7 Math through Algebra 2 as factors and participants' total scores on the test as the dependent variable to determine whether differences existed in the overall score of conceptual understanding of linear relationships between grade levels. To answer research question 1b, a sub-score was calculated for each of the three defined constructs for each participant. Then, a separate ANOVA was conducted for each construct, again using mathematics courses as factors and sub scores as the dependent variables. In addition to the ANOVA test, post-hoc Tukey or Games-Howell analyses for each ANOVA calculation were conducted to determine pairwise comparisons and extract more meaningful information about the differences in conceptual understanding of linear relationships between mathematics courses.

To address the second research question, a qualitative content analysis was conducted by item within each construct. For multiple choice items, the distribution of chosen answers was

examined for all students by grade level. For free-response items, the common strategies that were found within the responses for students overall were identified as well as the common strategies within each grade level. I grouped my analyses by each construct and drew inferences about common strengths and weaknesses within each construct for students as a whole and for each mathematics course.

Significance

Students are asked to model using mathematics in upper-level mathematics classes as well as on standardized tests. If they do not conceptually understand the meaning of a linear relation or how different representations of linear relations are connected, they will likely struggle later on in their mathematics careers and might not be able to think algebraically to solve real-world problems. If a lack of growth in mathematical understanding occurs from Grade 7 Math to Algebra 2, we are doing something wrong as mathematics educators, and students are somehow missing out or going backwards in their mathematical understandings.

This study is significant because it has the potential to show the effectiveness or areas for improvement in mathematics standards and curricula. If students do not have higher levels of conceptual understanding in later mathematics courses than they do in earlier courses, the mathematics teams at schools and mathematics researchers in general may want to do further investigation as to why this is the case. The findings from this study can help teachers, school and district leaders, and curriculum developers evaluate their mathematics curriculum as it relates to linear relationships and identify areas in need of improvement.

Summary

Conceptual understanding and being able to critically think through unfamiliar situations are essential to success in mathematics. Algebra in particular builds on prior knowledge and

understanding of linear relationships that students start learning in Grade 6. A student's conceptual understanding of linear relationships and ability to move fluidly between representations can greatly impact their future success, whether it is in mathematics classes such as algebra, calculus, and statistics, or in the workforce. The focus of this study was to determine how conceptual understanding of linear relations differs among students enrolled in middle and high school mathematics courses. In Chapter 2, a review of relevant literature is provided about conceptual understanding, mathematics standards, and student achievement in regards to algebra and linear relations. Chapter 3 provides a detailed overview of the methodology used to conduct this study. Chapter 4 provides the results from the data analysis for the study. In Chapter 5, I discuss my findings, along with the limitations of those findings and how they may be used for further research.

Definition of Terms

In order to ensure a consistent understanding of the terms that are used in this study, the following definitions will be referenced throughout this paper.

- **relation (relationship):** Two quantities that are related such that a change in one quantity causes a change in the other quantity.
- **linear relation:** Two quantities that are related such that a change in one quantity causes a change in the other quantity based on a constant rate of change.
- **equation:** A mathematical statement that two expressions are equal, with an equal sign in between the expressions.
- **linear equation:** A statement that two expressions are equal with a constant rate of change between them.
- **function:** A relationship in which every input value in the domain has exactly one output value that makes the relationship true.
- **linear function:** A function which is also a linear relationship.
- **conceptual understanding:** The ability to fluidly and precisely use and understand multiple representations to solve problems, based on a connected web of knowledge.

CHAPTER TWO

REVIEW OF THE LITERATURE

This chapter provides a synthesis of the literature related to the study. First, I will provide an overview of my literature search strategy. Then, I will provide background information on linear relationships, conceptual understanding, and representational fluency. Next, I will discuss the conceptual framework utilized in this study, as well as the rationale for each of the constructs within the framework. Finally, I will address equity in mathematics education and how the inequity of our education system may be impacting many of our students' conceptual understanding of linear relationships.

Literature Search Strategy

The search for relevant literature to this study was completed using a variety of methods. First, the University of Wisconsin-Milwaukee Library online databases were used to find published studies pertaining to linear relationships or conceptual understanding using the following keywords: linear relationships, linear relations, linear equations, conceptual understanding, conceptual understanding of linear relationships, middle school linear relationships, high school linear relationships, middle school mathematics understanding, high school mathematics understanding, unit rates, unit rates proportional, representations of linear relationships, and representational fluency. Next, the same databases were searched to find relevant literature about the cross-sectional study design using the keyword cross-sectional design. Finally, the University of Wisconsin-Milwaukee Library online databases were used to search for relevant literature pertaining to equity in mathematics education using the following keywords: equity in mathematics education, inequity in mathematics education, equity-based mathematics, racial gap in mathematics, and tracking in mathematics. Books from the National

Council of Teachers of Mathematics (NCTM) were also consulted in the literature search. The reference pages of relevant studies helped point me to additional sources of information for use in this study.

Linear Relationships

The identifying factor of a linear relationship is the existence of a constant rate of change between two variables. Students learn about the idea of a constant rate of change in a few different ways beginning in grade six. First, students learn about *ratio relationships* between two quantities, where as the first quantity increases by one, the other quantity increases or decreases by a constant rate, called the *unit rate* (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). In grade seven, students learn that ratio relationships are also called *proportional relationships* and can be represented using equations of the form $y = kx$, where k is a constant representing the unit rate, which is now called the *constant of proportionality*. For example, as stated in the Common Core State Standards for Mathematics (CCSSM), “if total cost t is proportional to the number n of items purchased at a constant price p , the relationship between the total cost and the number of items can be expressed as $t = pn$ ” (NGA & CCSSO, 2010). Since the quantities in a proportional relationship are related by a constant rate of change, graphing these relationships gives us a straight line, which provides a visual representation showing that proportional relationships are linear. In grade eight, students graph proportional relationships and learn that the *unit rate* is called the *slope* of the graph. They then derive the equation $y = mx + b$, where m represents the slope, which is the constant rate of change and b represents the value at which the line intercepts the vertical axis. This equation models a linear relationship and therefore all equations in the form $y = mx + b$ are represented graphically by straight lines. In high school, students work to solve and graph single linear

equations and systems of linear equations, as well as explore linear and nonlinear functions in more depth.

Conceptual Understanding of Linear Relationships

In mathematics, the same idea can often be represented in multiple ways, and students are exposed to multiple representations from very early in their mathematics education. Hiebert and Lefevre (1986) stated that conceptual knowledge is “knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information” (p. 3). Lesh et al. (1987) acknowledged the importance of understanding individual representations, but emphasized the importance of being able to translate between different representations as a prerequisite for understanding larger mathematical concepts. Moschkovich et al. (1993) acknowledged the importance of students understanding procedural skills, but asserted that “we should teach for connections and understanding, not merely for procedural skills” (p. 98). When looking through the lens of linear relationships, we should see students connecting symbolic representations through the use of variables and equations or tables with visual representations of graphs along with contextual representations of a real-world scenario to show a robust conceptual understanding of linear relationships.

Identifying the Unit Rate in Proportional Relationships

The knowledge and understanding of a unit rate is foundational to proportional reasoning, linear relationships, and advanced mathematical thinking (Beigie, 2016; Carney & Crawford, 2016; Herbert, 2010; Herbert & Pierce, 2011; Wieland et al., 2020). Ercole et al. (2011) shared that “to determine a unit rate, students must first recognize that a relationship exists between the two given quantities and then calculate the rate when one of the quantities is 1” (p. 484). When

students have gained an understanding of unit rates, they can use unit rates to solve a wide range of problems. For missing value problems involving proportional relationships, unit rates can be calculated and then used to scale and create equivalent ratios to find unknown values in a given situation (Carney et al., 2023). Students may use the unit rate to think additively and “build-up” to a missing quantity, however this strategy can “become cumbersome when the problem contains non-integer ratios” (Tourniaire & Pulos, 1985). The foundational understanding of unit rates can help students understand and move to using additional strategies with proportional relationships.

In the curricular materials used in the school of study, students are often asked to calculate unit rates for speed with distance per unit of time, cost per item, and other familiar contextual situations with units of time being present in most examples (College Board, 2014). Herbert (2010) conducted a phenomenographic investigation of twenty Australian Year 10 students’ conceptions of rate. They observed video-taped interviews, from which they organized students’ ideas of rates into eight categories. These categories led Herbert to focus on the impact of different contexts on understanding of rate. Their findings highlighted students’ familiarity with units of time. They found that more students were able to express a unit rate in a distance-time context than were able to express a unit rate in an area-height context. It was hypothesized that students brought their contextual knowledge of speed to the distance-time context while not being able to transfer that knowledge and apply it to the area-height context.

The constant rate of change in a linear relationship is fundamental to understanding what is happening in that relationship. In eighth grade, students learn how to graph a linear relationship in the form $y = mx + b$ and are expected to know that m represents the slope of the line and b represents the y-intercept of the line. Students should be able to utilize their

understanding of rates of change and different representations of linear relationships to solve problems (NGA & CCSSO, 2010).

Representational Fluency of Linear Relationships

As defined by Fonger (2019), representational fluency is “the ability to create, interpret, translate between, and connect multiple representations” within a mathematical topic. In order to be able to fluidly move between representations, students must understand the components of the representations. In the case of linear relationships, students would need to be able to identify where the constant rate of change or unit rate are used in each representation and be able to connect those parts between representations. NCTM (2000, 2014) highlighted that being able to use multiple representations in mathematics supports students’ conceptual and procedural understanding of mathematical topics. Specifically, NCTM (2000) shared that “an understanding of the meanings and uses of variables develops gradually as students create and use symbolic expressions and relate them to verbal, tabular, and graphical representations” (p. 225). In this study, if a student is able to show an understanding of different representations of linear relationships and is able to compare linear relationships that are represented in different ways, they will be showing that they have a strong conceptual understanding of linear relationships.

Linear relationships can be represented with equations, graphs, tables, and verbal descriptions. The ability to fluidly move between these representations when looking at the same linear relationship shows that students are able to identify key aspects such as the rate of change in all representations. Wells (2016) described a card sort activity where students were given cards with graphs, tables, equations, and written contexts on them, and asked to compile groups of cards which represented the same linear relationship. As Wells declared, “Students who deeply understand linear functions are able to make connections among representations and fluently

translate among them” (p. 311) While this study does not focus on identifying solutions to linear equations, prior research concluded that students struggle to use representations outside of an algebraic equation to solve problems (Birgin, 2012; Knuth, 2000).

Moschkovich et al. (1993) attempted to create a tutoring curriculum to help students gain a deeper understanding of the interconnectedness of representations of linear relationships through different investigations. Through their tutoring program, they found that students were able to work with a single representation of a linear relationship, but struggled when they were forced to change perspectives on a mathematical problem and use different representations to solve that problem. They highlighted that “procedural competence is a component of understanding, but it should not be mistaken for the real thing” (p. 98) and emphasized that students need to be engaged in learning experiences where they are developing an understanding of translating between multiple representations of linear relationships.

Representational Fluency in Classroom Learning Experiences

Brenner et al. (1997) shared that “symbol manipulation skills and word-problem representations skills are cognitive prerequisites for success in algebra, yet traditional instruction may focus on symbol manipulation skills at the expense of representation skills” (p. 684). This sentiment was echoed by Kieran (2007), who shared that in traditional algebra programs, the manipulation of polynomial and rational expressions often took precedence over working with different representations of functions. NCTM (2014) shared that teachers should “use and connect mathematical representations” as a practice of effective mathematics teaching. The five types of representations presented were physical, visual, symbolic, verbal, and contextual. The argument was made that if students can use and connect different representations of a mathematical concept, they will have a deeper understanding of those mathematics concepts. As

shared by NCTM (2014), “The depth of understanding is related to the strength of connections among mathematical representations that students have internalized” (p. 25).

Ozgun-Koca (1998) conducted a study observing college-level freshmen and their choices in representations in their mathematics course. They found that students favored whichever representations their instructors spent more time teaching. As a result of their findings, they suggested that “if we, as educators, provide an environment with multiple representations instead of favoring a particular representation, we will enable students to experience different representations and choose the representation that is most meaningful for them” (p. 13).

Arnold and Son (2011) concluded that the CCSSM emphasizes that students should be able to make connections between descriptions, tables, and graphs. When they analyzed five different sets of curricular materials used in the United States, all but one of the curricula put little to no emphasis on this type of reasoning and sense making to connect representations of linear relationships. Instead, many focused on using given equations and formulas to calculate slopes and y-intercepts. Four of the five texts analyzed presented over 80% of their problems for students without context. Their results “revealed a significant gap between learning goals as set forth in intended curricula, and the potentially implemented curricula contained in many current U.S. Textbooks” (p. 389).

Moseley (2005) discussed student work with understanding rational numbers in various representations. He found that when students were engaged in instruction using multiple perspectives (i.e., representations) of rational numbers, they demonstrated a greater ability to use multiple representations in their thinking on a final task than students who were instructed using a single representation. This supports conclusions of existing literature that students need to

receive instruction on multiple representations if they are expected to be able to solve problems involving multiple representations (Brenner et al., 1997).

Knuth (2000) studied 178 high school mathematics students enrolled in Algebra 1 through Calculus, examining their understandings of the connections between algebraic and graphical representations of functions. He found that when asked to find a solution to a problem, the majority of students preferred to use an algebraic equation, even when a different representation was presented or would have been more efficient to solve the problem. He shared that “perhaps the most significant influence on students’ choices of solution methods is a curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation” (p. 505). Wilkie and Ayalon (2018) conducted an investigation of the understanding of linear functions of 215 secondary Australian students from Years 7 to 12, where they were given three tasks with four representations of linear relationships. They found that students’ understanding of linear functions was context-specific and they were not able to make connections between representations. Wilkie and Ayalon called for future research to be done on students’ exposure to and ability represent the same linear relationship pattern with verbal, tabular, algebraic, and graphical representations. This study will contribute to the greater research pool on this topic.

Birgin (2012) studied Turkish students’ understanding of the slope of linear functions and found that most students memorized the procedure to identify the slope from an equation and graph the line to represent the equation, but they struggled to make the connection from graph to equation or between any other representations. Overall, he found that “much confusion existed regarding the connection between the algebraic and geometric aspects of slope of a line” (p. 155). This connection can be referred to as the “Cartesian connection,” which is “considered a

critical translation for students to learn to make – between algebraic equations and their graphs” (Wilkie & Ayalon, 2018, p. 504).

Davis (2007) studied eight students in a high school mathematics classroom who were investigating the concept of a y-intercept, or “start value” in various representations of linear functions. Students were developing their own terminology for real-world situations and creating tables, graphs, and equations to represent these situations. Davis noted that the majority of students used the term “y-intercept” when discussing a table or a graph whose line went outside the first quadrant, and used “starting point” when discussing a verbal real-world context or a graph whose line was strictly in the first quadrant. Only one of the eight students was able to make the connection that the “starting point” was the same thing as the y-intercept and translate between representations. This finding demonstrates that students do not always see the connections between representations of linear relationships and need to have more learning experiences with connecting various representations.

Amit and Fried (2004) videotaped lessons on systems of equations for fifteen days. The teacher whose classes were being filmed was purposely emphasizing different representations of linear equations as she was teaching. The results of the study showed that the students in the class saw each representation as a solution strategy, rather than representing the same linear relationships in different ways. It seemed as though the students did not understand the individual representations such as algebraic equations and graphs as being related to each other, but only knew they were able to get a solution to the system of equations by making a graph. These students were lacking the foundational understanding of the connection between representations of a single linear relationship.

Representational Fluency on Standardized Assessments and Mathematics Achievement

The ability to connect multiple representations of linear relationships may have an impact on standardized test achievement. Beyranevand (2010) analyzed the association between 443 seventh and eighth grade students' achievement on standardized tests and their abilities to work with multiple representations of linear relationships. Students were asked to solve problems from three different representations: verbal, pictorial, and algebraic. He found "a strong association between students' achievement on standardized tests and their ability to recognize different representations of the same linear relationship with one unknown" (p. 80). Beyranevand's study was focused on using multiple representations to solve linear equations with one variable, but similar findings could appear with being able to compare two linear relationships represented differently or being able to move fluidly between representations of a single linear relationship. In multiple studies, students who were "low-achieving" as evidenced by pre-tests and placement in remedial college mathematics classes shared that they preferred to focus on one representation when solving a problem instead of using different representations as a strategy to solve problems (Dreyfus & Eisenberg, 1982; Keller & Hirsch, 1998, Ozgun-Koca, 1998, 2001). Students' inability or lack of confidence in utilizing multiple representations seems to have a clear impact on their performance and ability to attain high levels of achievement in mathematics.

In their study assessing problem-representation skills of students from the United States, China, Taiwan, and Japan, Brenner et al. (1999) found that the Asian students outperformed the American students, although students across the study had more difficulty with identifying different representations of a given scenario than they did with simply finding a solution to a given problem. They shared that "the gap in performance between the American students and the three Asian groups was greater for the representational items than for the solution items" (p.

552). This supports the aforementioned studies' findings that students who are not able to move between representations see low achievement in mathematics.

Standards Assessed in this Study

The standards pertaining to linear relationships are primarily within the Expressions and Equations (EE) domain of the CCSSM (NGA & CCSSO, 2010), although some are in the Ratios and Proportions (RP) domain in sixth grade and in the Functions (F) domain in eighth grade. As I read through the CCSSM standards in the EE, RP, and F domains, I found common themes, which helped me create the framework for this study.

One large aspect of linear relationships in the CCSSM that was not included in this study is the idea of solving for a variable in a linear equation, and understanding what it means to be a solution to a linear equation. I chose not to include this because I wanted to focus on students' abilities to connect the various representations of linear relationships which include more than algebraic equations. Solving for a variable in a linear equation is also a relatively procedural task, so it does not fit with the purpose of this study addressing conceptual understanding of linear relationships. This being said, I chose to focus on the standards that address real world applications, analyzing relationships between different representations of linear relationships, and understanding the unit rate or constant rate of change in real world settings. Table 2.1 lists the standards that are assessed by each construct and the full standards with descriptions can be found in Appendix A.

Table 2.1

Constructs of Conceptual Understanding of Linear Relationships with Descriptions and Associated Standards

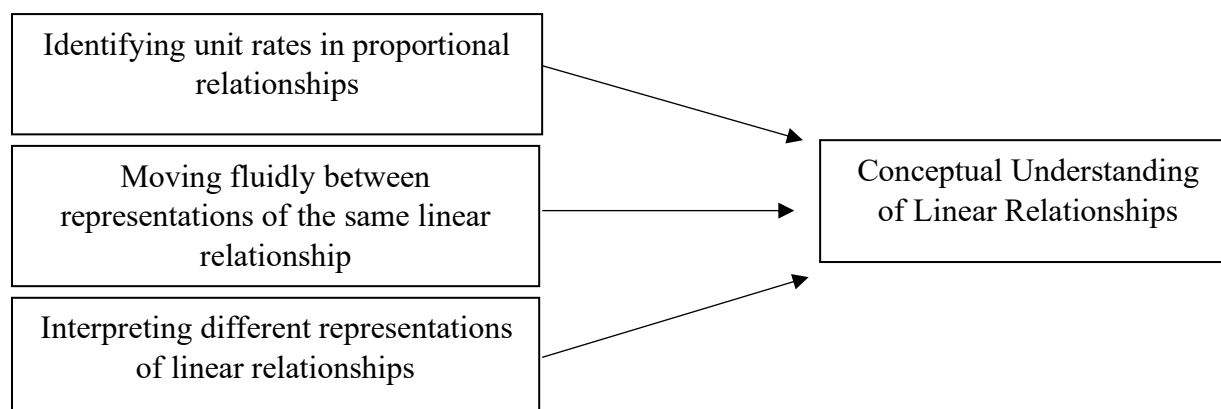
Construct	Description	Standards
1. Identifying unit rate in proportional relationships	Students can identify the unit rate/constant of proportionality and represent a proportional relationship algebraically with an equation	CCSS.MATH.CONTENT.7.RP.A.2 CCSS.MATH.CONTENT.7.RP.A.2.B CCSS.MATH.CONTENT.7.RP.A.2.C
2. Moving fluidly between representations of the same linear relationship	When given one representation of a linear relationship, students will be able to identify or provide at least one other representation of the same relationship.	CCSS.MATH.CONTENT.6.EE.B.7 CCSS.MATH.CONTENT.6.EE.C.9 CCSS.MATH.CONTENT.7.EE.B.4 CCSS.MATH.CONTENT.8.F.B.4
3. Interpreting different representations of linear relationships	Students can identify differences between relationships when given two different representations of linear relationships	CCSS.MATH.CONTENT.8.EE.B.5 CCSS.MATH.CONTENT.8.F.A.2 CCSS.MATH.CONTENT.8.F.B.5

Conceptual Framework

The conceptual framework being used in this study was created using the standards found in the CCSSM (NGA & CCSSO, 2010), the progressions documents (Common Core Standards Writing Team, 2011), as well as curricular materials and standardized assessments. A diagram showing the conceptual framework is shown in Figure 2.1. Three constructs were identified as leading to an overall conceptual understanding of linear relationships. A more detailed explanation of the conceptual framework and the justification for these three constructs is included in Chapter 3.

Figure 2.1

Conceptual Framework



Equity in Mathematics Education

Every student in America has to take mathematics courses throughout their school career. Unfortunately, not every student in America is given access to the same mathematical content or educational experiences. As Aguirre et al. (2017) shared, “there is a long-standing thoroughly documented, and seemingly intractable problem in mathematics education: inequity. Children of certain racial, ethnic, language, gender, ability, and socioeconomic backgrounds experience mathematics education in school differently, and many are disaffected by their mathematics education experience” (p. 125). The inequitable experiences in mathematics education compounded over thirteen years of K-12 schooling can have detrimental effects on students’ futures. As the NCTM (2018) shared,

each and every student’s access to a high-quality mathematics education is critical, given the importance of mathematics to each student’s continued postsecondary education and professional opportunities and to every student’s understanding of the wide variety of economic, social, medical, and environmental issues in society and in the student’s own life. (p. 15)

As research has shown, mathematics education has the power to impact a student's academic, personal, and professional lives (Burdman, 2018; Knudson, 2019; NCTM, 2018; National Research Council, 2001), so it is imperative that each and every student is given access to a high-quality mathematics education. Linear relationships can be used to represent a wide variety of situations in the real world, so if students are able to gain a conceptual understanding of linear relationships, they will be able to use them as tools to solve problems in their daily lives.

Providing inequitable mathematics experiences produces an unjust mathematics education system, which prohibits students from being able to take higher level mathematics courses and can limit postsecondary and career options (Burdman, 2018). The systems that have been in place in mathematics education at certain times in United States history have been blatantly racist, and awareness of this has impacted some policies and systemic changes, however some systemic barriers still impact Black and other minority students at disproportionately high levels, which maintains an implicit system of silent racism in mathematics education.

The population of the school where this study is taking place has a majority Black student population (Wisconsin Department of Public Instruction [DPI], 2021). Since the majority of the student population is Black, the focus of this equity section will primarily be on Black students. While many factors contribute to the unjust and inequitable mathematics experiences that so many students in the United States face, the focus of the discussion in this section will be systemic barriers in education, such as student population and funding, access to high quality instruction, levels of expectations, and classroom learning experiences that work together to provide inequitable and unjust mathematics experiences for Black students, and how these inequitable mathematics experiences may impact students' opportunities to gain a conceptual understanding of linear relationships.

The United States has a long history with racism. Many systems were put in place to keep Black children from accessing the same education as White children such as slavery and segregation, which were inherently racist. The ruling in the U.S. Supreme Court case *Plessy v. Ferguson* (1896) allowed schools to be segregated by race, under the idea that the facilities would be “separate but equal.” However, the experiences and facilities of the separate schools were anything but equal, and the educational experience of Black students suffered as a result. In 1954, the U.S. Supreme Court ruled that segregation of public schools was a violation of the 14th amendment, making it unconstitutional in the case *Brown v. Board of Education* (1954). Throughout the 1900s, systems of standardized tests were used to determine who was “able” to “do” mathematics and who was not able, which would determine which schools students could attend and which mathematics courses students could take. A disproportionately high number of Black and low-socioeconomic (SES) students were deemed as having low mathematics abilities and their access to meaningful mathematics learning opportunities was therefore very limited (Berry et al., 2014). Although these may not have been directly racist acts, the way the tests were designed clearly put Black and low-SES students at a disadvantage.

Since the 1900’s, a variety of mathematics education reforms have been put in place, which as Berry et al. (2014) explained have often been “designed and enacted from the perspectives of the dominant society” (p. 541). This means that the needs and perspectives of people of color and low socioeconomic status were largely disregarded. In their *Curriculum and Evaluation Standards for School Mathematics*, NCTM (1989) declared that equity had become an economic issue and that “the social injustices of past schooling practices can no longer be tolerated” (p. 4). This document pushed an “opportunity for all” agenda. Criticism of wishing to better the country’s economy being the catalyst to provide equitable instruction was presented by

Berry et al. (2013). They stated that “increasing Black children’s participation in mathematics education is based on ensuring that America’s economic interests are met rather than being concerned specifically with serving the families and communities of Black children” (pp. 546-547). In other words, leaders were interested in providing equitable mathematics instruction for Black students only when it would impact the country as a whole, without real concern for the individuals or the needs they possessed. Although the reforms may have been founded in well-intentions, they were still perpetuating the unjust mathematics education field and implicitly privileging White students and oppressing Black students, making the mathematics education system implicitly racist.

In 2000, NCTM published a revised set of standards called *Principles and Standards for School Mathematics*. The first principle listed in this document is the Equity principle. In this document, NCTM explained that “equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students” (NCTM, 2000) This principle has been expanded as the Access and Equity principle in *Principles to Action* (NCTM, 2011). National organizations continue to push for equitable mathematics instruction, but their suggestions have not yet been fully implemented across our country.

Student Population and Funding

According to the data collected from the NAEP 2017 test results, Black students are underperforming when compared with their White peers. The National Center for Education Statistics (NCES, 2019) shared that “the gap between White and Hispanic students in school readiness has narrowed, but the gap between White and Black students showed less movement”

(p. 1). If we were providing just, equitable mathematics educational experience for all students, these gaps should not continue to exist.

Bohrnstedt et al. (2015) found that students in high Black student density schools (where 60% or more of the student population is Black) earned lower scores overall than students who were at schools with lower Black student densities. The majority of these high Black student density schools are located in the south or in cities. Since funding for public schools is determined somewhat by the property values in the areas where schools are located, and property values in cities and the south are lower than those in affluent suburbs, the schools with higher proportions of Black students are receiving less funding than the schools with higher proportions of White students. Less funding means less money to attract and retain highly qualified teachers, as well as less money to be spent on classroom resources and learning material (Ostrander, 2015).

Access to High-Quality Instruction

As Berry et al. (2013) shared, Black learners are routinely put into classrooms of teachers with the least amount of preparation to teach. Not being well-prepared to teach is sure to have an impact on student success in mathematics. One chapter of the National Mathematics Advisory Panel (NMAP) 2008 report is dedicated to teachers and teacher education's impact on student achievement in mathematics. The panel found that "when teachers are ranked according to their ability to produce student achievement gains, there is a 10 percentile point difference across the course of a school year between achievement gains of students of top-quartile teachers versus bottom-quartile teachers" (p. 35). If students are enrolled in the same school for multiple years with teachers who are not well-prepared, that ten percentile point difference can grow to be much larger, putting these students very far behind where they should be in terms of mathematics

achievement. This has the potential to drastically impact students' conceptual understanding of linear relationships. Students are supposed to learn about linear relationships every year from Grade 6 through high school (NGA & CCSSO, 2010), so if students are not given the opportunity to engage with linear relationships in meaningful ways, they may never develop a conceptual understanding of linear relationships. Black students who are already at risk of being behind because of the systemic issues that have already been discussed are now being further hindered by their teachers' lack of preparation to teach them, thus perpetuating the unjust and inequitable mathematics education being provided to them.

Access to Professional Development Opportunities for Teachers

High-quality instruction starts with high-quality instructional resources, such as curricular materials, technology, and professional development opportunities for teachers. Ham and Walker (1999) described the implementation of the standards-based Equity 2000 initiative in a large, urban school district. This initiative initially supported professional staff development on the standards-based initiative. However, due to union contracts and a lack of funding, many newer teachers were not given access to these professional development opportunities. This meant that many teachers were not given access to the same resources and were therefore not able to implement the strategies addressed in the trainings. The teachers who were able to attend the trainings struggled to implement what they learned when they went back to their classrooms due to lack of technology resources such as calculators and instructional materials, as well as hesitancy to change the way they were teaching. Multiple participants mentioned not having the resources needed to implement the strategies they learned in their own classrooms, and that they went back to using their old textbook and teaching the way they were taught. One math resource trainer shared that “professional development that doesn’t have accountability and follow-up in

classroom support is not going to result in application” (p. 42). In practice, funding and time availability often prevent effective, ongoing professional development from occurring, which is detrimental to student learning experiences.

Foster et al. (2013) studied an in-service, grant-funded mathematics and science professional development program in Appalachian Kentucky that was targeted at historically low-achieving schools. They found that middle school mathematics student achievement significantly increased with more hours of teacher participation in the professional development. Teachers were able to implement research-based mathematics strategies they learned in the in-service sessions in their classrooms, and student outcomes improved. This study provides another example of the importance of effective professional development for mathematics teachers.

Level of Expectations

As The New Teacher Project (TNTP, 2018) shared, all students are able to meet appropriate expectations in school when given the opportunity. However, students of color and low-income students are frequently held to low expectations and are less likely to be provided with grade-level-appropriate assignments in their classes, even when they are in a class with the same title as their White counterparts. These students are often given work that is well below their current grade level, and are then given false comfort when they earn high scores on the easier work, because they have not even been given the opportunity to access grade-level content. The effect of only being given access to material that is below grade-level for multiple years in a row works to promote inequity in mathematics education. Low expectations in the mathematics classroom often leads to strictly procedural teaching and learning, which does not engage students in problem solving using multiple representations of linear relationships. If

students are not being held to high expectations, they may never have the opportunity to develop a conceptual understanding of linear relationships.

Classroom Learning Experiences

In addition to the content being presented and assessed, student activities and experiences in the classroom differ between students of color or low SES status and their White or high SES peers. TNTP (2018) found that “mostly white classrooms offered about three-and-a-half times as many strong instructional practices, and higher-income classrooms offered more than five times as many” (p. 46) as classrooms with mostly students of color or lower-income. Lee (2006) echoed this sentiment:

Particularly in low-income schools judged as failures, there often is a tendency to move into highly formulaic and rigidly programmed curriculum, boring to both students and teachers, and, worse yet, to spend time not on teaching their subjects but on drilling on test-taking strategies. (p. 7)

As Boston et al. (2017) shared, students in a mathematics classroom should be “active learners, constructing their knowledge of mathematics through exploration, discussion, and reflection” (p. 2). Boaler (2016) also stated that “mathematics tasks should offer plenty of space for learning. Instead of requiring that students simply give an answer, they should give students the opportunity to explore, create, and grow” (p. 180). Students should be able to explore their world through a mathematical lens, rather than simply memorizing facts and procedures. When Black or low SES students are only engaged in the procedural-level learning, without access to meaningful mathematics experiences, they are not going to see mathematics as useful to their lives or the world that they live in. Students engaged in rote memorization and procedural instruction will not have the opportunities to explore and connect multiple representations in

mathematics, which we have seen will impact their achievement on standardized assessments and future endeavors (Beyranevand, 2010; Brenner et al., 1999).

Between getting placed in underfunded classrooms with under-prepared teachers, being held to low expectations simply because of the color of their skin, and not having access to strong instructional practices in their classrooms, Black students are being presented with unnecessary barriers to their education that have nothing to do with their abilities or potential as young mathematicians. As TNTP (2018) shared, “the ‘achievement gap,’ then, isn’t inevitable. It’s baked into the system, resulting from the decisions adults make, consciously and unconsciously, about which students get what resources. It’s a gap of our own design” (p. 50). If all students were given access to high quality mathematics educational experiences that led them to having a robust conceptual understanding of linear relationships, we would see more equitable outcomes for all students. These equitable outcomes may lead to more students taking higher level mathematics courses such as calculus and statistics and enable students to have a greater selection of potential postsecondary and career opportunities.

Summary and Conclusions

Linear relationships are a foundational concept in algebra, and many factors may impact a student’s conceptual understanding of linear relationships. Some factors that were highlighted are understanding the concept of a constant rate of change, moving fluidly between different representations, showing representational fluency, and having access to high-quality, equitable mathematics instruction and experiences. If students have access to learning experiences where they are able to foster a conceptual understanding of linear relationships as shown in the conceptual framework for this study, they will be more likely to succeed in higher level

mathematics courses, will likely score higher on standardized tests (Beyranevand, 2010; Brenner et al., 1999), and will have a greater selection of postsecondary opportunities (Burdman, 2018)

CHAPTER THREE

METHODOLOGY

This study had multiple components. The first component was developing an instrument to use to collect data on students' conceptual understanding on linear relationships. The second component was collecting and analyzing the data from students enrolled in Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, and Algebra 2 courses. This chapter will provide detailed descriptions of the methodology behind both of these components, which were used to address the following research questions:

1. What differences exist in students' conceptual understanding of linear relationships across grade levels?
 - a. Is there a difference in overall conceptual understanding of linear relationships between grade levels for students in Grades 7 through 11?
 - b. Are there differences in students' conceptual understanding of constructs for linear relationships in Grades 7 through 11?
2. What are common strengths and weaknesses in students' conceptual understanding of linear relationships?
 - a. What are overall strengths and weaknesses in the conceptual understanding of linear relationships for middle and high school students?
 - b. What are grade-specific strengths and weaknesses in the conceptual understanding of linear relationships for students in Grades 7 through 11?

Research Design

A cross-sectional design was used in this study. Cross-sectional studies are a type of observational study, where the researcher is gathering information about different groups of

people at the same time. An important aspect of the cross-sectional design is that participants “are selected based on the inclusion and exclusion criteria set for the study” (Setia, 2016, p. 261). In this study, the inclusion criteria used was based on mathematics course enrollment. In a cross-sectional study, the researcher is studying a difference in outcome between groups. A cross-sectional design was appropriate for this study because I wanted to study the differences in conceptual understanding of linear relationships (the outcome) between students enrolled in different mathematics courses (the groups). In this study, students from five sections, or mathematics courses, all took the same assessment and their scores and overall conceptual understanding of linear relationships were compared between sections. This design was chosen because the goal of the study was to observe the current levels of conceptual understanding of linear relationships for students enrolled in Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, and Algebra 2 courses.

Research Setting

This study was conducted in a small school within a large, urban school district in the Midwest. At the time of the study, the district enrollment was over 65,000 students and the school’s enrollment was about 1,100, with approximately 800 of those students being in grades 7 through 12. The population of the school where this study is taking place has students in the following demographics: 59.8% Black or African American, 16.7% White, 13.4% Hispanic or Latino, 5.4% Asian, 3.8% Two or more races, 0.5% American Indian or Alaskan Native, and 0.4% Native Hawaiian or Pacific Islander (Wisconsin Department of Public Instruction [DPI], 2021) The levels of mathematics achievement for the state, district, and school of study for the 2022-2023 school year can be found in Table 3.1. Although the students in the school of study outperformed the overall district in mathematics, they underperformed when compared with the entire state in mathematics. The students in the school being studied have higher mathematics

achievement than the students in the overall district. The immediate population for this study is the population of all students enrolled in Grade 7 Math through Algebra 2 courses who attend the school, and the wider population is all students within the district who are enrolled in Grade 7 Math through Algebra 2 courses.

Table 3.1

Mathematics Achievement for State, School, and District of Study (2022-2023)

	Below Basic	Basic	Proficient	Advanced
State	31%	31%	29%	9%
School District	64%	25%	10%	2%
School of Study	31%	36%	27%	7%

It is important to note that the course sequence that is followed at the school of study is atypical. The high school mathematics courses follow the sequence of Geometry, Algebra 1, Algebra 2 rather than the typical Algebra 1, Geometry, Algebra 2 sequence. Students have the ability to be accelerated in mathematics in this school, which means some advanced students are in a mathematics class with students from higher grade levels. For example, the majority of students in Grade 8 Math are in eighth grade, but some advanced sixth and seventh grade students are also in Grade 8 Math. This continues through Algebra 2, where the majority of the students are in eleventh grade, but some advanced ninth and tenth grade students are enrolled in Algebra 2.

Participants

Approval was obtained from the graduate institution's Institutional Review Board (IRB) as well as the school district where the research was being conducted before attempting to recruit any participants (see Appendix B and Appendix C). Students from multiple mathematics courses (Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, and Algebra 2) were invited to participate in

the study. The teachers of each of the mathematics courses being studied were given a letter (see Appendix D) to read to students, explaining the purpose and design of the study and inviting students to participate by taking an assessment on linear relationships. After reading the letter out loud, the teachers gave students a copy of the letter for parents, a student assent form (see Appendix E) and a parent consent form (see Appendix F) that had to be completed to be able to participate in the study. Students were given one week to return the signed forms to their classroom mathematics teacher. Ideally, after running a power analysis, the sample should have contained a total of 200 participants. The sample size for this study was 195 students, just below the threshold for statistical power. The number of participants from each mathematics course can be found in Table 3.2.

Table 3.2

Number of Study Participants from Each Mathematics Course

	Number of Participants	Percent of Total Sample
7th Grade Math	24	12.3%
8th Grade Math	52	26.7%
Geometry	43	22.1%
Algebra 1	31	15.9%
Algebra 2	45	23.1%
Total	195	100.0%

Data Collection Procedures

The teachers were given a one-week window to administer the tests after receiving the assent and consent forms from students who chose to participate. They were given this window so that they could fit it into their schedule as they saw best. On their chosen day, teachers gave the assessment to all students in each class who returned their consent and assent forms. The students completed the assessment during their regularly scheduled mathematics class. At the

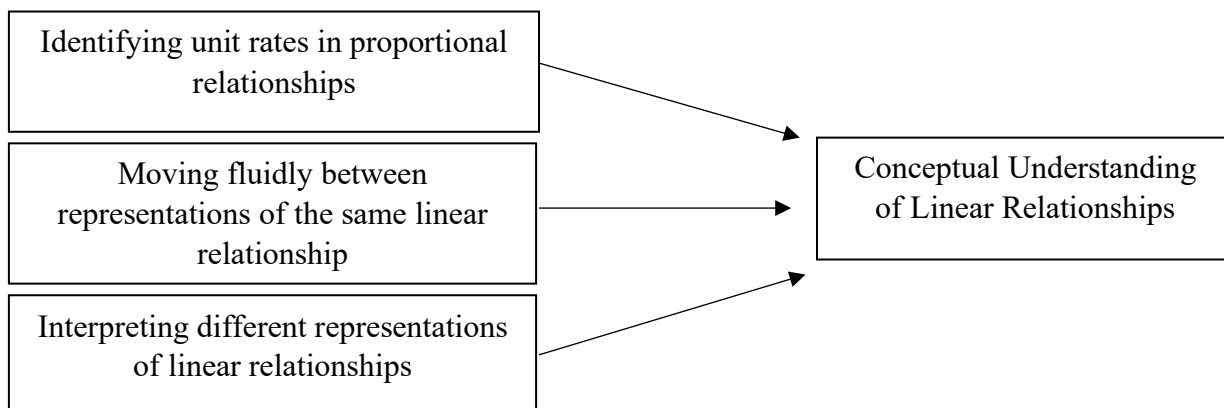
end of class, students returned the assessment to their classroom teacher, and the classroom teachers gathered all tests to return to the researcher for scoring.

Instrument Development

The instrument used in this study consists of 15 items, all assessing some aspect of conceptual understanding of linear relationships, and was designed based on the conceptual framework described in this section. The conceptual framework was created using the standards found in the CCSSM (NGA Center & CCSSO, 2010) and the progressions documents (Common Core Standards Writing Team, 2011), as well as curricular materials and standardized assessments. A diagram showing the conceptual framework is shown in Figure 3.1. Three areas of focus, or constructs, can be found leading to an overall conceptual understanding of linear relationships. If a student can successfully identify unit rates in proportional relationships, move fluidly between representations of the same linear relationship, and interpret different representations of linear relationships, that will demonstrate that they have a sound conceptual understanding of linear relationships.

Figure 3.1

Conceptual Framework



Standards

The primary factor in creating the assessment for this study was determining which *Common Core State Standards for Mathematics* (CCSSM) standards (NGA Center & CCSSO) to assess. I chose to focus on the Expressions and Equations (EE) domain in Grades 6 through 8 because the bulk of student learning expectations for linear relationships are included in this domain. I also used some important linear relationship ideas from the Grade 6 Ratios and Proportions (RP) and the Grade 8 Functions (F) domains. High school standards were not included in the assessment because they focus mainly on functions that may not be linear, which are not the focus of this study.

Constructs

As I looked at the CCSSM EE, RP, and F domains, I organized the main ideas into three constructs, which served as the overarching structure for my assessment. The constructs that I chose to use, along with which standards addressed by each construct are shown in Table 3.3. A complete list of the standards and their descriptions can be found in Appendix A.

Table 3.3

Constructs of Conceptual Understanding of Linear Relationships with Descriptions and Associated Standards

Construct	Description	Standards
1. Identifying unit rate in proportional relationships	Students can identify the unit rate or constant of proportionality and represent a proportional relationship algebraically with an equation.	CCSS.MATH.CONTENT.7.RP.A.2 CCSS.MATH.CONTENT.7.RP.A.2.B

2. Moving fluidly between representations of the same linear relationship	When given one representation of a linear relationship, students will be able to identify or provide at least one other representation of the same relationship.	CCSS.MATH.CONTENT.6.EE.B.7 CCSS.MATH.CONTENT.6.EE.C.9 CCSS.MATH.CONTENT.7.EE.B.4 CCSS.MATH.CONTENT.8.F.B.4
3. Interpreting different representations of linear relationships	Students can identify differences between relationships when given two different representations of linear relationships.	CCSS.MATH.CONTENT.8.F.B.4 CCSS.MATH.CONTENT.8.F.A.2 CCSS.MATH.CONTENT.8.F.B.5

Since the fundamental idea of a linear relationship is the constant rate of change, that also had to be a fundamental construct to demonstrate a conceptual understanding of linear relationships. I chose to have the first construct be “identifying unit rates in proportional relationships” because a proportional relationship is the most basic form of a linear relationship and I want to determine whether students can correctly identify the unit rate in isolated proportional relationships.

For the second construct, I looked at the middle school expectations in the CCSSM standards. I noticed that students were expected to write algebraic expressions to represent real-world situations, interpret graphs in a real-world context, and identify the constant of proportionality in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships” (NGA Center & CCSSO, 2010). All of these ideas involved being able to fluidly move between different representations of the same linear relationship. Since these skills are prevalent throughout the middle school mathematics standards, they are essential to having a conceptual understanding of linear relationships, so the second construct in the framework being used for this study is “moving fluidly between representations of the same linear relationships.”

Finally, in Grade 8 (NGA Center & CCSSO, 2010), students are expected to “compare two different proportional relationships represented in different ways. This is the most applicable aspect of the conceptual framework, because it would allow students to determine which company has a better rate, which car has a greater speed, or which plan is a better one to purchase. The essential idea here is that students are able to see different representations of linear relationships and identify what is happening in each representation so that they can answer questions about the two relationships being compared. Therefore, the third construct identified for this study is “interpreting different representations of linear relationships.”

If a student can demonstrate that they are able to identify the unit rate in a given proportional relationship, move fluidly between multiple representations of the same relationship, and interpret different representations of linear relationships, then they will be demonstrating that they have a solid conceptual understanding of linear relationships and their understanding is not simply procedural.

Item Collection

To create the assessment, I chose items that have been deemed reliable by their use on other standardized tests. First, I looked through sample test items from the National Assessment of Educational Progress (NAEP) for eighth grade and twelfth grade to find items that assessed the chosen standards for each construct. Next, I searched the item bank from the Partnership for Assessment of Readiness for College and Careers (PARCC) test item bank. I also searched through the various curricular materials for items.

Instrument Development

After I gathered a bank of test questions, I began to organize them by standards and constructs, then assembled a draft assessment with 20 items. I met with select committee

members as well as other mathematics teachers about the test items and ultimately developed the assessment that was used for the study. I also had students who attended schools outside of the school being studied pilot some of the items and the test as a whole to make sure questions were worded at an understandable level and that the test did not take too long to complete. When the pilot students took the assessment, they were asked to time themselves. All but one student took over 50 minutes to complete the assessment, which was too long to be given in a single class session for the actual study. I found five items that were assessing similar concepts to other items and removed them, resulting in a final assessment with fifteen items.

In the final version of the student assessment, each construct is assessed by five items. The test blueprint in Table 3.4 provides the specific item numbers assessing each construct as well as the percentage of the whole test that each construct represents. Table 3.5 presents the construct being assessed, a description, and the source of each item on the assessment. Seven items came from the NAEP, three items came from the PARCC, four items came from curricular materials, and one item was self-created. Appendix G includes the justification for each item along with correct responses, potential distractors or misconceptions, source location, and performance data from the original assessment where applicable. Figures 3.2, 3.3, and 3.4 show the actual items in each construct on the assessment, in condensed form. The full version of the assessment that students were given can be found in Appendix H.

Table 3.4

Test Blueprint Showing Assessment Items in Each Construct on Linear Relationships Assessment

Constructs	Item Numbers on Assessment	Number of Items	Percentage
1. Identifying unit rate in proportional relationships	4,5,7,11,14	5	33%

2. Moving fluidly between representations of the same linear relationship	1,2,3,6,10	5	33%
3. Interpreting Different Representations of Linear Relationships	8,9,12,13,15	5	33%
Total		15	100%

Table 3.5

Assessment Constructs, Descriptions, and Sources by Item

Item Number	Construct	Description of Item	Source of Item
1	2	This item presents students with a table of values showing the relationship between the number of cards sold and profit made and asks them to choose the correct equation that represents the relationship. The relationship is directly proportional.	Adapted (clarifying variables added into question text and answer choices re-ordered) from 2007 NAEP Grade 8 Assessment
2	2	This item asks students to select the correct graph that represents the symbolic equation that is given.	2007 NAEP Grade 8 Assessment
3	2	This item asks students to identify an equation that represents the same relationship shown in a table. This relationship has a nonzero y-intercept, so it is not directly proportional.	2013 NAEP Grade 8 Assessment
4	1	This item requires students to fill in missing values in a table representing hours walked (n) and miles walked (d). Students were told that the person walks at a constant speed, which should signal to them that this is a proportional relationship.	Adapted from Illustrative Mathematics Task: Walk-a-thon 2
5	1	This item explicitly asks students to state the walking rate in miles per hour. This is the unit rate, and I chose to include this item to see whether students were able to connect the idea of a unit rate to the "walking rate" in context.	Adapted from Illustrative Mathematics Task: Walk-a-thon 2
6	2	In this item, students are asked to move from the tabular representation of the relationship to an algebraic representation by writing an equation to relate the number of hours walked (n) with the distance walked (d).	Adapted from Illustrative Mathematics Task: Walk-a-thon 2

7	1	Students are asked to calculate the unit rate for two proportional, linear relationships. One is presented in a table and the other is present with an equation.	Adapted from MC ² PARCC Practice Test Items Grade 8 Mathematics
8	3	Students are asked to choose the least expensive company and explain their choice. This forces students to interpret the unit rate for two linear relationships, one being represented in a table and the other represented with an equation.	Adapted from MC ² PARCC Practice Test Items Grade 8 Mathematics
9	3	In this item, students are given two linear relationships, one represented with a table and the other with an equation. They are given four statements relating the two functions and are asked to choose the statement that is incorrect.	Adapted from PARCC Released Test Items 2017
10	2	This item asks students to choose an expression that represents the given verbal description of a linear relationship.	1996 Grade 8 NAEP Assessment
11	1	This item is a very simple unit rate problem. The students are given a proportional relationship verbally, and are asked to find the unit rate.	SpringBoard Course 1, Unit 4 Practice (2014) #21
12	3	This item asks students to interpret the meaning of values from a given equation in context of the real world situation presented. This particular problem addresses the y-intercept, or initial cost in the situation.	2017 Grade 8 NAEP Assessment
13	3	This item asks students to interpret the meaning of values from a given equation in context of the real world situation presented. This particular problem addresses the slope, or rate of change in the situation.	2011 Grade 8 NAEP Assessment
14	1	This item presents students with verbal information about two people driving for given times, and they are asked to tell who was driving faster, then explain how they know this. This item is a unit rate problem, but does not explicitly ask students to find the unit rate.	Self-created
15	3	This item gives students a graphical representation of two runners. They are asked to fill in a table with the verbal interpretation of different aspects of the graphical representations.	2009 Grade 12 NAEP Assessment

Figure 3.2

Items Assessing Construct 1: Identifying Unit Rates in Proportional Relationships

<u>Assessment Items #4 & 5</u>											
Julianna participated in a walk-a-thon to raise money for cancer research. She recorded the total distance she walked at several different points in time, but a few of the entries got smudged and can no longer be read. The times and distances that can still be read are listed in the table below. #4. Assume Julianna walked at a constant speed. Complete the table with the missing miles walked and time in hours. <table border="1" style="margin: 10px auto; border-collapse: collapse;"><thead><tr><th style="padding: 5px;">Time in Hours (<i>n</i>)</th><th style="padding: 5px;">Miles walked (<i>d</i>)</th></tr></thead><tbody><tr><td style="text-align: center; padding: 5px;">1</td><td style="padding: 5px;"></td></tr><tr><td style="text-align: center; padding: 5px;">2</td><td style="text-align: center; padding: 5px;">6.4</td></tr><tr><td style="padding: 5px;"></td><td style="text-align: center; padding: 5px;">8</td></tr><tr><td style="text-align: center; padding: 5px;">5</td><td style="padding: 5px;"></td></tr></tbody></table>		Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)	1		2	6.4		8	5	
Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)										
1											
2	6.4										
	8										
5											
#5. What was Julianna's walking rate in miles per hour?											
<u>Assessment Item #7</u>											
Two utility companies sell electricity in units of kilowatt-hours. The cost of electricity for company P is shown in the table. The cost of electricity for company M can be found by using the equation shown, where <i>y</i> represents the total cost in dollars for <i>x</i> kilowatt-hours of electricity. Electricity Costs Company P: <table border="1" style="margin: 10px auto; border-collapse: collapse;"><thead><tr><th style="padding: 5px;">Number of Kilowatt-hours</th><th style="padding: 5px;">Total Cost (dollars)</th></tr></thead><tbody><tr><td style="text-align: center; padding: 5px;">1,250</td><td style="text-align: center; padding: 5px;">150.00</td></tr><tr><td style="text-align: center; padding: 5px;">1,650</td><td style="text-align: center; padding: 5px;">198.00</td></tr></tbody></table> Company M: $y = 0.15x$		Number of Kilowatt-hours	Total Cost (dollars)	1,250	150.00	1,650	198.00				
Number of Kilowatt-hours	Total Cost (dollars)										
1,250	150.00										
1,650	198.00										
#7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.											
<u>Assessment Item #11</u> #11. A factory can produce 720 wagons in an 8-hour day. What is the unit rate of wagons made per hour? A. 45 B. 90 C. 728 D. 5760	<u>Assessment Item #14</u> #14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? How do you know?										

Figure 3.3

Items Assessing Construct 2: Moving Fluidly between Representations of the Same Linear

Relationship

Assessment Item #1

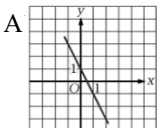
#1. Angela makes and sells special-occasion greeting cards. The table below shows the relationship between the number of cards sold (n) and her profit (p). Based on the data in the table, which of the following equations shows how the number of cards sold (n) and profit (p) (in dollars) are related?

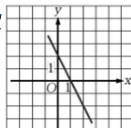
	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Number Sold, n	4	0	5	2	3	6
Profit, p	\$2.00	\$0.00	\$2.50	\$1.00	\$1.50	\$3.00

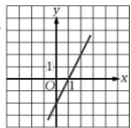
A. $p = 0.5n$ B. $p = 2n$ C. $p = n - 2$ D. $p = n + 1$ E. $p = 6 - n$

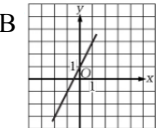
Assessment Item #2

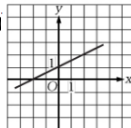
#2. Which of the following is the graph of the line with equation $y = -2x + 1$?

A 

C 

E 

B 

D 

Assessment Item #3

#3. The table below lists the coordinates of several points on a line.

x	y
1	3
2	5
3	7
4	9

Which of the following is an equation of the line?

A. $y = x + 2$
 B. $y = 3x$
 C. $y = 2x + 1$
 D. $y = 4x - 1$
 E. $y = 3x - 1$

Assessment Item #10

#10. A plumber charges customers \$48 for each hour worked plus an additional \$9 for travel. If h represents the number of hours worked, which of the following expressions could be used to calculate the plumber's total charge in dollars?

A. $48 + 9 + h$
 B. $48 \times 9 \times h$
 C. $48 + (9 \times h)$
 D. $(48 \times 9) + h$
 E. $(48 \times h) + 9$

Assessment Item #6

Julianna participated in a walk-a-thon to raise money for cancer research. She recorded the total distance she walked at several different points in time, but a few of the entries got smudged and can no longer be read. The times and distances that can still be read are listed in the table below.

Time in Hours (n)	Miles walked (d)
1	
2	6.4
	8
5	

#6. Write an equation for the distance d , in miles, that Julianna walked in n hours.

47

Figure 3.4

Items Assessing Construct 3: Interpreting Different Representations of Linear Relationships

Assessment Item #8

Two utility companies sell electricity in units of kilowatt-hours. The cost of electricity for company P is shown in the table. The cost of electricity for company M can be found by using the equation shown, where y represents the total cost in dollars for x kilowatt-hours of electricity.

Company P:

Number of Kilowatt-hours	Total Cost (dollars)
1,250	150.00
1,650	198.00

Company M: $y = 0.15x$

#8. Which is the *least* expensive company? Explain your answer.

Assessment Item #9

#9. Two linear functions are shown.

Function 1

Input (x)	Output (y)
-3	-2.5
-1	0.5
2	5
4	8

Function 2

$$y = \frac{4}{3}x + 2$$

Which statement about the functions is incorrect?

- A. The rate of change for function 1 is greater than the rate of change for function 2.
- B. The y-intercept of the line for function 1 is equal to the y-intercept of the line for function 2.
- C. The y-intercept of the line for function 1 is less than the y-intercept of the line for function 2.
- D. The line shown for the graph of function 2 is less steep than the line shown for the graph of function 1.

Assessment Item #12

#12. Jessica makes clay bowls.

The following equation relates the number of clay bowls, n , she makes and the total amount of money, d , in dollars, it costs for her to make them.

$$d = 375 + 50n$$

Based on the equation, which of the following statements is true?

- A. Without making any clay bowls, Jessica's total cost is \$50.
- B. Without making any clay bowls, Jessica's total cost is \$375.
- C. Each clay bowl Jessica makes decreases her total cost by \$50.
- D. Each clay bowl Jessica makes increases her total cost by \$375.
- E. Each clay bowl Jessica makes costs \$425.

Assessment Item #13

#13. The number of gallons of water, y , in a tank after x hours may be modeled by the linear equation $y = 800 - 50x$. Which of the following statements about the tank is true?

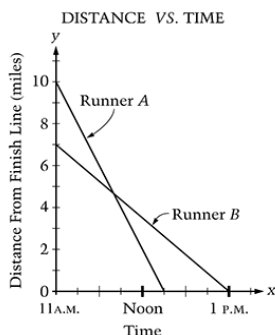
- A. It is filling at a rate of 800 gallons per hour.
- B. It is filling at a rate of 50 gallons per hour.
- C. It is emptying at a rate of 16 gallons per hour.
- D. It is emptying at a rate of 50 gallons per hour.
- E. It is emptying at a rate of 800 gallons per hour.

Figure 3.4 (Continued)

Items Assessing Construct 3: Interpreting Different Representations of Linear Relationships

Assessment Item #15

#15. The graph below shows distance *versus* time for a race between runners *A* and *B*. The race is already in progress, and the graph shows only the portion of the race that occurred after 11 A.M



The given table lists several characteristics of the graph. Interpret these characteristics in terms of what happened during this portion of the race. Include times and distances to support your interpretation. (A sample interpretation of the *y*-intercepts is given in the table.)

Characteristic of Graph	Interpretation in Terms of the Race
<i>y</i>-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	
Point of Intersection	
<i>x</i>-intercepts	

Validity

First, I assessed the content validity of my test. The content validity was verified by three groups of people: certified mathematics teachers, committee members who are experts in the field, and students who piloted the assessment. First, I created a construct table (see Table 3.5)

that shows the standards for linear relationships addressed by each item. Each item is also aligned to one of the three constructs of conceptual understanding in this table. Next, I met with three certified mathematics teachers to have them verify that I had correctly mapped each item to the standards and construct that it was assessing. An early draft of the assessment contained an item that two of the teachers thought were overlapping on standards, so the item was removed. I then met with committee members who are familiar with the CCSSM to examine the content validity of the test and they agreed with my alignment of the test items to the standards and constructs. Finally, I had six students in Grades 7, 8, 10, and 11 who attend different schools pilot my assessment. Their responses and strategies verified that the items on the assessment measured the intended content.

The student work from the pilot assessment was also used to verify the construct validity by comparing their responses for the items within each construct. This type of validity examines the relationships among items or parts of a test. For example, responses to items that are all meant to measure a certain type of skill or understanding, such as multiplication should be highly correlated, while responses to items that are not meant to measure similar things, such as multiplication and reading, should not be highly correlated (Zhang, 2019). When analyzing my data from the pilot, I found that if a student struggled on one item inside a construct, they also struggled on the other items in that construct. This told me that the test was valid from a construct perspective.

Reliability

Reliability is much harder to measure before administering the assessment. Test reliability refers to “the relative degree of consistency (scores and interpretation) between measurements” (Zhang, 2019). When referring to test reliability, we are looking primarily at measurement error. According to Kaplan and Saccuzzo (2013), “Theoretically, reliability is the

ratio of true score variance to observed score variance” (p. 643). After data was collected, Cronbach’s alpha was calculated to determine the internal consistency and reliability of the assessment. The reliability coefficient, Cronbach’s Alpha, for the overall assessment as well as the individual constructs can be found in Table 3.6. When the reliability of the overall assessment was measured, Cronbach’s Alpha was .87. The reliability threshold for this research study was .7, so the assessment as a whole proved to be reliable, meaning the assessment is internally consistent. Construct 1 also proved to be reliable, with Cronbach’s Alpha being .78. Constructs 2 and 3 were just below the threshold with Cronbach’s Alphas of .69 and .65 respectively. Although the reliability coefficients were just below the .70 threshold, Constructs 2 and 3 were still used for analysis in this study to provide valuable insight into student strengths and weaknesses in understanding. The decision to include these constructs was made because the overall assessment is not a high-stakes test and was developed for diagnostic purposes. Constructs 2 and 3 were subscores comprised of only five items each, however if more items were added to these constructs for future versions of this assessment, the reliability coefficient would likely increase.

Table 3.6

Tests of Reliability

Assessment Section	Cronbach's Alpha	N of Items
Overall Assessment	.87	15
Construct 1 – Identifying unit rates in proportional relationships	.78	5
Construct 2 – Moving fluidly between representations of the same linear relationship	.69	5
Construct 3 – Interpreting different representations of linear relationships	.65	5

Data Analysis

Each student's assessment was scored individually by item, construct, and total score, then the data were organized for analysis. The following provides a more detailed description of the data analysis plan used in this study.

Compiling and Preparing Data

After the teachers administered the assessment and returned the completed assessments, I scored each item and overall assessment. Each item on the assessment was worth one point, so each item carried the same weight for the total score of fifteen points. The total score for each construct was five points, so each construct also carried the same weight for the overall score. Each multiple-choice item was scored as either correct (1 point) or incorrect (0 points). Each open-response item was scored with a point value of 0, 0.5 or 1 point using the rubric in Appendix I.

Each student's test was assigned an identification code, where the first number represents their grade level, the letter represents their math course (a = Grade 7 Math, b = Grade 8 Math, c = Geometry, d = Algebra 1, e = Algebra 2), and the second number represents the sequentially ordered test number for students in the same grade level and math course. Next, I entered each student's grade level, mathematics course, and scores for each item into SPSS, with each row of values representing one student. I then conducted statistical analyses on students' overall scores and subscores for each construct.

Plan for Analyses

Since this study was comparing and looking for differences in scores on the same assessment between each grade level, the analysis of variance (ANOVA) was deemed most appropriate. An ANOVA determines whether significant differences in the dependent variable

exist between different groups, known as factors. If it is determined that a difference exists, other post-hoc comparisons, such as Tukey's test can be conducted to get more details about this difference.

To address the first research question, a one-way analysis of variance (ANOVA) was conducted along with post-hoc tests to determine whether differences exist in students' conceptual understanding of linear relationships across grade levels. To answer research questions 1a, an ANOVA was conducted using Grade 7 Math through Algebra 2 as factors and participants' total scores on the test as the dependent variable to determine whether differences exist in the overall scores of conceptual understanding of linear relationships between grade levels. Tukey's post-hoc analysis was conducted to determine pairwise comparisons and extract more meaningful information about the differences in conceptual understanding of linear relationships between grade levels. To answer research question 1b, a subscore was calculated for each of the three defined constructs for each participant. Then, a separate ANOVA was conducted for each construct, using grade levels as factors and subscores as the dependent variables. For Construct 1, Tukey's post hoc analysis was used to determine pairwise comparisons for each mathematics course. Constructs 2 and 3 did not meet the homogeneity of variances assumption required for ANOVA. Therefore, the Games-Howell post hoc analysis which adjusts for heterogeneity of variances was used to determine pairwise comparisons for Constructs 2 and 3.

Content and Item Analysis

To address the second research question, a qualitative content analysis was conducted by item within each construct. For multiple choice items, the distribution of chosen answers is presented for all students and by mathematics course. For free-response items, the common

strategies that were found within the responses for students overall are discussed, with student work samples of correct responses and common errors being shared in Chapter 4. I grouped the analyses by construct and drew inferences from the data about common strengths and weaknesses within each construct for students as a whole and for each mathematics course.

Researcher Background

It is important to acknowledge the biases and background that I bring to this study as the researcher. I come to this study as a white, cisgender, female, heterosexual educator, and graduate student. I started my career teaching high school mathematics in a large, urban district. I taught Algebra 1, Geometry, College Algebra, and IB Math Studies for four years. I saw the inequities that my students were experiencing due to a lack of resources and limited access to high-quality instructional materials, which inspired me to pursue a doctoral degree in urban education with the hopes of working to lessen the inequitable mathematics educational experiences for students in large, underfunded, urban school districts. As I worked on my doctoral degree, I taught high school math for two years in a small district just outside a major metropolitan area that serviced about one thousand students. Again, I realized that my students were not being given access to high-quality mathematics education, as all curricular materials being used in this school were teacher-created, and were not aligned to research-based best practices. I then moved to a school in a large, urban district where I have taught math intervention for the last three years. This district prioritizes research-based, high-quality mathematics curriculum, but teachers do not receive adequate training to implement the curriculum with fidelity with their students. This has often led back to teachers teaching the way they learned mathematics, rendering the high-quality curricular materials irrelevant.

My past experiences led me to the topic of this study. I noticed that my students in each of the schools in which I worked struggled with linear relationships and seeing the connection between different representations of linear relationships. This perceived gap in conceptual understanding of linear relationships inspired me to dig deeper into what may be causing confusion for students. My goal was to be able to use the results of this study to offer suggestions for equitable mathematics instructional practices to help students build a strong conceptual understanding of linear relationships.

CHAPTER FOUR

RESULTS

This study examined the conceptual understanding of linear relationships for students in mathematics courses from Grade 7 Math through Algebra 2. To learn more about students' conceptual understanding of linear relationships throughout middle and high school, the following questions were investigated:

1. What differences exist in students' conceptual understanding of linear relationships across mathematics courses?
 - a. Is there a difference in overall conceptual understanding of linear relationships between mathematics courses for students enrolled in in Grade 7 Math through Algebra 2?
 - b. Are there differences in students' conceptual understanding of constructs for linear relationships in Grade 7 Math through Algebra 2?
2. What are common strengths and weaknesses in students' conceptual understanding of linear relationships?
 - a. What are overall strengths and weaknesses in the conceptual understanding of linear relationships for middle and high school students?
 - b. What are course-specific strengths and weaknesses in the conceptual understanding of linear relationships for students in Grade 7 Math through Algebra 2?

The participants in this study were 195 students from a single school who were enrolled in Grade 7 Math ($n = 24$), Grade 8 Math ($n = 52$), Geometry ($n = 43$), Algebra 1 ($n = 31$), and Algebra 2 ($n = 45$). The sequence of high school mathematics courses at this school is atypical, with students taking Geometry, Algebra 1, then Algebra 2 rather than the traditional Algebra 1,

Geometry, Algebra 2 sequence. All participants took the same assessment on linear relationships containing 15 items, including eight multiple choice items and seven open-response items.

This chapter first provides the results of the statistical analyses at the test level to discuss overall conceptual understanding of linear relationships between students in Grade 7 Math through Algebra 2. Then, the construct-level results are presented. Each construct was analyzed using ANOVAs with associated post hoc tests and item analysis to provide insight into the students' understanding of each construct's topic as it relates to linear relationships. Finally, the item-level content analysis is presented that highlights common strengths and weaknesses in students' conceptual understanding of linear relationships.

Overall Test Results

The assessment measured students' conceptual understanding of linear relationships. The descriptive statistics for the overall test score for each mathematics course can be found in Table 4.1. The mean score in Grade 7 Math was the lowest ($M = 5.02$, $SD = 2.78$), and the mean score for students in Algebra 2 was the highest ($M = 9.52$, $SD = 4.11$). This shows that the Algebra 2 students were able to answer an average of 4.5 more items correctly than the Grade 7 Math students, meaning the Algebra 2 students have a much stronger overall conceptual understanding of linear relationships. The mean scores increased by an average of two points from Grade 7 Math to Grade 8 Math, but then stayed about the same for Grade 8 Math, Geometry, and Algebra 1 students. This means that the conceptual understanding of linear relationships remained stagnant for three years of mathematics courses. These results indicate that students in higher-level mathematics courses have a greater conceptual understanding of linear relationships than those in lower-level mathematics courses.

Table 4.1*Total Score Descriptive Statistics by Mathematics Course*

	N	Mean	SD	Minimum	Maximum
Grade 7 Math	24	5.02	2.78	1.00	9.50
Grade 8 Math	52	7.02	3.66	1.00	14.00
Geometry	43	7.91	4.02	2.00	15.00
Algebra 1	31	7.63	4.05	1.50	14.00
Algebra 2	45	9.52	4.11	1.00	15.00
Total	195	7.64	4.01	1.00	15.00

A one-way analysis of variance (ANOVA) was conducted to determine if the observed differences in students' conceptual understanding of linear relationships across mathematics courses were significant. The homogeneity of variances assumption was confirmed (see Table 4.2), with all significance values above .05.

Table 4.2*Tests of Homogeneity of Variances*

		Levene Statistic	df1	df2	Sig.
Total Score	Based on Mean	1.85	4	190	.12

The one-way ANOVA results in Table 4.3 show that a significant difference ($p < .001$) existed between total test scores across the middle and high school mathematics courses, meaning that the conceptual understanding of linear equations is different between some mathematics courses. Since a significant difference existed, a post hoc analysis was performed to identify which mathematics courses demonstrated significantly different levels of conceptual understanding. Tukey HSD was chosen as the post hoc test.

Tukey's comparison test revealed three significant differences among the groups. The results of this test are shown in Table 4.4. The results showed that the mean total test score for students enrolled in Grade 7 Math ($M = 5.02$, $SD = 2.78$) was significantly different from the

students enrolled in Geometry ($M = 7.91$, $SD = 4.02$) and Algebra 2 ($M = 9.52$, $SD = 4.01$). A significant difference in mean total test scores also existed between students in Grade 8 Math ($M = 7.02$, $SD = 3.66$) and Algebra 2 ($M = 9.52$, $SD = 4.01$). Taken together, these results suggest that students in a higher-level mathematics class (Algebra 2) have a significantly greater conceptual understanding of linear relationships than students in middle school mathematics courses and students in Geometry have a significantly greater conceptual understanding of linear relationships than students in Grade 7 Math. The lack of significant difference between the other mathematics courses suggests that the overall conceptual understanding of linear relationships is about the same for students enrolled in Grade 8 Math, Geometry, and Algebra 1 courses. It is important to note that the sample size of 195 students does not provide enough statistical power to potentially detect statistical differences between these groups. Ideally, the sample would have included 200 students, with 40 in each mathematics course to have sufficient statistical power.

Table 4.3

Total Test Score ANOVA Results

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	347.17	4	86.80	5.94	<.001
Within Groups	2777.31	190	14.62		
Total	3124.48	194			

Table 4.4

Total Test Score Multiple Comparisons – Tukey's Post Hoc Test

(I) Mathematics Course	(J) Mathematics Course	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Grade 7 Math	Grade 8 Math	-2.00	.94	.22	-4.60	.60
	Geometry	-2.89*	.97	.03	-5.57	-.20

Grade 8 Math	Algebra 1	-2.61	1.04	.09	-5.47	.25
	Algebra 2	-4.50*	.97	<.001	-7.16	-1.84
	Geometry	-.89	.79	.79	-3.06	1.28
	Algebra 1	-.61	.87	.96	-3.00	1.78
	Algebra 2	-2.50*	.78	.01	-4.65	-.36
	Geometry	.28	.90	1.0	-2.20	2.76
	Algebra 2	-1.62	.82	.28	-3.86	.63
Algebra 1	Algebra 2	-1.89	.89	.22	-4.35	.56

Construct Results

The linear relationships assessment was comprised of three constructs, each measuring a critical feature of conceptual understanding of linear relationships, with five items assessing each construct. The three constructs are:

Construct 1: Identifying unit rates in proportional relationships

Construct 2: Moving fluidly between representations of the same linear relationship

Construct 3: Interpreting different representations of linear relationships

An analysis of each construct was conducted in an effort to identify the extent to which each area contributed to the differences in conceptual understanding of linear relationships across the mathematics courses.

Construct 1: Identifying Unit Rates in Proportional Relationships

Construct 1 is the most basic of the three constructs, as it lays the foundation of understanding linear relationships by calculating and analyzing unit rates. The descriptive statistics for Construct 1 for each mathematics course are listed in Table 4.5 and displayed in Figure 4.1. The maximum score for Construct 1 was five points, one point for each of the five items assessing this construct. The overall mean score for Construct 1 was 3.01 ($SD = 1.52$), which shows 60.2% accuracy. Algebra 2 students had the highest mean score ($M = 3.63$, $SD =$

1.48), and Grade 7 Math students had the lowest mean score ($M = 2.08$, $SD = 1.49$), showing that Algebra 2 students have a stronger understanding of unit rates than Grade 7 Math students.

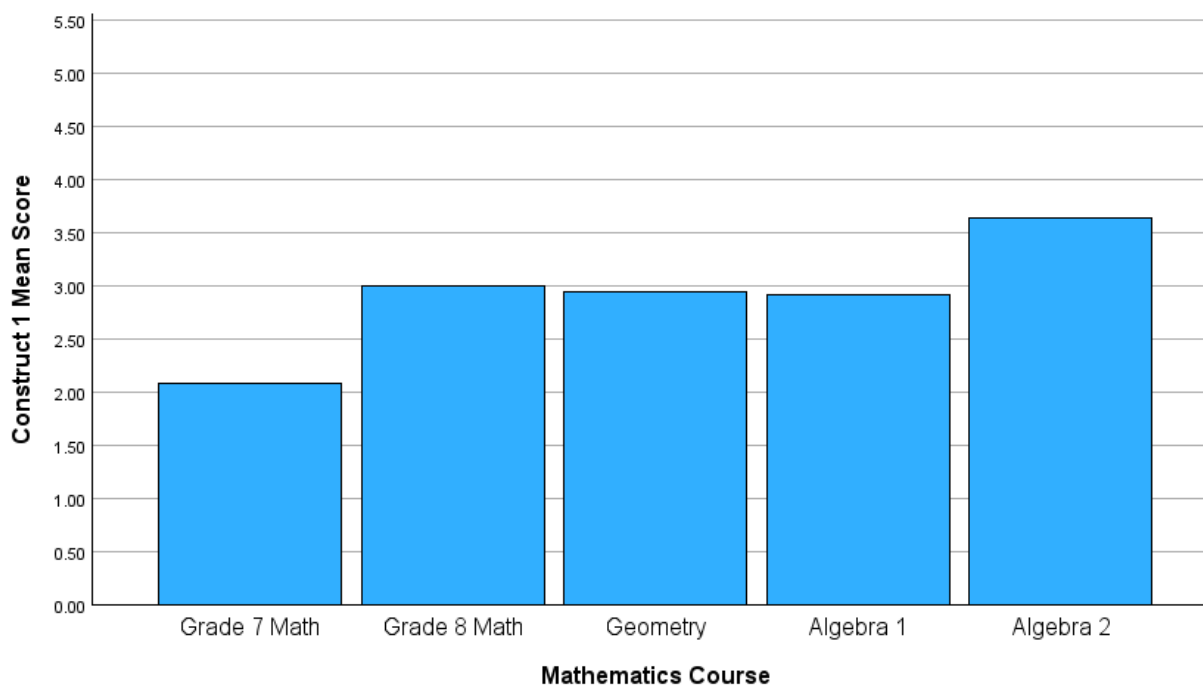
Table 4.5

Construct 1 Descriptives

	N	Mean	Std. Deviation	Minimum	Maximum
Grade 7 Math	24	2.08	1.49	.00	4.50
Grade 8 Math	52	3.00	1.43	.00	5.00
Geometry	43	2.94	1.42	.00	5.00
Algebra 1	31	2.92	1.54	.00	5.00
Algebra 2	45	3.63	1.48	.00	5.00
Total	195	3.01	1.52	.00	5.00

Figure 4.1

Construct 1 Mean Score by Mathematics Course



According to the CCSSM, students are introduced to unit rates in Grade 6 and explore them more in Grade 7. This could explain the lower score for students in Grade 7 Math when

compared with the older students because they may not have been exposed to this topic at the point in the school year when the assessment was given, or it may still be a relatively new topic that they need more experience with to fully understand. The mean scores increased between Grade 7 Math and Grade 8 Math, then they remain about the same for students in Grade 8 Math, Geometry, and Algebra 1, then jump up again between Algebra 1 and Algebra 2. These results could indicate that unit rates are not discussed as much in the learning experiences in Grade 8 Math, Geometry, and Algebra 1. By being older and taking more mathematics courses, students in Algebra 2 have had more opportunities to engage in learning activities involving real-world scenarios, which likely involve unit rates at some point. The students in Algebra 2 are therefore able to apply various concepts that they have learned to determine unit rates, compared to students in Grade 7 Math who are only at the beginning of learning about unit rates and have not yet learned multiple strategies for finding unit rates.

Overall, the mean scores were the highest for Construct 1 among the three constructs in every mathematics course. It is encouraging that students were able to identify unit rates correctly over half of the time in the assessment. However, students should be able to identify a unit rate nearly every time they are asked, so the mean scores should be much closer to five for this construct. Unit rates are foundational in the understanding of ratios and proportional reasoning, which are in turn foundational for understanding linear relationships. If students are not able to calculate and recognize unit rates, they will struggle with interpreting linear relationships and moving fluidly between different representations of linear relationships. Many students in this study lacked the foundational understanding of unit rates, and they will likely struggle with the concepts being assessed in Constructs 2 and 3 of this study.

The total score for Construct 1 was analyzed using an ANOVA to determine whether there is a difference in abilities to identify unit rates in proportional relationships across the mathematics courses. The scores in Construct 1 met the homogeneity of variances assumption, as shown in Table 4.6, and the initial one-way ANOVA results in Table 4.7 show that a significant difference ($p = .002$) exists, meaning there is a significant difference in abilities to identify unit rates in proportional relationships between some of the mathematics courses.

Table 4.6

Construct 1 - Tests of Homogeneity of Variances

	Levene Statistic	df1	df2	Sig.
Construct 1 Total Based on Mean	.32	4	190	.86

Table 4.7

Construct 1 ANOVA

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	38.56	4	9.64	4.49	.002
Within Groups	407.69	190	2.15		
Total	446.24	194			

Given that a significant difference existed and the data's variances were homogenous, Tukey's post hoc analysis was conducted to compare Construct 1 scores between each pair of mathematics courses. The results of Tukey's post hoc analysis in Table 4.8 show that a significant difference existed in the Construct 1 score for students enrolled in Grade 7 Math and Algebra 2. This means that students in Algebra 2 showed a significantly stronger ability to identify unit rates in proportional relationships than the students in Grade 7 Math. The mean score in Construct 1 for students enrolled in Algebra 2 was 1.55 points higher than the mean score for students enrolled in Grade 7 Math, with Algebra 2 students answering an average of

72% of the items correctly, and Grade 7 Math students answering and average of 42% of the items correctly. No other significant differences existed between math courses in Construct 1 scores, so an understanding of unit rates in proportional relationships is likely not a contributing factor to the significant differences in overall conceptual understanding of linear relationships between students in Grade 8 Math and Algebra 2 or Grade 7 Math and Geometry.

Table 4.8

Construct 1 - Multiple Comparisons – Tukey's Post Hoc Test

(I) Mathematics Course	(J) Mathematics Course	Mean Difference (I-J)	Std. Error	Sig.
Grade 7 Math	Grade 8 Math	-.92	.36	.09
	Geometry	-.86	.37	.15
	Algebra 1	-.84	.40	.23
	Algebra 2	-1.55*	.37	<.001
Grade 8 Math	Geometry	.06	.30	1.00
	Algebra 1	.08	.33	1.00
	Algebra 2	-.63	.30	.21
Geometry	Algebra 1	.02	.35	1.00
	Algebra 2	-.69	.31	.18
Algebra 1	Algebra 2	-.71	.34	.23

The content of Construct 1 focused on identifying unit rates in proportional relationships. This construct was assessed by five items (Items 4, 5, 7, 11, and 14). The mean scores for each mathematics course on each of these items are shown in Table 4.9. Given the significant difference in Construct 1 scores between students enrolled in Grade 7 Math and students enrolled in Algebra 2, focus should be given to the scores for these two mathematics courses within this construct. The largest differences occurred in the mean scores for Grade 7 Math and Algebra 2 students for items four, five, and seven, with a difference of about 0.4 points between the two

courses for each item. These items likely contributed to the significant difference in unit rate understanding between Grade 7 Math and Algebra 2 students. Item 14 may have had a small impact on the difference between Grade 7 Math and Algebra 2 students' scores, with a mean difference of .25 for this item. The scores for item 11 were similar across all mathematics courses, so this item likely did not impact the difference in unit rate understanding.

Table 4.9

Mean Item Score by Mathematics Course for Construct 1 – Identifying Unit Rates

Mathematics Course	Q4	Q5	Q7	Q11	Q14
Grade 7 Math	.29	.33	.13	.79	.54
Grade 8 Math	.54	.64	.24	.81	.78
Geometry	.43	.56	.31	.93	.71
Algebra 1	.44	.55	.32	.90	.71
Algebra 2	.67	.71	.56	.91	.79
Total	.50	.59	.33	.87	.72

Construct 2: Moving Fluidly Between Representations of the Same Linear Relationship

The items in Construct 2 assessed students' abilities to move fluidly between representations of the same linear relationships, such as equations, graphs, tables, and verbal descriptions. This is slightly more complex than identifying the unit rate in Construct 1, because students needed to be able to identify the unit rate in various representations to determine which were representing the same data. Construct 2 was assessed by five items. The mean Construct 2 scores for each mathematics course are listed in Table 4.10 and Figure 4.2. The maximum score for Construct 2 was five points, one point for each of the five items assessing this construct. The overall mean score for Construct 2 was 2.74 ($SD = 1.61$), which shows 54.8% accuracy. Algebra 2 students had the highest mean score ($M = 3.39$, $SD = 1.47$), and Grade 7 Math students had the

lowest mean score ($M = 1.92$, $SD = 0.99$). This means that the Algebra 2 students were able to move fluidly between representations of the same linear relationships with greater success than students in other classes.

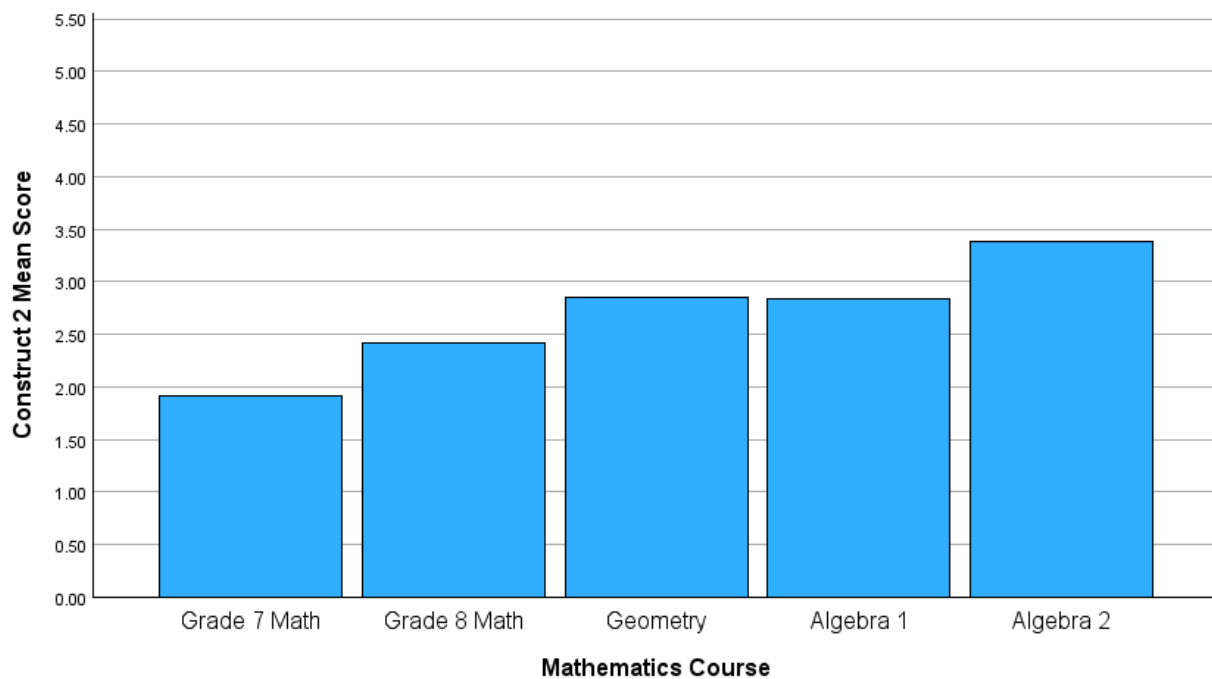
Table 4.10

Construct 2 Descriptives

	N	Mean	Std. Deviation	Minimum	Maximum
Grade 7 Math	24	1.92	.99	.00	4.00
Grade 8 Math	52	2.41	1.81	.00	5.00
Geometry	43	2.85	1.51	.00	5.00
Algebra 1	31	2.84	1.64	.00	5.00
Algebra 2	45	3.39	1.47	.50	5.00
Total	195	2.74	1.61	.00	5.00

Figure 4.2

Construct 2 Mean Score by Mathematics Course



The items that assess Construct 2 address CCSSM standards from Grades 6, 7, and 8. Given the range of standards, it would be expected that students in Grade 7 Math should score the lowest due to lack of exposure to some of the content covered in these items. Students enrolled in Grade 8 Math may also have had limited learning experiences with some of the content assessed in this Construct depending on the time of year these particular Grade 8 standards are taught in that course. The mean scores for Construct 2 increased with each mathematics course from Grade 7 Math through Algebra 2, except between Geometry and Algebra 1, whose mean scores were almost the same. The mathematical ideas of different representations of linear relationships are not present in a typical Geometry course, but should definitely be present in an Algebra 1 course, so it was surprising that the mean score for students enrolled in Algebra 1 was slightly lower than the mean score for students enrolled in Geometry. Similar to the analysis of Construct 1, Algebra 2 students may have been able to use a variety of strategies to compare different representations of linear relationships that they have learned in different mathematics courses and were therefore able to see more success on Construct 2 items.

Overall, the mean scores for students in each mathematics course were lower for Construct 2 than they were for Construct 1. If students struggled with identifying unit rates in isolation in Construct 1, it would be expected that they would have a difficult time identifying different representations of a given linear relationship. The ability to recognize and move fluidly between different representations of a linear relationship is essential for analyzing and interpreting linear relationships, which we will see in Construct 3.

The Construct 2 data did not meet the homogeneity of variances criteria for an ANOVA, as shown in Table 4.11. However, the Games-Howell post hoc analysis, which adjusts for heterogeneity of variances was conducted for the Construct 2 to determine whether any

significant pairwise differences existed in Construct 2. The ANOVA results for Construct 2 are shown in Table 4.12. The results of the Games-Howell post hoc analysis can be found in Table 4.13 and show that a significant difference in Construct 2 scores exists between the following courses: Grade 7 Math and Geometry, Grade 7 Math and Algebra 2, and Grade 8 Math and Algebra 2. These pairs of courses are the same pairs that had significantly different overall test scores, so the Construct 2 items may highlight the overall differences in conceptual understanding of linear relationships.

Table 4.11

Construct 2 - Tests of Homogeneity of Variances

		Levene Statistic	df1	df2	Sig.
Construct 2	Based on Mean	5.73	4	190	<.001
Total					

Table 4.12

Construct 2 ANOVA

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	41.57	4	10.39	4.28	.002
Within Groups	461.10	190	2.43		
Total	502.67	194			

Table 4.13

Construct 2 Games-Howell Post Hoc Analysis

(I) Mathematics Course	(J) Mathematics Course	Mean Difference (I-J)	Std. Error	Sig.
Grade 7 Math	Grade 8 Math	-.50	.32	.539
	Geometry	-.93*	.31	.027
	Algebra 1	-.92	.36	.089

	Algebra 2	-1.47*	.30	<.001
Grade 8 Math	Geometry	-.44	.34	.706
	Algebra 1	-.43	.39	.807
	Algebra 2	-.98*	.33	.034
Geometry	Algebra 1	.01	.37	1.000
	Algebra 2	-.54	.32	.438
Algebra 1	Algebra 2	-.55	.37	.566

The content of Construct 2 assessed students' abilities to move fluidly between representations of the same linear relationship. This construct was assessed by five items (Item 1, 2, 3, 6, and 10). The mean scores for each item by mathematics course are shown in Table 4.14. Given the significant difference in Construct 2 scores between students enrolled in Grade 7 Math and Geometry, Grade 7 Math and Algebra 2, and Grade 8 Math and Algebra 2, focus should be given to the scores for Grade 7 Math, Grade 8 Math, Geometry, and Algebra 2 courses within this construct.

When comparing Grade 7 Math scores with Algebra 2 scores, the Algebra 2 students outscored the Grade 7 Math students by at least .23 points on every item in construct 2. This suggests that all items in Construct 2 contributed to the difference in ability to move fluidly between representations of the same linear relationship for Grade 7 Math and Algebra 2 students. The largest difference in scores between Grade 7 Math and Geometry occurred for items one, three, and ten. These items likely contributed to the difference in ability to fluidly move between representations of linear relationships for students in Grade 7 Math and Geometry. Items one and ten are the items with the greatest difference in score for Grade 8 Math and Algebra 2 students, and likely contributed to their significant difference in Construct 2 score.

Table 4.14

Mean Item Score by Mathematics Course for Construct 2 – Moving Fluidly between Representations of the Same Linear Relationship

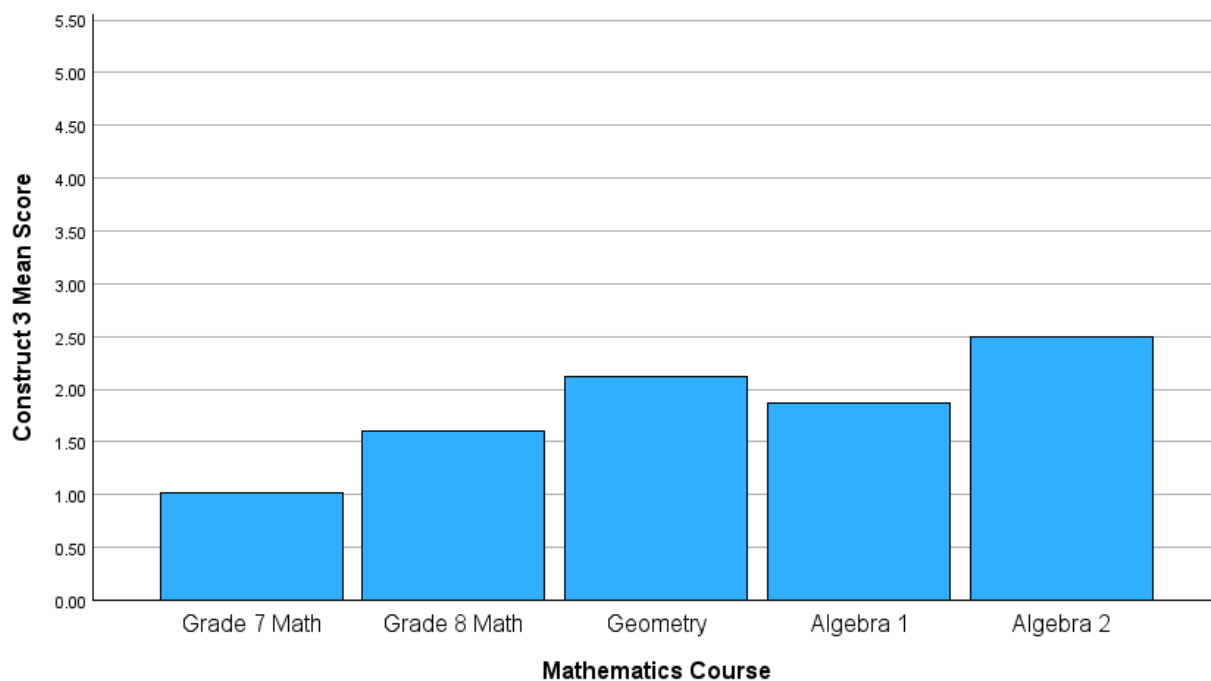
Mathematics Course	Q1 Points	Q2 Points	Q3 Points	Q6 Points	Q10 Points
Grade 7 Math	.54	.25	.46	.25	.42
Grade 8 Math	.54	.37	.54	.39	.58
Geometry	.74	.35	.65	.36	.74
Algebra 1	.65	.48	.71	.32	.68
Algebra 2	.80	.53	.69	.57	.80
Total	.66	.41	.62	.40	.66

Construct 3: Interpreting Different Representations of Linear Relationships

The items in Construct 3 asked students to interpret different representations of linear relationships, such as equations, graphs, tables, and verbal descriptions. Items in Construct 3 rely on the foundations of being able to identify a unit rate in Construct 1 and move fluidly between different representations of linear relationships in Construct 2. Construct 3 was assessed by five items. The mean Construct 3 scores for each mathematics course are listed in Table 4.15 and Figure 4.3. The maximum total score for Construct 3 was five points, one point for each of the five items assessing this construct. The overall mean score for Construct 3 was 1.89, which shows 37.8% accuracy in this construct. Algebra 2 students had the highest mean score, which was 2.50 points (50% accuracy), and Grade 7 Math students had the lowest mean score, with 1.02 (20.4% accuracy) points. These mean scores show that students in all mathematics courses struggled with interpreting different representations of linear relationships.

Table 4.15*Construct 3 Descriptives*

	N	Mean	Std. Deviation	Minimum	Maximum
Grade 7 Math	24	1.02	.94	.00	3.00
Grade 8 Math	52	1.61	1.25	.00	4.00
Geometry	43	2.12	1.55	.00	5.00
Algebra 1	31	1.87	1.32	.00	4.50
Algebra 2	45	2.50	1.58	.00	5.00
Total	195	1.89	1.44	.00	5.00

Figure 4.3*Construct 3 Mean Score by Mathematics Course*

The mean scores in Construct 3 were the lowest for students in all mathematics courses out of the three constructs. This was to be expected, as the content being assessed was more complex and required more prior knowledge and understanding than items in the other constructs. The mean scores, and therefore the ability to interpret different representations of

linear relationships, did increase gradually for each mathematics course from Grade 7 Math through Algebra 2, with the exception of the mean score dropping from students in Geometry to students in Algebra 1. It appears that opportunities to engage in more mathematical experiences and different types of problems throughout the years most likely helped students more accurately interpret linear relationships. Students should have a solid understanding of linear relationships by the time they are enrolled in Algebra 2 and would ideally be able to interpret different representations of linear relationships at a higher rate than 50%, which is the mean score of Algebra 2 students on items in this construct. Students may have struggled with items that ask them to interpret different representations of linear relationships if they have not been given many opportunities to engage with these types of problems in their mathematics courses, or if they have a strictly procedural understanding of linear relationships. Students across all mathematics courses demonstrated a weakness in Construct 3.

The Construct 3 data did not meet the homogeneity of variances criteria for an ANOVA, as shown in Table 4.16. However, the Games-Howell post hoc analysis, which adjusts for heterogeneity of variances was conducted to determine whether any pairwise differences existed in Construct 3. The results of the ANOVA are shown in Table 4.17. The results of the Games-Howell post hoc analysis can be found in Table 4.18 and show that a significant difference in Construct 3 scores exists between the following courses: Grade 7 Math and Geometry, Grade 7 Math and Algebra 2, and Grade 8 Math and Algebra 2. These pairs of courses are the same pairs that had significantly different overall test scores, so the ability to interpret different representations of linear relationships likely contributed to the overall differences in conceptual understanding of linear relationships.

Table 4.16*Construct 3 - Tests of Homogeneity of Variances*

		Levene Statistic	df1	df2	Sig.
Construct 3 Total	Based on Mean	3.96	4	190	.004

Table 4.17*Construct 3 ANOVA*

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	41.29	4	10.32	5.41	<.001
Within Groups	362.81	190	1.91		
Total	404.10	194			

Table 4.18*Construct 3 Games-Howell Post Hoc Analysis*

(I) Mathematics Course	(J) Mathematics Course	Mean Difference (I-J)	Std. Error	Sig.
Grade 7 Math	Grade 8 Math	-.58	.26	.170
	Geometry	-1.10*	.30	.005
	Algebra 1	-.85	.30	.054
	Algebra 2	-1.48*	.30	<.001
Grade 8 Math	Geometry	-.51	.29	.415
	Algebra 1	-.27	.29	.894
	Algebra 2	-.89*	.29	.024
Geometry	Algebra 1	.25	.33	.948
	Algebra 2	-.38	.33	.780
Algebra 1	Algebra 2	-.63	.33	.335

The content of Construct 3 assessed students' abilities to interpret different representations of linear relationships. It was assessed by five items (Items 8, 9, 12, 13, and 15).

The mean scores for each item by mathematics course are shown in Table 4.19. Given the significant difference in Construct 3 scores between students enrolled in Grade 7 Math and Geometry, Grade 7 Math and Algebra 2, and Grade 8 Math and Algebra 2, focus should be given to the scores for Grade 7 Math, Grade 8 Math, Geometry, and Algebra 2 courses within this construct.

The scores for item 9 were roughly the same across all mathematics courses, so this item likely did not contribute to the difference in abilities to interpret different representations of linear relationships between mathematics courses. Items 8, 12, 13, and 15 each showed great differences between the pairs of mathematics courses with significant differences in scores, suggesting they all contributed to the differences in students' abilities to interpret different representations of linear relationships between Grade 7 Math and Algebra 2, Grade 7 Math and Geometry, and Grade 8 Math and Algebra 2.

Table 4.19

Mean Item Score by Mathematics Course for Construct 3 – Interpreting Different Representations of Linear Relationships

Mathematics Course	Q8 Points	Q9 Points	Q12 Points	Q13 Points	Q15 Points
Grade 7 Math	.25	.29	.17	.29	.02
Grade 8 Math	.33	.27	.33	.46	.22
Geometry	.42	.30	.51	.49	.40
Algebra 1	.29	.36	.45	.48	.29
Algebra 2	.58	.29	.60	.60	.43
Total	.39	.30	.43	.48	.30

Item Content Analysis

The mean scores for each assessment item broken down by mathematics course is shown in Table 4.20. The highest mean score for each assessment item is bolded. Students enrolled in

Algebra 2 had the highest mean score on 12 of the 15 items on the assessment, Algebra 1 students had the highest mean score on two items, and Geometry students had the highest mean score on one item. The following section will provide an analysis of the results of each individual item on the assessment.

Table 4.20

Mean Item Score by Mathematics Course

Mathematics Course	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15
Grade 7 Math	.54	.25	.46	.29	.33	.25	.13	.25	.29	.42	.79	.17	.29	.54	.02
Grade 8 Math	.54	.37	.54	.54	.64	.39	.24	.33	.27	.58	.81	.33	.46	.78	.22
Geometry	.74	.35	.65	.43	.56	.36	.31	.42	.30	.74	.93	.51	.49	.71	.40
Algebra 1	.65	.48	.71	.44	.55	.32	.32	.29	.36	.68	.90	.45	.48	.71	.29
Algebra 2	.80	.53	.69	.67	.71	.57	.56	.58	.29	.80	.91	.60	.60	.79	.43
Total	.66	.41	.62	.50	.59	.40	.33	.39	.30	.66	.87	.43	.48	.72	.30

Item 1: Greeting Cards Connecting Table to Equation Task

Item 1 presented students with a story about Angela making and selling greeting cards. They were then given a table of values representing a linear relationship between the number of greeting cards sold (n) and profit made (p) and were asked to choose the linear equation that represented that relationship. This relationship was directly proportional, with the cost per card being \$0.50. This item was chosen to determine whether students would be able to identify the equation of a simple, proportional relationship from given values. This would show whether students were able to fluently move between a tabular and algebraic representation of a directly proportional linear relationship. This item was aligned to Common Core State Standards for Mathematics (CCSSM) from Grade 6, so all students in this study have had exposure to this content and should have been able to identify the correct equation to represent the situation.

In each mathematics course, the majority of the students chose the correct answer, *A*: $p=0.5n$, as shown in Table 4.21 and Figure 4.4. The second most popular answer choice in every mathematics course was *B*: $p=2n$. Students who chose this answer may have looked at the relationship backwards, seeing that the number of cards sold was twice the profit shown. The high school students outperformed the middle school students on this task, which may be as a result of having multiple learning experiences with tables and equations over time in multiple mathematics courses. The distribution of responses from Grade 8 students on the 2007 National Assessment of Educational Progress (NAEP) for this item can be found in Table 4.21 for comparison to students in this study. The distribution of answer choices for Grade 8 Math students in this study closely resembles those from the 2007 NAEP for students in Grade 8, which shows that the students in this study have had similar experiences with writing equations from tables as the random sample of eighth grade students who took the NAEP in 2007.

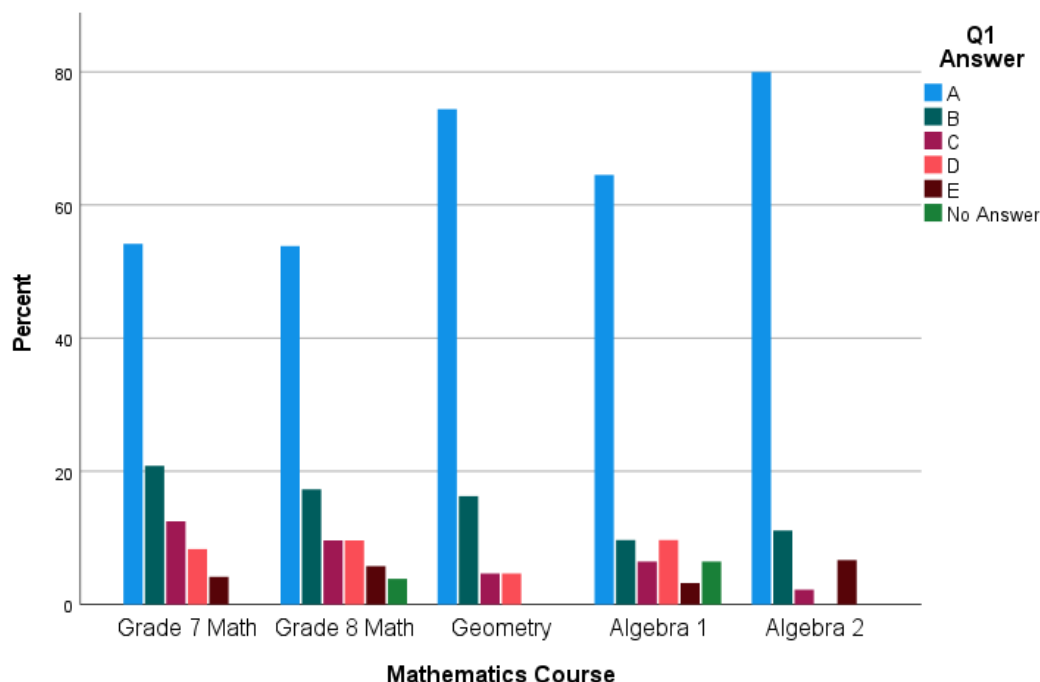
Table 4.21

Item 1 Responses by Mathematics Course

		2007 NAEP Grade 8	Mathematics Course										Total	
			Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
			N	%	N	%	N	%	N	%	N	%	N	%
A		54.0%	13	54.2%	28	53.8%	32	74.4%	20	64.5%	36	80.0%	129	66.2%
B		17.0%	5	20.8%	9	17.3%	7	16.3%	3	9.7%	5	11.1%	29	14.9%
C		13.0%	3	12.5%	5	9.6%	2	4.7%	2	6.5%	1	2.2%	13	6.7%
D		6.0%	2	8.3%	5	9.6%	2	4.7%	3	9.7%	0	0.0%	12	6.2%
E		9.0%	1	4.2%	3	5.8%	0	0.0%	1	3.2%	3	6.7%	8	4.1%
No Answer		1.0%	0	0.0%	2	3.8%	0	0.0%	2	6.5%	0	0.0%	4	2.1%
Total		100.0%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.4

Item 1 Responses by Mathematics Course, Correct Answer: A



Item 2: Connecting Equation to Graph Task

The second item asks students to identify the correct graph of a line when given the linear equation. This item did not provide a context, as it was intended to highlight whether students understand the relationship between the symbolic and graphical representations of the same linear relationship by identifying the graph of a linear equation. The equation given in this item was $y = -2x + 1$, which requires students to correctly recognize and interpret the slope and rate of change when choosing the correct graph.

The correct answer for this item was graph A, which was the most frequently selected answer for students in all mathematics courses except Grade 7 Math, as shown in Table 4.22 and Figure 4.5. The scores in Grade 7 Math should have been lower on this item, because graphing

linear equations in the form $y=mx+b$ is not something that is introduced in the CCSSM standards until Grade 8. Students in Algebra 1 and Algebra 2 had the highest percentage of correct answer choices, which also makes sense because at that point in their mathematics education journey, students have been graphing linear equations for multiple years and should be relatively comfortable with the concept. The students in Grade 7 Math had about the same distribution of answer choices as the Grade 8 students who took the 2007 NAEP, and the students in Grade 8 Math outscored those on the 2007 NAEP, so graphing linear equations must have been focused on heavily in the Grade 8 Math course at the school being studied.

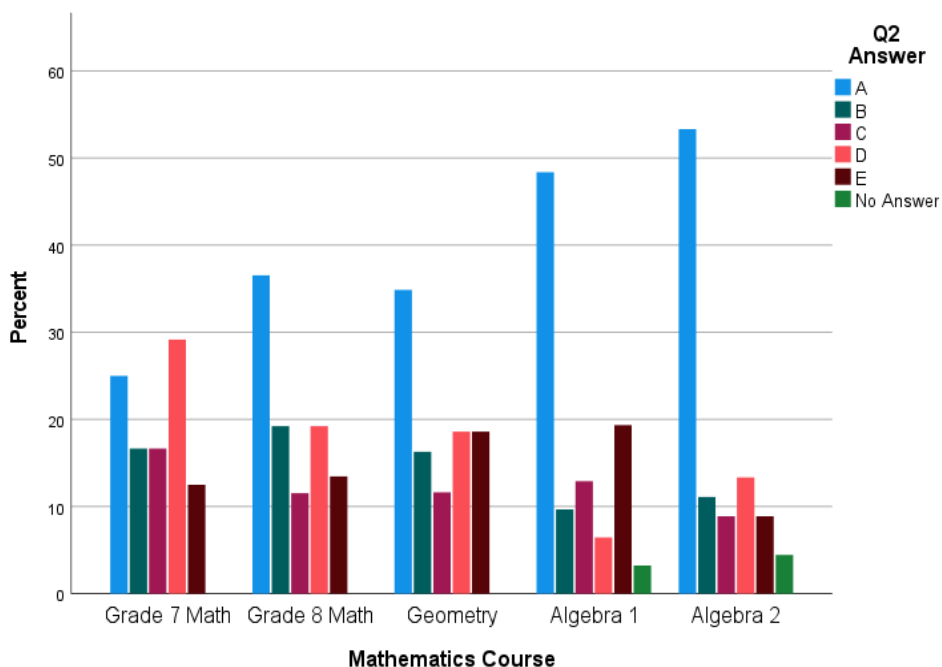
Table 4.22

Item 2 Responses by Mathematics Course

	2007 NAEP Grade 8	Mathematics Course										Total	
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2		N	%
		N	%	N	%	N	%	N	%	N	%		
A	25%	6	25.0%	19	36.5%	15	34.9%	15	48.4%	24	53.3%	79	40.5%
B	15%	4	16.7%	10	19.2%	7	16.3%	3	9.7%	5	11.1%	29	14.9%
C	15%	4	16.7%	6	11.5%	5	11.6%	4	12.9%	4	8.9%	23	11.8%
D	31%	7	29.2%	10	19.2%	8	18.6%	2	6.5%	6	13.3%	33	16.9%
E	13%	3	12.5%	7	13.5%	8	18.6%	6	19.4%	4	8.9%	28	14.4%
No Answer	1%	0	0.0%	0	0.0%	0	0.0%	1	3.2%	2	4.4%	3	1.5%
Total	100.0%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.5

Item 2 Responses by Mathematics Course, Correct Answer: A



Item 3: Connecting Table to Equation Task

The third item on the assessment was similar to the first item because it presented students with a table of values and asked them to choose the correct equation of the line represented in the table. This problem provided no context, students were only presented with a table showing a set of x and y values. The other difference between this item and the first is that it is not directly proportional, it had a nonzero y -intercept. This item was intended to determine whether students were able to move fluidly between the tabular and algebraic representation of a linear relationship that is not directly proportional. Students are asked to write equations from real-world situations and from tables according to the CCSSM starting in sixth grade, so identifying a correct equation from a table is something that we should expect most students to be able to do. However, students are exposed to very few relationships that are not directly

proportional in Grade 6 and Grade 7, so students in Grade 7 Math were expected to score the lowest on this item.

The correct answer was $C: y=2x+1$ and the majority of students enrolled in every mathematics course chose the correct answer (Table 4.23, Figure 4.6). The second most frequent answer choice in all grade levels was $A: y=x+2$. This would be the correct equation if students only looked at the first line of the table, when x was 1 and y was 3, but the equation does not accurately represent the rest of the data in the table. As predicted when looking at curricular materials, students in Grade 7 had the lowest percentage of correct responses, indicating they do not have a complete understanding of representing non-directly proportional relationships. Students in Grade 8 did not score substantially higher than those in Grade 7, but students in the high school courses all scored substantially higher, which may be due to having more practice with these types of relationships throughout multiple mathematics courses.

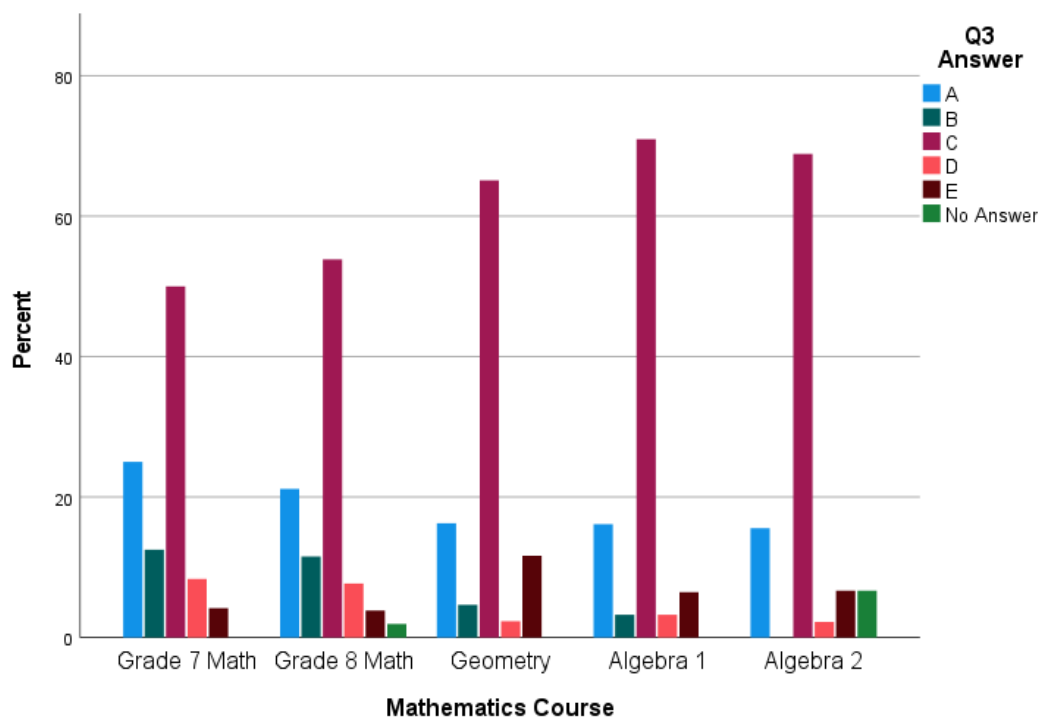
Table 4.23

Item 3 Responses by Mathematics Course

	2013 NAEP Grade 8	Mathematics Course										Total	
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
		N	%	N	%	N	%	N	%	N	%	N	%
A	16%	6	25.0%	11	21.2%	7	16.3%	5	16.1%	7	15.6%	36	18.5%
B	7%	3	12.5%	6	11.5%	2	4.7%	1	3.2%	0	0.0%	12	6.2%
C	67%	12	50.0%	28	53.8%	28	65.1%	22	71.0%	31	68.9%	121	62.1%
D	5%	2	8.3%	4	7.7%	1	2.3%	1	3.2%	1	2.2%	9	4.6%
E	4%	1	4.2%	2	3.8%	5	11.6%	2	6.5%	3	6.7%	13	6.7%
No Answer	1%	0	0.0%	1	1.9%	0	0.0%	0	0.0%	3	6.7%	4	2.1%
Total	100.0%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.6

Item 3 Responses by Mathematics Course, Correct Answer: C



Item 4: Walk-a-Thon Task – Missing Table Entries

Item four on this assessment was part of a three-item set that are all based on the same real-world situation. Students were presented with a story about Julianna participating in a walk-a-thon and recording the total distance she walked at various points in time. In item four, students are told that Julianna’s paper got smudged and she is missing a few of the entries in her table. They were asked to fill in three missing entries for time walked and miles walked for the partially completed table. The relationship presented was directly proportional, but the independent variable (time in hours) was not presented in uniform increments, so students needed to know how to calculate and apply a unit rate to be able to correctly complete the table.

This item was an open-response item and was therefore scored using a rubric to award each student 0, 0.5, or 1 point. If a student entered zero of the three missing values into the table,

they earned 0 points. If the student entered one or two of the three values correctly, they earned 0.5 points, and if they entered all three missing values correctly, they earned 1 point. The number and percentage of students in each mathematics course who earned each score level is presented in Table 4.24.

The students enrolled in Algebra 2 showed the most correct understanding of this task with 62.2% of students earning a full point for a completely correct table. The students in Grade 7 Math showed the least correct understanding with only 25% of students earning a full point for a correct table. Two-thirds of the students enrolled in Grade 7 Math earned 0 points on this item, which shows that completing tables with unfamiliar missing values is an area of mathematics in which they need more practice. Two to three students in each mathematics course left this item blank or wrote “IDK” as their response, which earned them 0 points.

Table 4.24

Item 4 Scores by Mathematics Course

	Mathematics Course											
	Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
	N	%	N	%	N	%	N	%	N	%	N	%
0.0	16	66.7%	18	34.6%	20	46.5%	17	54.8%	13	28.9%	84	43.1%
0.5	2	8.3%	12	23.1%	9	20.9%	1	3.2%	4	8.9%	28	14.4%
1.0	6	25.0%	22	42.3%	14	32.6%	13	41.9%	28	62.2%	83	42.6%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Students who earned a full point on this item either calculated the unit rate using one row of values, or used proportions to find the missing values in the table. Student work samples of correct responses are shown in Figure 4.7. Each student’s test was assigned an identification code, where the first number represents their grade level, the letter represents their math course (a = Grade 7 Math, b = Grade 8 Math, c = Geometry, d = Algebra 1, e = Algebra 2), and the

second number represents the sequentially ordered test number for students in the same grade level and math course. Students 7a6 and 7b12 both calculated the unit rate by dividing 6.4 miles walked by 2 hours to determine that Julianna walked 3.2 miles per hour. These two students multiplied that unit rate by the given hours in the table to determine the number of miles walked for one and five hours, then used the same logic to work backwards and find that it took Julianna 2.5 hours to walk 8 miles. These students used the same strategy, with Student 7a6 writing their work in the workspace and Student 7b12 writing their work within the table. Student 9c29 used proportions to find the missing values in the table. This student used the given complete row of the table showing it took Julianna 2 hours to walk 6.4 miles to set up three separate proportions to calculate the times and miles that were missing in the table. These three student work samples were chosen to highlight two strategies that many students used to correctly complete this table. Additional examples of student work using these strategies can be found in Appendix F.

Figure 4.7

Item 4 Correct Responses and Strategies

Student 7a6 <table border="1" data-bbox="544 352 820 562"> <thead> <tr> <th>Time in Hours (<i>n</i>)</th><th>Miles walked (<i>d</i>)</th></tr> </thead> <tbody> <tr> <td>1</td><td>3.2</td></tr> <tr> <td>2</td><td>6.4</td></tr> <tr> <td>2.5</td><td>8</td></tr> <tr> <td>5</td><td>16</td></tr> </tbody> </table>	Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)	1	3.2	2	6.4	2.5	8	5	16	<u>Workspace (if needed):</u> $6.4 / 2 = 3.2$ $8 / 3.2 = 2.5$ $5 \cdot 3.2 = 16$
Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)										
1	3.2										
2	6.4										
2.5	8										
5	16										
Student 7b12 <table border="1" data-bbox="535 640 820 861"> <thead> <tr> <th>Time in Hours (<i>n</i>)</th><th>Miles walked (<i>d</i>)</th></tr> </thead> <tbody> <tr> <td>1 $\times 3.2$</td><td>3.2</td></tr> <tr> <td>2 $\times 3.2$</td><td>6.4</td></tr> <tr> <td>2.5 $\times 3.2$</td><td>8</td></tr> <tr> <td>5</td><td>16</td></tr> </tbody> </table>	Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)	1 $\times 3.2$	3.2	2 $\times 3.2$	6.4	2.5 $\times 3.2$	8	5	16	<u>Workspace (if needed):</u> $\begin{array}{r} 3.2 \\ 2 \overline{) 6.4} \end{array}$
Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)										
1 $\times 3.2$	3.2										
2 $\times 3.2$	6.4										
2.5 $\times 3.2$	8										
5	16										
Student 9c29 <table border="1" data-bbox="548 928 824 1129"> <thead> <tr> <th>Time in Hours (<i>n</i>)</th><th>Miles walked (<i>d</i>)</th></tr> </thead> <tbody> <tr> <td>1</td><td>3.2</td></tr> <tr> <td>2</td><td>6.4</td></tr> <tr> <td>2.5</td><td>8</td></tr> <tr> <td>5</td><td>16</td></tr> </tbody> </table>	Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)	1	3.2	2	6.4	2.5	8	5	16	<u>Workspace (if needed):</u> $\frac{2}{6.4} = \frac{x}{8} \rightarrow \frac{16}{6.4} = \frac{6.4x}{6.4} \rightarrow x = 2.5$ $\frac{1}{x} = \frac{2}{6.4} \rightarrow \frac{6.4}{2} = \frac{2x}{2} \rightarrow x = 3.2$ $\frac{5}{x} = \frac{2}{6.4} \rightarrow \frac{32}{2} = \frac{2x}{2} \rightarrow x = 16$
Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)										
1	3.2										
2	6.4										
2.5	8										
5	16										

Many students seemed to have assumed the time in hours was increasing by one for each entry and just subtracted the miles given to find the difference for each entry of miles walked. This common error was made by three students in Grade 8 Math, 11 students in Geometry, eight students in Algebra 1 and six students in Algebra 2. Examples of this error in student work is shown in Figure 4.8. Students 8b14, 9c15, and 10d7 each subtracted 6.4 from 8 to get a rate of change of 1.6. Student 8b14 incorrectly calculated a unit rate of 1.6 using this strategy, but they showed one understanding that the other students did not. This student wrote 1.6 for the number of miles walked in one hour for the top row of the table, showing that they know that the unit rate should be the number of miles walked in one hour. The other two students used the unit rate

to subtract from 6.4 to get 4.8 as the number of miles walked in the top row of the table, and all three students added 1.6 to the given 8 miles to find 9.6 as the number of miles for the bottom row of the table. These students did not use the number of hours at all in their calculations, which indicates they were assuming the hours were increasing by one for each row, as shown by students 9c15 and 10d7 continuing the hours column of the table by placing a “3” in the empty cell, following 1 and 2 hours. This error shows that students are accustomed to tables increasing by one unit and not skipping values, which means they may have a strictly procedural view of finding a unit rate from tables and are not yet able to understand the relationship between all of the values in a table.

Figure 4.8

Item 4 Common Errors

Student
8b14

Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)
1	1.6
2	6.4
4.8	8
5	9.6

Workspace (if needed):

$$8 - 6.4 = 1.6$$
~~$$6.4 - 1.6 = 4.8$$~~

$$8 - 6.4 = 1.6$$

$$8 + 1.6 = 9.6$$

Student
9c15

Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)
1	4.8
2	6.4
3	8
5	9.6

$$\begin{array}{r} 5 \times 1.6 \\ - 1.6 \\ \hline 4.8 \end{array}$$

Workspace (if needed):

1.6

Student
10d7

Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)
1	4.8
2	6.4
3	8
5	9.6

Workspace (if needed):

$$8 - 6.4 = 1.6$$

$$6.4 - 1.6 = 4.8$$

$$1 + 1 = 2 \quad 1 + 2 = 3$$

$$2 + 3 = 5 \quad 5 - 2 = 3$$

Overall, the students in this study were not able to show proficiency with using a unit rate to complete a table of values. Students in higher-level mathematics courses performed better than those in lower-level courses, but over 25% of the students in each mathematics course earned zero points on this item. Students need to engage in more learning and practice opportunities around using a unit rate to complete a table of values representing a linear relationship to develop a more solid understanding.

Item 5: Walk-a-Thon Task – Stating Unit Rate

Item five was the second open-response item in the three-item Walk-a-Thon task. After completing the table in item four, students were now asked to state Julianna’s walking rate in miles per hour. The purpose of this item was to determine whether students were able to use the information given in a table to calculate the constant of proportionality or unit rate in a real-world situation, which is a seventh-grade standard in the CCSSM and curricular materials.

This item was scored using a rubric from zero to one point, and the scores for each mathematics course are shown in Table 4.25. Students earned 0 points if they had an incorrect unit rate with no thought process shown. They received 0.5 points if they had an incorrect unit rate but a clear thought process was shown or they used the incorrect unit rate from item four. They earned 1 point if they provided the correct unit rate of 3.2 miles per hour. The Algebra 2 students had the highest mean score ($M = .71$, $SD = .42$), and Grade 7 Math students had the lowest mean score ($M = .33$, $SD = .48$), which makes sense because this item was so closely connected to item 4. Most students did not show work for their response on this item, so it is not clear how students arrived at their incorrect responses. Since student thinking was not shown, no further analysis was done on this item, but calculating the unit rate for a situation when given a table was not an area of strength for many students.

Table 4.25

Item 5 Scores by Mathematics Course

		Mathematics Course										Total	
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
		N	%	N	%	N	%	N	%	N	%	N	%
0.0	16	66.7%	17	32.7%	14	32.6%	10	32.3%	10	22.2%	67	34.4%	
0.5	0	0.0%	4	7.7%	10	23.3%	8	25.8%	6	13.3%	28	14.4%	
1.0	8	33.3%	31	59.6%	19	44.2%	13	41.9%	29	64.4%	100	51.3%	

Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%
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Item 6: Walk-a-Thon Task – Writing Equation for Given Situation

The sixth item on this test was the final item in the set about Julianna’s walk-a-thon. In this item, students were asked to write an equation for the distance d , in miles, that Julianna walked in n hours. This item was scored on the same 0, 0.5, 1 scale and students were awarded 0.5 points if they had an equation based on the incorrect unit rate they found in item five. Students earned a full point if they provided the correct equation, $d=3.2n$. This item was included to determine whether students were able to move from a tabular representation to an algebraic representation of a proportional relationship.

The distribution of points for students in each mathematics course can be found in Table 4.26. Once again, students in Algebra 2 had the highest percentage of correct responses. When comparing the percentage of students who earned a full point for correctly identifying the unit rate in item five with those who earned a full point for providing a correct equation in this item, they were lower for item six in every mathematics course. The decline in scores on this item highlights a gap in understanding of how variables and equations are used to represent real-world situations algebraically. This is a topic that students are introduced to in elementary school with unknown values and more concretely with multiple variables in Grade 6, so all students who participated in this study should have had multiple learning experiences and opportunities to practice writing equations to represent a relationship. A disconnect seems to exist between correctly identifying the unit rate and using that unit rate to write an equation to represent the situation.

Table 4.26*Item 6 Scores by Mathematics Course*

Mathematics Course													
Grade 7 Math			Grade 8 Math		Geometry		Algebra 1		Algebra 2		Total		
	N	%	N	%	N	%	N	%	N	%	N	%	
0.0	17	70.8%	30	57.7%	26	60.5%	20	64.5%	17	37.8%	110	56.4%	
0.5	2	8.3%	3	5.8%	3	7.0%	2	6.5%	5	11.1%	15	7.7%	
1.0	5	20.8%	19	36.5%	14	32.6%	9	29.0%	23	51.1%	70	35.9%	
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%	

Item 7: Utility Companies – Finding Unit Rate from Given Values and Equation

The seventh item on the assessment was part of a two-item set based on the same real-world scenario. Students were presented with information about two different utility companies selling electricity in units of kilowatt-hours. For Company P, students were given a table of values with two different numbers of kilowatt-hours sold and the total cost in dollars for each of those amounts. For Company M, they were given an equation, $y=0.15x$, and were told that y represented the total cost in dollars for x kilowatt-hours of electricity. Item seven asks students to use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company. This item was included to determine whether students would be able to calculate the unit rate from a table of values, and recognize and interpret the unit rate in an equation for directly proportional relationships. Students are expected to have an understanding of unit rates in Grade 7 according to the CCSSM, so all students in this study should have had experiences with identifying the unit rate or constant of proportionality in tables and equations of proportional relationships.

The scores by mathematics course can be found in Table 4.27. The highest scoring mathematics class was Algebra 2, but only 44.4% of the Algebra 2 students earned a full point on

this item, which required a correct unit rate for both companies. From all the mathematics courses combined, 52 students earned 0.5 points on this item. To earn that score, students had one company's unit rate correct and one company's unit rate incorrect. All but one of the 52 students had Company M's unit rate correct and Company P's unit rate incorrect. Company M was the company whose cost was presented as an equation, so this indicates that students were able to identify the unit rate from a given equation much easier than they were able to calculate a

Table 4.27

Item 7 Scores by Mathematics Course

Mathematics Course													
Grade 7 Math			Grade 8 Math		Geometry		Algebra 1		Algebra 2		Total		
	N	%	N	%	N	%	N	%	N	%	N	%	
.0	19	79.2%	32	61.5%	23	53.5%	16	51.6%	15	33.3%	105	53.8%	
.5	4	16.7%	15	28.8%	13	30.2%	10	32.3%	10	22.2%	52	26.7%	
1.0	1	4.2%	5	9.6%	7	16.3%	5	16.1%	20	44.4%	38	19.5%	
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%	

unit rate from a table. Of the 105 total students who earned 0 points on this item, 57 students either left the item blank or wrote “IDK” indicating that they did not know what to do. The lack of student success on this item shows that students are not comfortable with identifying the unit rate from different representations of linear relationships.

Samples of correct student responses are shown in Figure 4.9 and additional student work samples can be found in Appendix F. Students 7a19 and 10e4 divided one of the given costs by the number of kilowatt-hours given for Company P to determine their unit rate was \$0.12 per kilowatt-hour. Student 10e2 also divided the cost by the number of kilowatt-hours, but they calculated it for both of the given prices for Company P, showing that both had a unit rate of \$0.12 per kilowatt-hour. Student 10d9 saw the given information as ordered pairs and calculated the slope between the two points. They calculated the change in y and the change in x , then

divided them to get a rate of change of \$0.12 per kilowatt-hour. This student was enrolled in Algebra 1, so they may have been more familiar with procedurally using the slope formula to find a rate of change than thinking about the context of the situation and simply dividing the values of cost and kilowatt-hours. Students 7a19, 10e2, and 10d9 all knew the rate of change was the coefficient of x in the given equation for Company M. Student 10d19 labeled 0.15 as the rate on their paper, and the others simply wrote the equation and stated that \$0.15 was the unit rate. Student 10e4 substituted 1650, one of the given number of kilowatt-hours from Company P into the given equation for Company M as the x -value to find Company M's cost for that number of kilowatt-hours. Then, this student used the same strategy as they used for Company P and divided the cost by the number of kilowatt-hours to get \$0.15 per kilowatt-hour. This student does not seem to have an understanding that the unit rate is the slope of the given linear equation, but they were still able to determine the unit rate using a different strategy.

Figure 4.9

Item 7 Correct Responses and Strategies

Student 7a19	7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company. <table style="width: 100%; border: none;"> <tr> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company P</u> Workspace: $\frac{1250}{150} = x$ $x = 0.12$ Unit Rate: <u>0.12</u> </td> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company M</u> Workspace: $y = 0.15x$ 247.5 Unit Rate: <u>0.15</u> </td> </tr> </table>	<u>Company P</u> Workspace: $\frac{1250}{150} = x$ $x = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ 247.5 Unit Rate: <u>0.15</u>
<u>Company P</u> Workspace: $\frac{1250}{150} = x$ $x = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ 247.5 Unit Rate: <u>0.15</u>		
Student 10e2	7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company. <table style="width: 100%; border: none;"> <tr> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company P</u> Workspace: $1250/150 = 8.3$ $1650/198 = 8.3$ $y = 8.3x$ $150/1250 = 0.12$ $198/1650 = 0.12$ Unit Rate: <u>0.12</u> </td> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company M</u> Workspace: $y = 0.15x$ Unit Rate: <u>0.15</u> </td> </tr> </table>	<u>Company P</u> Workspace: $1250/150 = 8.3$ $1650/198 = 8.3$ $y = 8.3x$ $150/1250 = 0.12$ $198/1650 = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ Unit Rate: <u>0.15</u>
<u>Company P</u> Workspace: $1250/150 = 8.3$ $1650/198 = 8.3$ $y = 8.3x$ $150/1250 = 0.12$ $198/1650 = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ Unit Rate: <u>0.15</u>		
Student 10e4	7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company. <table style="width: 100%; border: none;"> <tr> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company P</u> Workspace: $\frac{198}{1650} = 0.12$ $y = 0.12x$ Unit Rate: <u>0.12 per hour</u> </td> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company M</u> Workspace: $y = 0.15x$ $y = 0.15(1650) = 247.5$ $\frac{247.5}{1650} = 0.15$ Unit Rate: <u>0.15 per hour</u> </td> </tr> </table>	<u>Company P</u> Workspace: $\frac{198}{1650} = 0.12$ $y = 0.12x$ Unit Rate: <u>0.12 per hour</u>	<u>Company M</u> Workspace: $y = 0.15x$ $y = 0.15(1650) = 247.5$ $\frac{247.5}{1650} = 0.15$ Unit Rate: <u>0.15 per hour</u>
<u>Company P</u> Workspace: $\frac{198}{1650} = 0.12$ $y = 0.12x$ Unit Rate: <u>0.12 per hour</u>	<u>Company M</u> Workspace: $y = 0.15x$ $y = 0.15(1650) = 247.5$ $\frac{247.5}{1650} = 0.15$ Unit Rate: <u>0.15 per hour</u>		
Student 10d9	7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company. <table style="width: 100%; border: none;"> <tr> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company P</u> Workspace: $150 - 198 = -48 = y$ $1250 - 1650 = -400 = x$ $\frac{-48}{-400} = 0.12$ Unit Rate: <u>0.12</u> </td> <td style="width: 50%; vertical-align: top; padding: 5px;"> <u>Company M</u> Workspace: $y = 0.15x$ ↑ Rate Unit Rate: <u>0.15</u> </td> </tr> </table>	<u>Company P</u> Workspace: $150 - 198 = -48 = y$ $1250 - 1650 = -400 = x$ $\frac{-48}{-400} = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ ↑ Rate Unit Rate: <u>0.15</u>
<u>Company P</u> Workspace: $150 - 198 = -48 = y$ $1250 - 1650 = -400 = x$ $\frac{-48}{-400} = 0.12$ Unit Rate: <u>0.12</u>	<u>Company M</u> Workspace: $y = 0.15x$ ↑ Rate Unit Rate: <u>0.15</u>		

Students did not show much work on this item overall, so it was difficult to determine the various errors in their understanding. Multiple students incorrectly calculated Company P's unit rate by dividing the number of kilowatt-hours by cost instead of the other way around. These students found that the unit rate for Company P was \$8.33 per kilowatt-hour, which was much higher than the \$0.15 for Company M. Samples of this common error in student thinking are shown in Figure 4.10. Students 9c29 and 10e1 both divided the number of kilowatt-hours by the cost for Company P, but notated it differently. Student 10e18 wrote an equation, but did not show their calculation to arrive at 8.33 as the unit rate, so they likely divided 1,250 by 150. All three of these students were able to correctly identify the unit rate for Company M from the given equation, but they struggled to identify the unit rate when given specific values for the situation.

Figure 4.10

Item 7 Common Error

7. Use the information provided to find the unit rate, in <u>dollars per kilowatt-hour</u>, for each company.		
Student 9c29	<u>Company P</u> Workspace: $1,250 \div 150 = 8.\overline{3}$ $1,650 \div 198 = 8.\overline{3}$	<u>Company M</u> Workspace: $y = 0.15x$
	Unit Rate: <u>$8.\overline{3}$ per kilowatt-hour</u>	Unit Rate: <u>0.15 per kilowatt-hour</u>
7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.		
Student 10e1	<u>Company P</u> Workspace: $1,250 / 150 = 8.33$ $.63$ $y = 8.33x$	<u>Company M</u> Workspace:
	Unit Rate: <u>~~100~~ ~~10~~ 8.33</u>	Unit Rate: <u>.15</u>
7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.		
Student 10e18	<u>Company P</u> Workspace: $1.250h = 150d$ $8.\overline{3} \text{ unit rate}$	<u>Company M</u> Workspace:
	Unit Rate: <u>~~8.3~~</u>	Unit Rate: <u>.15x</u>

Item 8: Utility Companies – Interpreting Unit Rate to Determine Least Expensive Option

Item eight on the assessment was a continuation of the utility companies item set that started in item seven. Students were asked which company was the least expensive, and were asked to explain their answer. This item was included to determine whether students could successfully interpret and compare the unit rates of linear relationships when presented with a real-world situation. The correct answer to this item was that Company P is the least expensive company, since their unit rate was \$0.12 per kilowatt-hour, while the unit rate for Company M was \$0.15 per kilowatt-hour. Students earned a half of a point if they had the correct company but no justification, or if they stated the incorrect company, but used their incorrectly calculated unit rates from item seven to justify which company was less expensive.

The results for item eight are shown in Table 4.28. A total of 46 students answered this item correctly and earned the full point. There were 38 students who were able to interpret and compare the unit rates in a real-world situation, although their original calculations of the unit rates in item seven were incorrect. Six students wrote “IDK” on this item, and 45 students left it blank. The high number of students who did not attempt to answer this item, combined with only 46 students answering it correctly shows that overall, students are not able to interpret unit rates in real-world scenarios. This item didn’t have work to be shown, and largely depended on a student’s answer from item 7, so no student work was analyzed for this item.

Students who were enrolled in Algebra 2 had the highest success rate with this item, with 46.7% of students correctly identifying the less expensive company and justifying their choice, compared to the Grade 7 Math and Algebra 1 students, which each had under 13% of students answer the item successfully. Interestingly, students in Grade 8 Math and Geometry had better success with this item than Algebra 1 students. This may be due to the learning experiences and

problems students see in these classes being more real-world focused than those in Algebra 1, which would be something to further research.

Table 4.28

Item 8 Scores by Mathematics Course

	Mathematics Course										Total	
	Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
	N	%	N	%	N	%	N	%	N	%	N	%
.0	15	62.5%	26	50.0%	17	39.5%	17	54.8%	14	31.1%	89	45.6%
.5	6	25.0%	18	34.6%	16	37.2%	10	32.3%	10	22.2%	60	30.8%
1.0	3	12.5%	8	15.4%	10	23.3%	4	12.9%	21	46.7%	46	23.6%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Item 9: Interpreting Slopes and Y-Intercepts of an Equation and Graph

In the ninth item on this assessment, students were given two linear relationships, one represented with a table and the other with an equation. They were given four statements relating the two functions and were asked to choose the statement that is incorrect. Each statement is either comparing the slopes or y-intercepts of each relationship. This item was chosen for the assessment because it forces students to interpret different aspects of the given relationships such as slope and y-intercept and compare these aspects of the two relationships to each other. This item assesses an eighth-grade standard, so the Grade 7 Math students were expected to struggle more than other students due to not being exposed to this content yet.

The distribution of answer selections for students in each mathematics course are shown in Table 4.29 and Figure 4.11 below. The correct answer choice for this item was *C: The y-intercept of the line for function 1 is less than the y-intercept of the line for function 2*, which 30.3% of the total students selected. Choice *C* was the most frequently selected answer choice

for each mathematics course. Answer choices *A* and *D* compare the slope by talking about the rate of change and steepness of the lines. Students are taught to calculate or identify the slope from a table, equation, or graph starting in Grade 8 according to the CCSSM, so those should have been easy for students to calculate and eliminate as answer choices if they have a solid understanding of slope in various representations. Students in Geometry and Algebra 1 had the highest rate of success with this item, which could be attributed to multiple years in a row of calculating and exploring slope.

Out of the 195 students who took this test, 16 students left this item blank, despite it being a multiple-choice item. This could indicate that students were confused by the wording of the problem, or struggled to find an access point to start thinking about the problem. The spread of answer choices in general for each of the grade levels is relatively close to 25% for each answer choice, which would show that students may have been guessing. The lack of success on this item indicates that students have a poor understanding of interpreting slope and y-intercepts of different representations of linear relationships at all mathematics course levels.

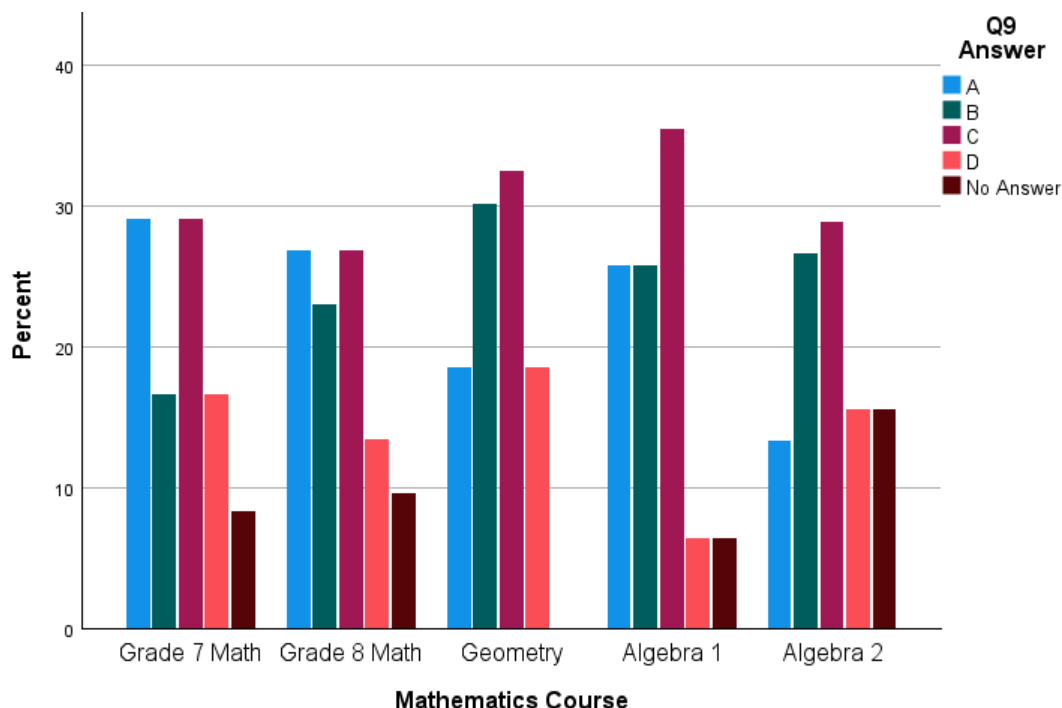
Table 4.29

Item 9 Responses by Mathematics Course

	Mathematics Course										Total	
	Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
	N	%	N	%	N	%	N	%	N	%	N	%
A	7	29.2%	14	26.9%	8	18.6%	8	25.8%	6	13.3%	43	22.1%
B	4	16.7%	12	23.1%	13	30.2%	8	25.8%	12	26.7%	49	25.1%
C	7	29.2%	14	26.9%	14	32.6%	11	35.5%	13	28.9%	59	30.3%
D	4	16.7%	7	13.5%	8	18.6%	2	6.5%	7	15.6%	28	14.4%
No Answer	2	8.3%	5	9.6%	0	0.0%	2	6.5%	7	15.6%	16	8.2%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.11

Item 9 Responses by Mathematics Course, Correct Answer: C



Item 10: Plumber's Cost – Connecting Verbal Description to Equation

The tenth item on this assessment presents students with a verbal description of a linear relationship associated with a plumber's charges for working, then asks them to choose the correct algebraic expression to represent the plumber's total charge for working h hours. This item addresses a seventh-grade standard of being able to use variables to represent unknown quantities and construct equations to represent real-world situations, which students start working towards in earlier grades as well. The answer selections by mathematics course can be found below in Table 4.30 and Figure 4.12.

The correct answer choice to this problem was $E: (48 \times h) + 9$, which the majority of students in each mathematics course chose. In Algebra 2, 80% of students chose the correct answer, showing that they have a solid understanding of assigning variables and using

expressions to model real-world situations, which makes sense because they have practiced solving this type of problem for many years through each mathematics course's curriculum. This item was on the 1996 Grade 8 NAEP assessment, and 58% of the Grade 8 students who took the test chose the correct answer. In this study, 57.7% of the Grade 8 students who took the assessment answered this item correctly, so the mathematics experiences that these students have had seem to be on par with the national average from the NAEP test.

Answer choices *C*: $48 + (9 \times h)$ and *A*: $48 + 9 + h$ were the most frequently chosen incorrect answers across the grade levels. Students who chose *C* likely have gaps in their understanding of unit rates in context. They knew that one of the charges needed to be multiplied by the number of hours worked, but chose the travel fee for this instead of the charge per hour. These students would need additional practice with identifying rates of change in real-world scenarios and translating them into algebraic expressions. Students who chose *A* do not seem to understand that anything is being multiplied in this real-world situation, so they likely do not understand rate of change in general. They may have seen the word "plus" in the verbal description and decided to add all values from the description together. These students would need additional instruction and practice in proportional relationships, unit rates, and rates of change.

Table 4.30

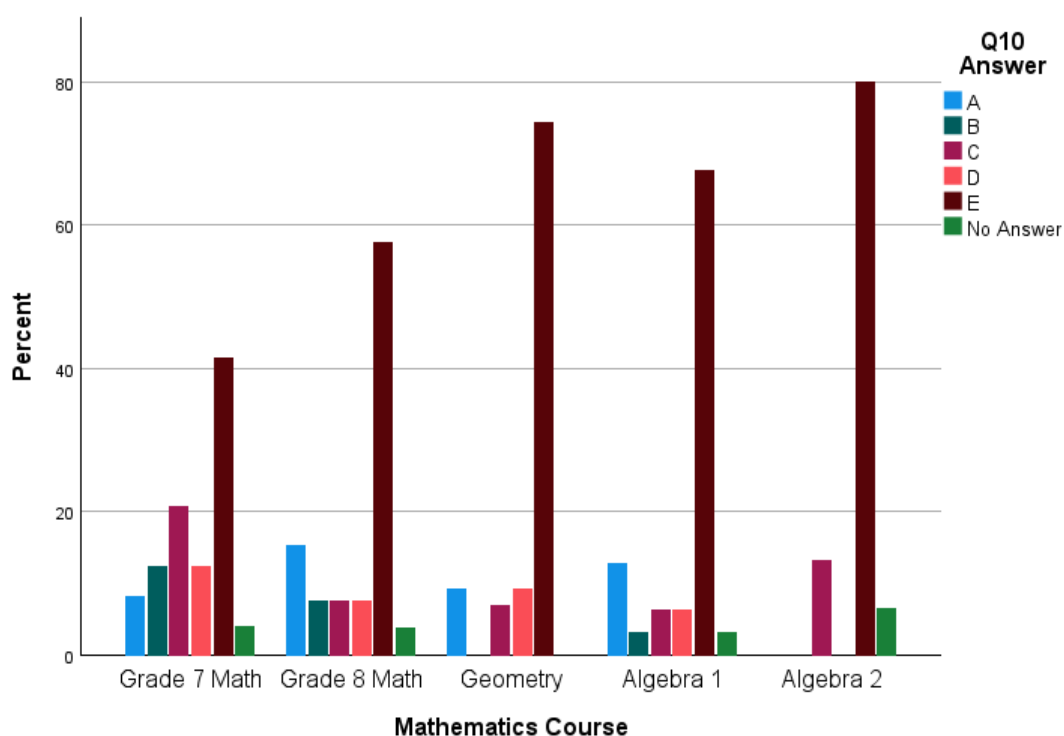
Item 10 Responses by Mathematics Course

		Mathematics Course											
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2		Total	
1996 NAEP		N	%	N	%	N	%	N	%	N	%	N	%
Grade 8													
A	15%	2	8.3%	8	15.4%	4	9.3%	4	12.9%	0	0.0%	18	9.2%
B	8%	3	12.5%	4	7.7%	0	0.0%	1	3.2%	0	0.0%	8	4.1%

C	11%	5	20.8%	4	7.7%	3	7.0%	2	6.5%	6	13.3%	20	10.3%
D	8%	3	12.5%	4	7.7%	4	9.3%	2	6.5%	0	0.0%	13	6.7%
E	58%	10	41.7%	30	57.7%	32	74.4%	21	67.7%	36	80.0%	129	66.2%
No Answer		1	4.2%	2	3.8%	0	0.0%	1	3.2%	3	6.7%	7	3.6%
Total	100%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.12

Item 10 Responses by Mathematics Course, Correct Answer: E



Item 11: Wagon Factory – Finding Unit Rate from Verbal Description

In item eleven, students were presented with a story, saying “A factory can produce 720 wagons in an 8-hour day” and are then asked to find the unit rate of wagons made per hour. This item came from the sixth-grade curricular materials that are used at the school being studied, and assesses a Grade 7 standard on ratios and proportional relationships. This item is a very simple unit rate problem. The students are given a proportional relationship verbally, and are asked to

find the unit rate. This item was included to see if students understand the concept of a unit rate in a basic proportional relationship.

Most students were able to calculate the unit rate correctly and choose answer choice *B*, as shown in Table 4.31 and Figure 4.13. Of the 195 students who took this assessment, only 17 chose an incorrect answer, and eight students left the item blank. This may be due to running out of time and not getting to the end of the test, or they may have been confused. The students in each mathematics course had a success rate of over 79%, so this tells us that when students are presented with a simple situation, they are able to find the unit rate. The answer choice that was most commonly chosen incorrectly was *D*: 5760, which would have been found by multiplying 720 wagons times eight hours. These students likely do not fully understand the concept of a unit rate and could use additional practice.

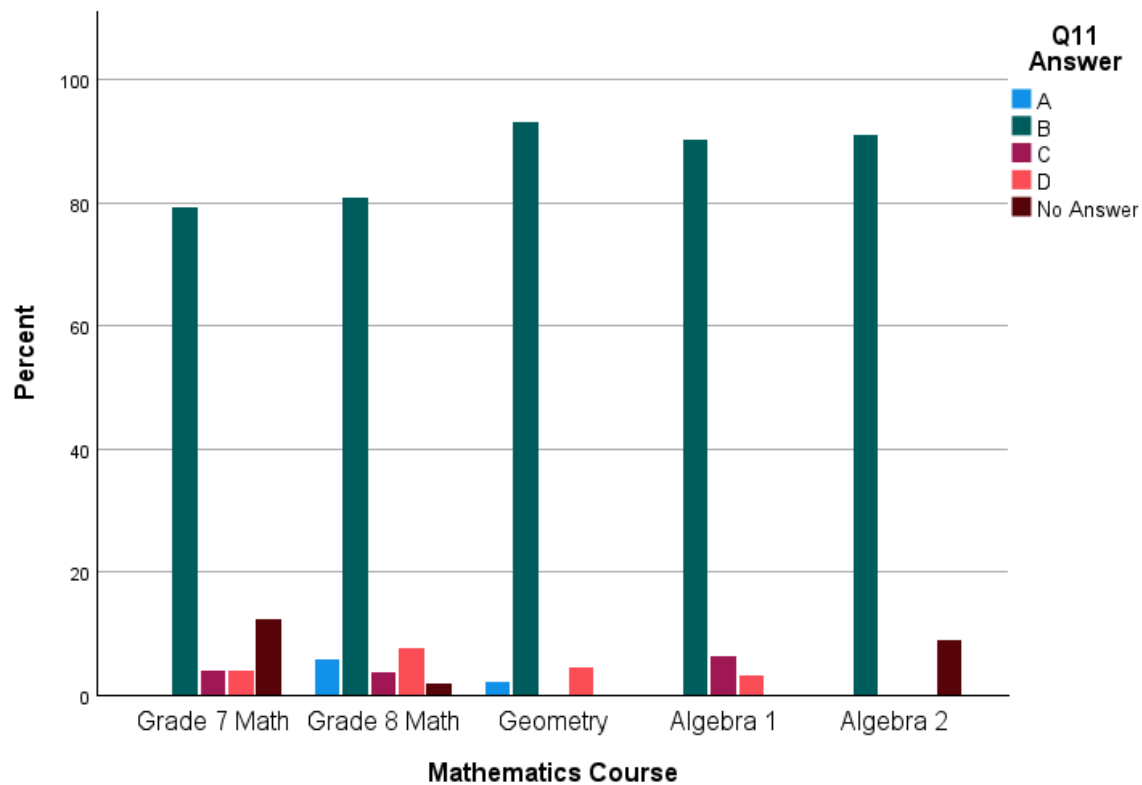
Table 4.31

Item 11 Responses by Mathematics Course

	Mathematics Course										Total	
	Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2			
	N	%	N	%	N	%	N	%	N	%	N	%
A	0	0.0%	3	5.8%	1	2.3%	0	0.0%	0	0.0%	4	2.1%
B	19	79.2%	42	80.8%	40	93.0%	28	90.3%	41	91.1%	170	87.2%
C	1	4.2%	2	3.8%	0	0.0%	2	6.5%	0	0.0%	5	2.6%
D	1	4.2%	4	7.7%	2	4.7%	1	3.2%	0	0.0%	8	4.1%
No Answer	3	12.5%	1	1.9%	0	0.0%	0	0.0%	4	8.9%	8	4.1%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.13

Item 11 Responses by Mathematics Course, Correct Answer: B



Item 12: Jessica’s Clay Bowls – Interpreting Equation in Context

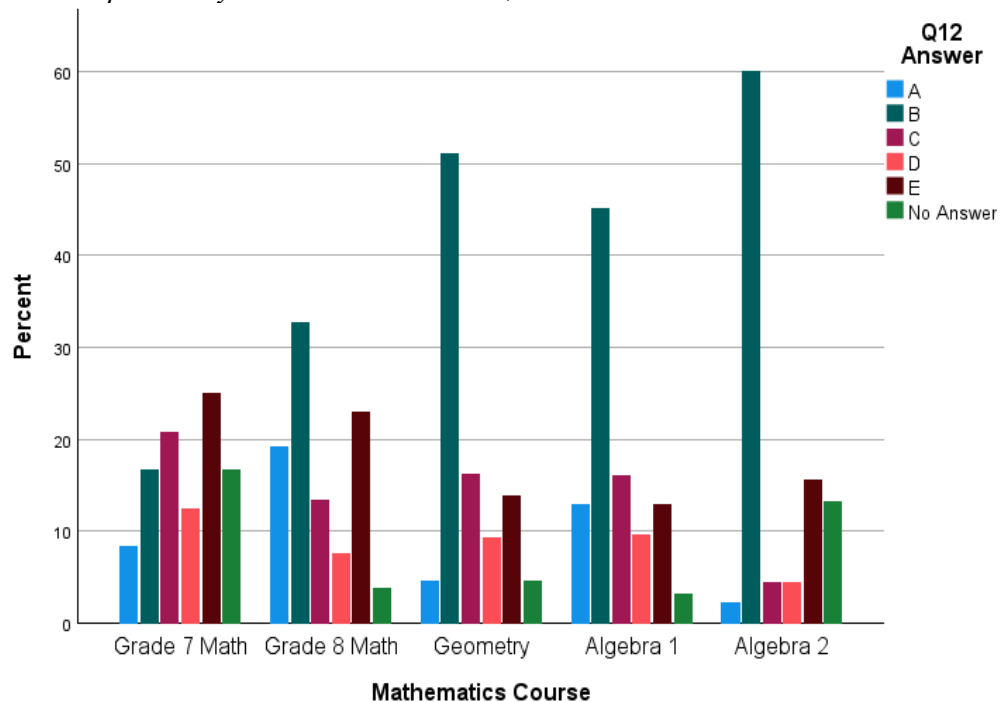
Item twelve introduced students to a situation about a person named Jessica and her costs associated with making clay bowls. Students are presented with an equation with two variables to represent the situation ($d = 375 + 50n$). This item asks students to interpret the meaning of values from the given equation in context of the real-world situation presented by choosing which of the given statements is true. This particular problem addresses the y-intercept, or initial cost in the situation. The purpose of including this item was to assess if students were able to interpret the meaning of numerical values in an algebraic representation of a linear relationship. This item assesses an eighth-grade standard, and the concept of interpreting values of linear relationships in context is covered in curricular materials starting in Grade 8.

The correct statement and answer choice for this item was *B: Without making any clay bowls, Jessica's total cost is \$375*. As shown in Table 4.32 and Figure 4.14 below, the majority of students in each mathematics course chose the correct answer with the exception of students in Grade 7 Math. On the 2017 NAEP, 37% of Grade 8 students chose the correct answer on this item. The students who were enrolled in Grade 8 Math in this study had similar success to the NAEP students, with 32.7% of students selecting the correct answer, and students enrolled in Grade 7 Math were much less successful with only 16.7% of students answering correctly. The Geometry, Algebra 1, and Algebra 2 students in this study outperformed the 2017 NAEP Grade 8 students with accuracy rates of 51.2%, 45.2%, and 60% respectively.

The most frequently selected incorrect answer across all students was *E: Each clay bowl Jessica makes costs \$425*. To arrive at this conclusion, students likely added the \$375 and \$50 from the equation together to get \$425, and ignored the variable completely. These students do not have a strong understanding of algebraic representations of linear relationships and may not know what the variable means at all. This answer was more commonly chosen by students in Grade 7 Math and Grade 8 Math, which is understandable because they have had fewer learning experiences and opportunities to interpret equations in context, but multiple students in the high school mathematics courses also chose this answer. Students may be used to writing linear equations from real-world scenarios, but have not had as many opportunities to interpret given equations in a real-world context.

Table 4.32*Item 12 Responses by Mathematics Course*

	2017 NAEP Grade 8	Mathematics Course										Total	
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2		N	%
		N	%	N	%	N	%	N	%	N	%		
A	12%	2	8.3%	10	19.2%	2	4.7%	4	12.9%	1	2.2%	19	9.7%
B	37%	4	16.7%	17	32.7%	22	51.2%	14	45.2%	27	60.0%	84	43.1%
C	18%	5	20.8%	7	13.5%	7	16.3%	5	16.1%	2	4.4%	26	13.3%
D	16%	3	12.5%	4	7.7%	4	9.3%	3	9.7%	2	4.4%	16	8.2%
E	16%	6	25.0%	12	23.1%	6	14.0%	4	12.9%	7	15.6%	35	17.9%
No Answer	1%	4	16.7%	2	3.8%	2	4.7%	1	3.2%	6	13.3%	15	7.7%
Total	100%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.14*Item 12 Responses by Mathematics Course, Correct Answer: B*

Item 13: Water in Tank – Interpreting Equation in Context

Item 13 was similar to item 12, as students were given an equation to represent the number of gallons of water, y , in a tank after x hours ($y = 800 - 50x$) and were asked to select which statement was true about the tank. While item 12 had students interpret the y-intercept accurately, this item has students interpret the slope, or rate of change in the given situation. The correct statement in this problem was *D: It is emptying at a rate of 16 gallons per hour*. This item assesses standards targeted at Grade 8 students, and the content does not show up in curricular materials until Grade 8 Math.

Overall, students had more success with this item than they did with item 12. The responses by mathematics course can be found in Table 4.33 and Figure 4.15 below. With the exception of students enrolled in Grade 7 math, over 46% of students enrolled in each mathematics course chose the correct answer. Of the students who took the NAEP in 2011, 48% chose the correct answer, and that was almost matched or exceeded by the students in this study who were enrolled in each mathematics course except Grade 7 Math. The most commonly selected incorrect answer by all students in this study as well as the 2011 NAEP students was *B: It is filling at a rate of 50 gallons per hour*. Students who chose this statement had an understanding that something was happening with 50 gallons per hour, but they did not understand that the $-50x$ meant the tank was losing water, or emptying at the rate of 50 gallons per hour. This is a slight misunderstanding that could likely be corrected with having students model an equation with a table or graph so they can see that the water in the tank is going down as more hours pass. Students who chose answer choices *A: It is filling at a rate of 800 gallons per hour* or *E: It is emptying at a rate of 800 gallons per hour* do not understand that the rate of change is connected to the variable, and they interpreted the y-intercept as the rate of change

instead of the amount of water originally in the tank. Students who chose answer choice *C: It is emptying at a rate of 16 gallons per hour* probably guessed, or divided 800 by 50 to get 16.

These students have a very limited understanding of algebraic representations of linear relationships.

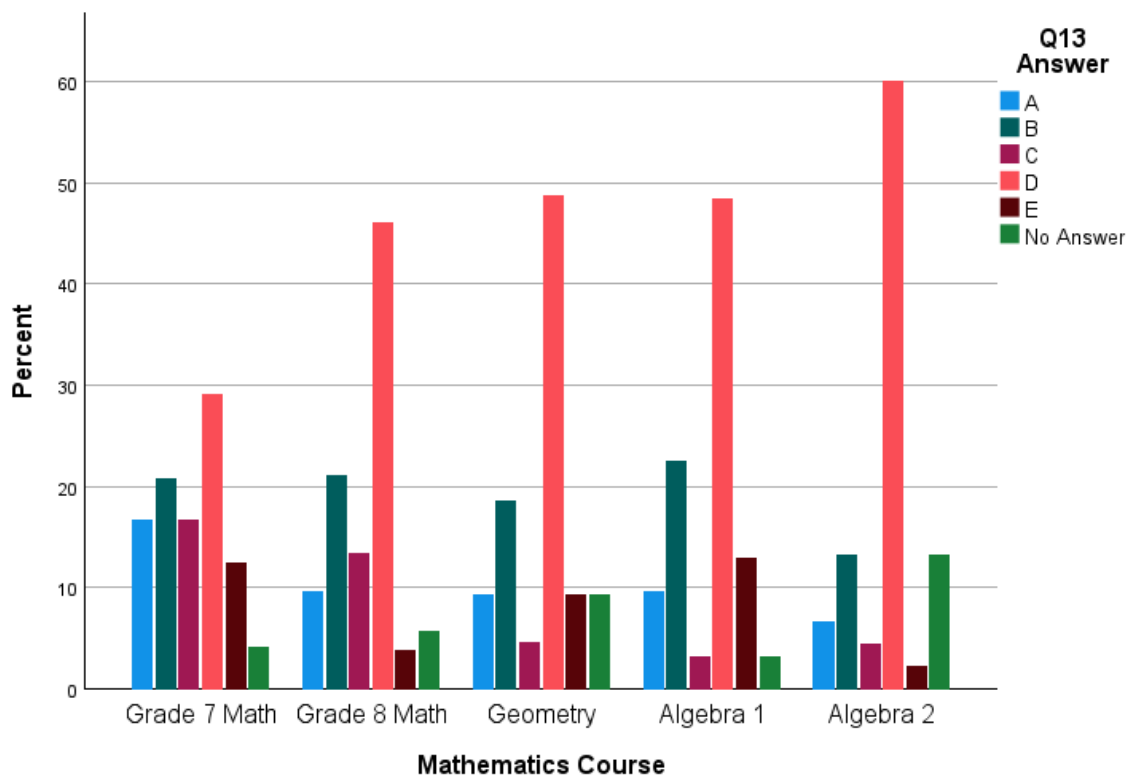
Table 4.33

Item 13 Responses by Mathematics Course

		Mathematics Course										Total	
		Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2		N	%
		N	%	N	%	N	%	N	%	N	%		
A	11%	4	16.7%	5	9.6%	4	9.3%	3	9.7%	3	6.7%	19	9.7%
B	21%	5	20.8%	11	21.2%	8	18.6%	7	22.6%	6	13.3%	37	19.0%
C	13%	4	16.7%	7	13.5%	2	4.7%	1	3.2%	2	4.4%	16	8.2%
D	48%	7	29.2%	24	46.2%	21	48.8%	15	48.4%	27	60.0%	94	48.2%
E	6%	3	12.5%	2	3.8%	4	9.3%	4	12.9%	1	2.2%	14	7.2%
No Answer	1%	1	4.2%	3	5.8%	4	9.3%	1	3.2%	6	13.3%	15	7.7%
Total	100%	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Figure 4.15

Item 13 Responses by Mathematics Course, Correct Answer: D



Item 14: Alisha and Julius Driving Task – Identifying and Interpreting Unit Rate

Item 14 on the assessment presented students with the following information: “*Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours,*” then asked students who was driving faster and to explain how they knew who was driving faster. This item assesses a Grade 7 standard on identifying the constant of proportionality, or unit rate, in a verbal description of a proportional relationship.

As shown in Table 4.34, 68.7% of all students identified Julius as the faster driver and provided an explanation, earning a full point. Students enrolled in Algebra 2 and Grade 8 Math had the highest rate of success with this item, with at least 75% of the students in each course earning full points. Students enrolled in Grade 7 Math were least successful on this item, which

makes sense given it is assessing a Grade 7 standard. Although they had the least success, 45.8% of Grade 7 Math students still earned a full point for this item, compared to 37.5% of Grade 7 Math students who earned no points. This shows that most students had at least some knowledge of unit rates to compare situations in the real world.

Table 4.34

Item 14 Scores by Mathematics Course

	Mathematics Course										Total	
	Grade 7 Math		Grade 8 Math		Geometry		Algebra 1		Algebra 2		N	%
	N	%	N	%	N	%	N	%	N	%		
.0	9	37.5%	10	19.2%	11	25.6%	7	22.6%	9	20.0%	46	23.6%
.5	4	16.7%	3	5.8%	3	7.0%	4	12.9%	1	2.2%	15	7.7%
1.0	11	45.8%	39	75.0%	29	67.4%	20	64.5%	35	77.8%	134	68.7%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

The majority of students who earned a full point on this item divided each person's number of miles driven by the hours it took them to drive those miles to get unit rates in miles per hour, and then chose the person who drove the higher number of miles per hour, Julius. Overall, students seem to have sufficient understanding of calculating a unit rate in context. Correct student work samples can be found in Figure 4.16 and Appendix F. Student 7a19 divided each person's number of miles driven by the hours it took them to drive those miles to find each person's driving rate in miles per hour. This student used the actual miles per hour values in their explanation, saying that Julius's rate was 65 miles per hour which is higher than Alicia's 62 miles per hour. Student 7a8 also divided the miles driven by the hours driven, but did not explicitly state the miles per hour rates. They used an inequality to show that Alisha's rate was lower than Julius's rate, so Julius was driving faster. These two students along with many others, used the

same strategy which shows they understand how to calculate a unit rate in a familiar situation of driving speed.

Student 7b7 used a novel approach to answer this item, and was the only student to use this strategy. This student seems to have calculated the unit rate for Alisha, then used that rate to calculate how many miles Alisha would have driven in the same amount of time that Julius drove. The student determined that Alisha would have driven fewer miles than Julius in the same 11 hours, so Julius was driving faster. This student demonstrated an understanding of calculating a unit rate and then using that unit rate to compare other situations.

Figure 4.16

Item 14 Correct Responses

Student 7a19	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Julius</u> How do you know? <u>Because I divided both of the people's miles and hours to get my answer. 62 is greater than 65 so therefore Julius was driving faster.</u>
Student 7a8	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Julius</u> How do you know? <u>496/8 < 715/11.</u> <u>Divide the miles driven by the number of hours and you get the MPH. Julius' MPH was greater than Alisha's.</u>
Student 7b7	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Julius</u> How do you know? <u>I know because I found the unit rate then multiplied it by 11 to see how many miles Alisha drove in 11 hours, it was only 682 miles compared to Julius's 715</u>

Five students across all math courses correctly calculated the unit rate for each person, but then chose Alisha as driving faster because she had the lower unit rate (see Figure 4.17). Students 8b8, 11e1, and 11e11 used the correct strategy of dividing each person's miles driven by their hours driven to calculate their driving speed, but they concluded that Alisha was driving faster. These students have a strong understanding of how to calculate a unit rate in this setting, but they seem to have a misunderstanding of the meaning of those rates in the driving context. In curricular materials, students often calculate the unit rate with prices of items, and compare which store has a better deal on said item. In those scenarios, students choose the lower unit rate as the better deal, so they may have been thinking along the same lines in this situation and thought that a lower number of miles per hour was faster than a higher number.

Figure 4.17

Item 14 Common Error 1: Choosing Lower Unit Rate as Faster Driver

Student 8b8	14. Alisha drove 496 miles in 8 hours. = 62 Julius drove 715 miles in 11 hours. = 65 Who was driving faster? <u>Alisha</u> How do you know? <u>Because when you divide</u> <u>the numbers Alisha miles</u> <u>per hour is faster by</u> <u>3 62, 65.</u>	Workspace (if needed): $\begin{array}{r} 62 \\ 8 \overline{)496} \\ \underline{-48} \\ 16 \\ \underline{-16} \\ 0 \end{array}$ $\begin{array}{r} 65 \\ 11 \overline{)715} \\ \underline{-66} \\ 55 \\ \underline{-55} \\ 0 \end{array}$
Student 11e1	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Alisha</u> How do you know? <u>Alisha drove 62 mph (496:8)</u> <u>Julius drove 65 mph (715:11)</u>	Workspace (if needed):
Student 11e11	14. Alisha drove 496 miles in 8 hours. 62 Julius drove 715 miles in 11 hours. 65 Who was driving faster? <u>Alisha</u> How do you know? <u>dividing the miles by</u> <u>the hours</u>	Workspace (if needed):

As shown in Figure 4.18, multiple students chose Alisha as driving faster because she spent less time driving. These students were mostly focused on time spent by each person in the car and they do not understand that there is a useful relationship between the number of miles driven and the time it takes to drive those miles (the unit rate), or how they can use the unit rate to compare two things in context. The students who used the justification of being in the car for a shorter period of time did not do any calculations on their papers, and instead made a decision based on their perception of being in the car for a long versus short amount of time. Student 7a18 concluded that “Julius took way longer” and student 10d19 justified Alisha driving faster “because she got a lot of miles in less hours.” Student 8b19 explicitly said “I know because she 496 in 8 hours which is less than 11 hours.” These students seemed to only be focused on the time and did not take the number of miles for each person into consideration which demonstrates that they do not understand what a unit rate is or why unit rates are helpful when comparing real-world contexts.

Figure 4.18

Item 14 Common Error 2: Choosing Faster Driver Based on Amount of Time

Student 7a18	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Alisha</u> How do you know? <u>Alisha only drove 8 hours</u> <u>and 496 miles which is way</u> <u>less than Julius. Which she</u> <u>had 715 miles in 11 hours.</u> <u>Julius took way longer.</u>	<u>Workspace (if needed):</u>
Student 8b19	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Alisha</u> How do you know? <u>I know because she</u> <u>496 in 8 hours which</u> <u>is less than 11 hours.</u>	<u>Workspace (if needed):</u>
Student 10d19	14. Alisha drove 496 miles in 8 hours. Julius drove 715 miles in 11 hours. Who was driving faster? <u>Alisha</u> How do you know? <u>because she got to alot</u> <u>of miles in less hours</u>	<u>Workspace (if needed):</u>

Item 15: Two Runners Task – Interpreting Elements of Graph in Context

The fifteenth and final item on this assessment presented students with a graphical representation of two runners in a race. They were asked to provide verbal interpretations of different characteristics of the graphical representations such as slopes, point of intersection, and x-intercepts. An example was provided for the y-intercepts, stating that “At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.” For the slope, students were expected to either calculate each runner’s running speed or describe which runner was running faster based on the steepness of the lines. For the point of intersection, students should have noted that this was where the runners crossed paths, or one runner passed the other, and may have estimated the time of intersection or distance from the finish line of the two runners at this point. The x-intercepts should have been interpreted as the time that each runner finished the race, or when they were zero miles from the finish line. As a whole, this item tested students’ abilities to interpret a graph in a real-world situation, which connects two representations of linear relationships.

Overall, students did not demonstrate much success with this item. As shown in Table 4.34, 54.4% of all the students who took the test earned 0 points on this item, and only 13.3% of students earned a full point. This may have been due to a lack of understanding of slope, intersection, and intercepts in general, or the inability to connect those aspects of the graph to a real-world context. This was also the last item on the test, so some students may have run out of time or were mentally exhausted and gave up at the end. Of the 106 students who earned zero points, 48 left all three parts of the item blank or wrote “IDK” in all three boxes. In total, 41 students left one or two of the three boxes blank on this item.

Students had the most success with interpreting the point of intersection as the time the two runners crossed paths, or when they were in the same place at the same time. This may be because the graphs are “crossing paths” and that makes the most sense in the context of a race. Interpreting the slope seemed to be the most difficult for students. Many students left this box blank, or tried to write some sort of equation instead of commenting on the steepness of the line, or calculating the slope from the graph. In curricular materials, students are taught that the slope-intercept form of a line is $y=mx+b$ and they learn that m represents slope or rate of change, but they do not always graph equations in real-world contexts past seventh and eighth grade, so this lack of learning opportunities to practice making real-world connections to graphs could have been the cause of some confusion on this item.

Table 4.34

Item 15 Scores by Mathematics Course

Mathematics Course											Total	
Grade 7 Math			Grade 8 Math		Geometry		Algebra 1		Algebra 2		N	%
N		%	N	%	N	%	N	%	N	%		
.0	23	95.8%	32	61.5%	19	44.2%	16	51.6%	16	35.6%	106	54.4%
.5	1	4.2%	17	32.7%	14	32.6%	12	38.7%	19	42.2%	63	32.3%
1.0	0	0.0%	3	5.8%	10	23.3%	3	9.7%	10	22.2%	26	13.3%
Total	24	100.0%	52	100.0%	43	100.0%	31	100.0%	45	100.0%	195	100.0%

Student work samples from item 15 are shown in Figure 4.19. Since the table for this item was open-ended, students provided a variety of responses and explanations for each of the graph’s features. Some students calculated the slope of each line and interpreted it in context (Students 10e2, 10e7), while others commented on the steepness of each line and interpreted the steepness as one runner running faster or slower than the other runner (Students 9c30, 9c28). For

the point of intersection, students had a variety of interpretations. Students 10e2, 9c30, and 9c28 all noted that the runners met at around 11:45am, giving various estimations for the distance the runners were from the finish line. Students 10e7 and 9c30 described the point as where the runners caught up to each other, or where Runner A passed Runner B. The majority of students who answered this item understood that the point of intersection represented a time when the runners were the same distance from the finish line at the same time. The x-intercepts section of this item was the section that was answered correctly most often. The majority of students who answered this item explained that the x-intercepts represented the times at which each runner crossed the finish line or completed the race. Students 10e2, 9c30, and 9c28 accurately stated the times at which each runner crossed the finish line, while Student 10e7 explained that the x-intercepts represented the runner crossing the finish line “because they had no distance from the finish line.”

Figure 4.19

Item 15 Correct Student Work

Student 10e2	Characteristic of Graph	Interpretation in Terms of the Race
	y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
	Slopes	Runner A has a steeper slope, so they were running faster than Runner B
	Point of Intersection	They met each other at 11:45, 4.5 miles from the finish line
	x-intercepts	runner B reached the finish line at 1pm and runner A reached the finish line at 12:15pm.

Student 10e7	Characteristic of Graph	Interpretation in Terms of the Race
	y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
	Slopes	RUNNER A IS RUNNING FASTER THAN RUNNER B BECAUSE THEIR SLOPE IS MUCH SHARPER THAN RUNNER B'S. A = HIGHER RISE/LOWER RUN VS. B = LOWER RISE/HIGHER RUN, MAKING B TAKE LONGER
	Point of Intersection	SINCE RUNNER A STARTED FARTHER (3 MILES) FROM THE FINISH LINE THAN RUNNER B, THEIR INTERSECTION IS THE POINT AT WHICH THEIR INCREASED SPEED OVERTAKES THE DISTANCE THAT B HAS ALREADY RUN BUT SLOWER
	x-intercepts	THE X INTERCEPTS REPRESENT THE POINTS AT WHICH THE RUNNERS FINISHED THEIR RACE, BECAUSE THEY HAD NO DISTANCE FROM THE FINISH LINE

Student
9c30

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A ran at a constant speed of 8 MPH while Runner B ran at a constant speed of 3.5 MPH
Point of Intersection	Just before 11:45, is when Runner A and Runner B caught up to each other.
x-intercepts	At 12:15, Runner A had completed the race and crossed the finish line. At 1 P.M., Runner B also finished and crossed the finish line.

Student
9c28

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A runs at a constant speed of 8mph towards the finish line and Runner B runs at a constant speed of 3.5 mph towards the finish line.
Point of Intersection	At 11:45 am Runner A and B meet 4.75 miles from the finish line.
x-intercepts	At 12:15 pm Runner A reaches the finish line and at 1 pm Runner B reaches the finish line.

Students who attempted this item were able to demonstrate their abilities to correctly interpret various aspects of a graphical representation of linear relationships. However, not many students actually attempted this item. Therefore, the overall understanding of what key features of graphical representations of linear relationships represent in a real-world context is not sufficient among students in this study.

Summary

This chapter presented the results of an assessment that was created to measure and compare students' conceptual understanding of linear relationships in middle and high school mathematics courses. Students in Algebra 2 had significantly higher total scores of conceptual understanding of linear relationships than students in Grade 7 Math and Grade 8 Math. Students in Geometry also scored significantly higher than Grade 7 Math students. However, the overall proficiency of student conceptual understanding of linear relationships was low across all courses.

For Construct 1, students in Algebra 2 scored significantly higher in identifying unit rates in proportional relationships than Grade 7 Math students. No other significance differences were found for Construct 1. For Construct 2, Algebra 2 students performed significantly better than Grade 7 Math and Grade 8 Math students in moving fluidly between representations of the same linear relationship. Geometry students performed significantly better than Grade 7 Math students in this construct as well. Similarly for Construct 3, Algebra 2 students performed significantly better than Grade 7 Math and Grade 8 Math students and Geometry students performed significantly better than Grade 7 Math students in interpreting different representations of linear relationships.

Student work was analyzed for each item to identify trends in student understanding. Algebra 2 students scored the highest on most items, and students in Grade 7 Math scored the lowest. Students were most successful with items from Construct 1 with just over half of the participants in the study able to identify the unit rate in a given situation. Students struggled to move fluidly between different representations of linear relationships (Construct 2) and struggled to interpret different representations of linear relationships (Construct 3), which is most likely due to not having the foundational understanding of unit rates from Construct 1. Overall, the data from this assessment showed that students across all mathematics courses in this study do not have a strong conceptual understanding of linear relationships, with the highest scoring mathematics course only answering about 60% of the items assessment accurately. The implications of these findings will be explored in Chapter 5.

CHAPTER FIVE

DISCUSSION AND CONCLUSIONS

The purpose of this cross-sectional study was to determine the level of conceptual understanding that students have of linear relationships and their applications through the use of multiple representations. The study was designed as a cross-sectional examination of student understanding in each of five sequential mathematics courses—Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, and Algebra 2. The following questions guided this study:

1. What differences exist in students' conceptual understanding of linear relationships across mathematics courses?
 - a. Is there a difference in overall conceptual understanding of linear relationships between mathematics courses for students enrolled in Grade 7 Math through Algebra 2?
 - b. Are there differences in students' conceptual understanding of constructs for linear relationships in Grade 7 Math through Algebra 2?
2. What are common strengths and weaknesses in students' conceptual understanding of linear relationships?
 - a. What are overall strengths and weaknesses in the conceptual understanding of linear relationships for middle and high school students?
 - b. What are course-specific strengths and weaknesses in the conceptual understanding of linear relationships for students in Grade 7 Math through Algebra 2?

This study included 195 middle and high school students from a mid-sized public school in an urban, midwestern school district. These students opted in and received parental consent to participate in this study. Each student was enrolled in one of five mathematics courses: Grade 7

Math ($n = 24$), Grade 8 Math ($n = 52$), Geometry ($n = 43$), Algebra 1 ($n = 31$), and Algebra 2 ($n = 45$). In this school, students enroll in mathematics courses in the following sequence: Grade 7 Math, Grade 8 Math, Geometry, Algebra 1, then Algebra 2. This is different from the typical high school sequence of Algebra 1, Geometry, Algebra 2. This school decided that they wanted Algebra 1 and Algebra 2 to be more connected, so they placed Geometry before Algebra 1. The proportion of students participating in the study was between one-fifth and one-third of the total enrollment of each course. A 15-item assessment was designed to assess students' conceptual understanding of linear relationships. All participants took this linear relationship assessment during a single class period (45 minutes) of their regular mathematics course. Their assessments were scored and then analyzed using SPSS software. Additionally, content analysis was conducted on their written work. These analyses generated six major findings that summarize the students' conceptual understanding of linear relationships.

This chapter first presents the major findings of the study, which are discussed with connections to the existing literature. Next, conclusions are presented to address the research questions. The chapter then discusses implications for practice in the wider field of mathematics education. Finally, limitations of this study and implications for future research are discussed.

Major Findings

The qualitative and quantitative data from this study was analyzed and organized into six major findings. Some findings are specific to mathematical concepts, and others are related to the overall conceptual understanding of linear relationships. These findings have potential to impact mathematics instruction in regards to linear relationships. The findings are presented below.

Finding #1: Students are Able to Calculate a Simple Unit Rate from a Familiar Context

When presented with a familiar real-world context, a majority of students across all mathematics courses were able to accurately calculate the unit rate. This finding supports Herbert's (2010) conclusion that students are generally comfortable with calculating rates in familiar contexts involving units such as distance and time. Of the five items on the assessment that asked students to find unit rates, two items presented students with a sentence describing a familiar context, both including time. Item 11 told students that "A factory can produce 720 wagons in an 8-hour day" and asked them to calculate the unit rate of wagons made per hour. Item 14 told students that "Alisha drove 496 miles in 8 hours" and "Julius drove 715 miles in 11 hours" and asked them to determine who was driving faster. Nearly all students were able to accurately calculate the unit rate for each of these two items from the given scenarios, which further shows that students feel comfortable calculating unit rates from familiar contexts, especially those involving time. This success may be contributed to the learning experiences that students are given in relation to unit rates. According to the Common Core State Standards for Mathematics (CCSSM; NGA & CCSSO, 2010), students are introduced to unit rates in Grades 6 and 7 using real-world contexts. Curricular materials and classroom learning experiences in this school are informed by CCSSM, so students have likely interacted with unit rates in real-world contexts numerous times, which would enable them to achieve fairly well when finding a unit rate from a familiar context.

Finding #2: Students are Able to Move Fluidly to an Equation from Other Representations

When students were asked to select or write an equation as an algebraic representation of a linear relationship, they were relatively successful when given a different given representation (e.g., table of values, verbal representation). This aligns with Knuth (2000) that students

preferred to work with an algebraic representation of linear equations. When students were given a table and asked to choose which equation represented the relationship shown in the table, they were able to identify the correct equation over 60% of the time (Item 1, Item 3). When they were given a verbal representation and asked to choose the equation that modeled the given situation, 61.5% of students across all mathematics courses were able to identify the correct equation (Item 10). The students in this study were more successful on items that involved creating or using an algebraic equation than on items that required them to use other representations, supporting existing research that students struggle to use representations outside of an algebraic equation to solve problems (Birgin, 2012; Knuth, 2000).

Finding #3: Students Struggle to Connect Equations and Graphical Representations

Students struggled to move from an equation to a graph of the same linear relationship. They did not seem to have an understanding of what each of the numerical values in the equation represented in the graph. This aligns with Birgin (2012) who found that “much confusion existed regarding the connection between the algebraic and geometric aspects of slope of a line” (p. 155). When given an equation and asked to choose the correct graph for that equation, the students in this study chose an incorrect graph about 60 percent of the time (Item 2). Students enrolled in Algebra 2 had the highest rate of success with connecting an equation to a graph, but they were only successful half of the time. This lack of success demonstrates that students are confused about the connection between the algebraic (equation) and geometric (graph) representations of a linear relationship. Wilkie and Ayalon (2018) argued that the translation between algebraic equations and their graphs is critical for students in order to have a full understanding of linear relationships. To further support the need for fluid connections between representations, NCTM (2000, 2014) stated that students’ conceptual understanding of

mathematical topics is supported by their ability to use multiple representations in mathematics. The participants of this study did not yet have this critical understanding.

Finding #4: Students Struggle to Interpret Linear Relationships in Any Representation

When asked to interpret a linear relationship that was presented as an equation, graph, or table in context, students were successful less than 45% of the time overall. For example, Item 13 presented students with a linear equation ($y = 800 - 50x$) representing the number of gallons of water, y , in a tank after x hours and were asked to choose a correct statement about the tank based on the equation. Only 48.2% of all students in this study chose the correct statement: “it is emptying at a rate of 50 gallons per hour.” Students enrolled in Grade 7 Math were the least successful with less than one-third of students choosing the correct statement, while 60% of students enrolled in Algebra 2 were able to identify the correct statement. Fonger (2019) concluded that representational fluency requires students to be able to interpret multiple representations in addition to creating, translating between, and connecting the representations. In this study, students struggled to interpret linear relationships. While some students in the study were able to demonstrate an ability to make connections between some representations, they were not able to make connections between all representations nor interpret various representations. The inability to interpret linear relationships in multiple representations shows that some of the connections students made may have been tenuous. Overall, students in this study did not have a strong level of representational fluency.

Finding #5: Unit Rate Understanding Plateaus, then Increases in Algebra 2

Calculating the unit rate for a given situation was a relative strength for the participants of this study. Each mathematics course had a higher mean score for the unit rate construct (Construct 1) than they did for the other two constructs. While this was a relative strength for all

mathematics courses, an interesting pattern emerged in the scores between courses. The mean score in Construct 1 improved from the students enrolled in Grade 7 Math ($M=2.0$) to the students enrolled in Grade 8 Math ($M=3.0$), but plateaued with no improvements from Grade 8 Math to Geometry ($M=2.9$) and Algebra 1 ($M=2.9$). Then, the mean score jumped up for students enrolled in Algebra 2 ($M=3.6$). This pattern was particularly interesting because at this school, students learn about unit rates and engage in learning activities involving unit rates in Grade 7 Math, Grade 8 Math, and Algebra 1 courses. Unit rates are not a topic that is specifically addressed in Algebra 2, which does not explain the increase in scores for those students. This plateau pattern of understanding for the middle classes has not been previously documented in the literature, but it is an interesting finding. The scores show that students' abilities to identify unit rates increase after Grade 7 Math, then neither increases or decreases for three years, then increases again in Algebra 2.

Finding #6: Increased Learning and Engagement Opportunities Help Students be More Successful with Topics Regarding Linear Relationships

Students enrolled in Algebra 2 scored the highest of all mathematics courses on 12 of the 15 items on the assessment. Students enrolled in Algebra 1 scored highest on two items, and Geometry students scored highest on one item. Grade 7 Math students scored the lowest on 13 of the 15 items, and Grade 8 math students scored the lowest on two items. Algebra 2 is the most advanced mathematics course taken by participants in this study, which means those students have taken more mathematics courses in their school career than anyone else in the study. Linear relationships are addressed in some capacity in each of the courses in this study at this school, with some courses spending more time on them than others, thus students should engage with linear relationships during each of these mathematics courses.

The students enrolled in Algebra 2 have had the greatest opportunity to engage in learning activities with linear relationships, which seems to have solidified a deeper understanding of linear relationships for these students. NCTM (2014) shared that “the depth of understanding is related to the strength of connections among mathematical representations that students have internalized” (p. 25). The Algebra 2 students in this study seem to have internalized more connections among mathematical representations than those in the other courses, while the Grade 7 Math students, who have had the least amount of engagement with linear relationships showed very little internalized understanding of connections among mathematical representations. Along with time spent learning about linear relationships, the type of learning experiences that students have been given also impact their conceptual understanding. The Algebra 2 students have had teachers with differing teaching philosophies and have engaged in a variety of learning experiences including procedural, conceptual, and investigation related experiences. The higher scores on this assessment are likely a result of access to at least some conceptual learning experiences across their five years of mathematics in these courses. This aligns with conclusions from the literature that students are more successful in mathematics when they have learning opportunities where they are able to explore and connect multiple representations in mathematics versus rote memorization and procedural instruction (Beyranevand, 2010; Brenner et al., 1999).

Conclusions

This study was designed to investigate the conceptual understanding of linear relationships for a cross-section of middle and high school students. A conceptual framework was created with three areas of focus leading to an overall conceptual understanding of linear relationships—identifying unit rates in proportional relationships, moving fluidly between

representations of the same linear relationship, and interpreting different representations of the same linear relationships. This framework was inspired by prior research on conceptual understanding (Hiebert & Lefevre, 1986; NCTM, 2014) and multiple representations in mathematics (Birgin, 2012; Knuth, 2000; NCTM, 2014; Wells, 2016). The following conclusions are meant to contribute to the emerging research area of conceptual understanding of linear relationships.

Question One: What differences exist in students' conceptual understanding of linear relationships across mathematics courses?

a. Is there a difference in overall conceptual understanding of linear relationships between mathematics courses for students enrolled in Grade 7 Math through Algebra 2?

An analysis of variance (ANOVA) was conducted on the total test scores as a measure of conceptual understanding, and a statistically significant difference was found in the conceptual understanding of linear relationships between mathematics courses. Tukey and Games-Howell post hoc analyses revealed that significant differences existed between students enrolled in some mathematics courses. The students in Algebra 2 scored significantly higher on overall conceptual understanding of linear relationships compared to students in Grade 7 Math and Grade 8 Math. The students in Geometry scored significantly higher on overall conceptual understanding of linear relationships than students in Grade 7 Math. Statistically significant differences in overall conceptual understanding of linear relationships were not present between any other mathematics courses.

b. Are there differences in students' conceptual understanding of constructs for linear relationships in Grade 7 Math through Algebra 2?

The first construct assessed students' abilities to identify unit rates in proportional relationships. Analyses revealed a significant difference between students in Grade 7 Math and students in Algebra 2, with Algebra 2 students scoring higher than Grade 7 Math students. The second construct assessed students' abilities to move fluidly between representations of the same linear relationship. The third construct assessed students' abilities to interpret different representations of linear relationships. In both Constructs 2 and 3, three pairs of courses showed significantly different understanding: Grade 7 Math and Algebra 2, Grade 8 Math and Algebra 2, and Grade 7 Math and Geometry. The students in the higher-level mathematics course scored higher than the lower-level course in each of the three pairs for both constructs.

Question Two: What are common strengths and weaknesses in students' conceptual understanding of linear relationships?

a. What are overall strengths and weaknesses in the conceptual understanding of linear relationships for middle and high school students?

The overall strength amongst all middle and high school students was the ability to successfully identify unit rates when provided a context presented in a sentence. Students in all mathematics courses were able to answer these items correctly over 70% of the time. An overall weakness for students across all mathematics courses was the ability to interpret linear relationships from different representations. Another area of weakness was the ability to move fluidly between representations of linear relationships.

b. What are course-specific strengths and weaknesses in the conceptual understanding of linear relationships for students in Grade 7 Math through Algebra 2?

Students enrolled in Grade 7 Math were strongest with identifying a unit rate when given a context presented in a sentence. The Grade 7 Math students' abilities to interpret linear

relationships and move fluidly between representations of linear relationships were very weak. The Grade 8 Math students showed a relatively strong understanding of identifying unit rates overall, however, their abilities to move fluidly between representations and interpret linear relationships were weak. Geometry students scored the highest of all mathematics courses on an item asking them to calculate a unit rate from a given context presented in a sentence, which highlights a strength in finding unit rates. Geometry students showed weakness on items involving equations of linear relationships. They struggled to write an equation from a table and to choose a graph that represented a given equation. Algebra 1 students showed strength in writing an equation from a given table and interpreting linear relationships in regards to their rate of change and y-intercepts. The Algebra 1 students showed weakness in interpreting linear relationships in context. Algebra 2 students showed the highest level of understanding on most items in the assessment, with a strength in writing equations when given a verbal representation or tabular representation.

Limitations

This study has some limitations that should be noted. First, the sample was from a single school. The sequence in which students take high school mathematics courses at this school does not follow the traditional sequence of Algebra 1, Geometry, then Algebra 2. This school places Geometry before Algebra 1, so the sequence of courses is Geometry, Algebra 1, then Algebra 2. This unusual order of courses may have contributed to the plateau in understanding of linear relationships that exists from Grade 8 Math through Algebra 1. Students from another school in the same district, or another school elsewhere in the country or world may have performed differently on the assessment. Second, the sample was chosen by students opting in and returning a signed parental consent form, which may not have gathered an accurate representation of the

students in the school. No identifying information was collected from the participants in the study other than their grade level and mathematics course, so demographic information was not available to determine whether the sample was representative of the school. However, given the large sample size ($N = 195$), with at least 20% of the students in each mathematics course taking the assessment, the results of this study are likely reflective of the learning and understandings that occur within each mathematics course at this school.

Despite having a relatively large sample of 195 students, the power analysis indicated that the sample size should have been larger with at least 200 total participants and 40 participants per mathematics course. With the sample size being below the 200-participant threshold, the data in this study is lacking in statistical power. The lack of sufficient statistical power may have limited the detection of significant differences between scores in some of the mathematics courses.

The assessment used for this study presents additional limitations. The assessment only had 15 items on it, so all translations of representations of linear relationships could not be included in the assessment. I methodically chose which representations and items to include on the assessment based on the Common Core State Standards for Mathematics (CCSSM), but I was inevitably missing some representation translations due to the brevity of the assessment. Additionally, the assessment had a combination of multiple choice and free-response items. Many students left the free-response items blank or wrote “IDK” (i.e., I Don’t Know) so I was not able to determine whether those students understood the content being assessed on those items. Follow-up interviews with the students would have allowed student understanding to be investigated further to determine whether these students truly did not know how to answer the question or if they simply did not want to try it. Similarly, the multiple-choice items do not allow

one to see student thinking, and it is impossible to know whether students were working through the problem or guessing if they did not show work on their paper without conducting follow-up interviews.

Using a single method of data collection presents an additional limitation to this study. Students took the assessment in a single class period, then turned it in to their teacher without their names on the papers. This meant that as the researcher, I did not have the ability to interview students about their thought processes on certain items on the assessment and I had to instead hypothesize what they may have been thinking based on what I saw written on their papers. This may have caused some understandings to have been misinterpreted. I also was not able to determine how much effort students put into this assessment, so I was not sure if the students who left items blank or wrote “IDK” chose not to attempt the items or if they truly did not know where to start to answer the question being asked. Ideally, student interviews or another form of additional data could have been used to provide a more extensive analysis of the student thinking.

Implications for Practice

The results of this study showed that middle and high school students have a limited conceptual understanding of linear relationships. Their lack of success in moving fluidly between representations and interpreting various representations of linear relationships may be due to the learning experiences provided in their mathematics courses.

Using a table of values to represent a linear relationship is routine in many mathematics courses. Students should begin using a table of values to represent quantities that are in equivalent ratios in Grade 6 (NGA & CCSSO, 2010) and eventually learn to make connections between a table of values with graphs and algebraic equations as they go through middle school. When students are using tables to represent equivalent ratios, they are engaging in multiplicative reasoning when the values of the tables are

not uniformly incremental (i.e., input values increasing by 1 for each row). When students start using tables of values for linear relationships to create ordered pairs to be graphed, curricular materials usually start with x-values of -2, -1, 0, 1, and 2, which no longer encourages students to think multiplicatively. As found in this study, when an incomplete table representing a linear relationship was presented without the input values increasing by 1, the majority of students were not able to correctly identify the missing values in the table. The most common error appeared to be that students were thinking additively, because they seemed to be assuming the input values in the table were increasing by 1. I would recommend a review of curricular materials and supplemental teacher-created materials to ensure that students continue to see tables with inconsistent increments in values. Being exposed to tables of consistent and inconsistent increments would help students develop their abilities to think additively as well as multiplicatively when interacting with linear relationships.

Students showed strength in being able to move from a table of values or a verbal representation of a linear relationship to an algebraic equation representing the relationship. Standards and curricular materials ask students to write equations with unknown values starting in first grade, and with two variables in sixth grade (NGA & CCSSO, 2010). Students are familiar with the structure of equations because they have engaged with them for almost the entirety of their school careers, which may have contributed to students' success in moving from a table of values to an equation and from a verbal representation to an equation in this study. Students in this study struggled to move between the following representations: equation to graph, verbal to table, and graph to verbal. The learning experiences that our students receive should deepen their understanding of how different representations of the same relationship are connected. Curricular materials should have students practice translating between tables, graphs, equations, and verbal representations of linear relationships in all directions. The same directional connections (i.e., table to equation, verbal to equation, equation to graph) should not be the only focus, otherwise students develop procedural understanding of these connections and miss out on the conceptual understanding of how all representations of a linear relationship are related.

The concept of a unit rate is heavily present in Grades 6, 7, and 8 mathematics standards (NGA & CCSSO, 2010), but it is not mentioned in the high school standards. Students engage in calculating and interpreting unit rates in proportional relationships throughout middle school, then are expected to build on this knowledge of proportional relationships in high school mathematics courses. In this study, having an understanding of unit rates was foundational to having a conceptual understanding of linear relationships. I recommend that connections to a unit rate be explicitly made in all mathematics courses when discussing linear relationships, instead of only in middle school courses. This would help students maintain their foundational understanding of proportional relationships and facilitate connections to new topics they are learning, increasing their overall conceptual understanding of linear relationships.

Implications for Equitable Mathematics Learning Experiences

The students in this study were not able to demonstrate proficiency in a conceptual understanding of linear relationships, even in the highest-level mathematics course being assessed. The school of study is in a large, underfunded, urban district, and has a majority Black student population. Since the district and school are underfunded and lack resources, students may not be receiving access to a high-quality mathematics education, which would prevent them from having the opportunity to gain a conceptual understanding of a mathematical topic such as linear relationships. As Lee (2006) shared, teachers and students in low-income schools tend to use formulaic, procedural curriculum. This often leads to student disinterest in the subject of mathematics, as well as not providing access to developing a conceptual understanding of mathematics.

If the students in this school have only been engaged in procedures-based instruction, they have not had the opportunities to explore and connect multiple representations in mathematics, which impacts their achievement on standardized assessments and future mathematical endeavors (Beyranevand, 2010; Brenner et al., 1999). The lack of access to higher-

level thinking and conceptually-based mathematics instruction prevents students from having a chance to deepen their understandings of mathematics.

Teacher training and instructional decisions play a large role in determining student learning experiences because teachers become the gatekeepers of access to high-quality instruction, sometimes unintentionally. As Berry et al. (2013) shared, Black learners are routinely put into classrooms of teachers with the least amount of preparation to teach. This does not mean the teachers are bad teachers, but they are underprepared in many cases. One way that teachers can be unprepared is due to a lack of professional development opportunities. If teachers are given a new curriculum to use in their classes, but do not receive adequate training on how to effectively implement this curriculum, they are not likely to teach it as it was intended to be taught. Teachers may become frustrated with trying to understand the new curriculum and return to teaching the mathematics topics the way they learned the topics, using strategies they are most comfortable implementing, which are likely not the conceptual-based strategies the new curriculum encourages. The lack of training and professional development in research-based best practices for mathematics instruction directly impacts the learning opportunities they give their students and perpetuates the unjust and inequitable mathematics education being provided to them.

It is essential that all students be given the opportunity to learn mathematics on a deeper level than the procedural level. The students in this study did not seem to move past the procedural level in any mathematics course over five sequential courses, as evidenced by their abilities to only calculate a unit rate from a familiar real-world context and their inability to move fluidly between representations of a linear relationship or interpret any representation of a

linear relationship. Districts and schools must prioritize an equitable mathematics education for their students, which starts with an equitable professional development system for their teachers.

Recommendations for Future Research

While this study produced some interesting results and trends that may be useful to think about in future planning for mathematics courses, further research would provide more specific insight into student understanding of linear relationships on a larger scale. Some recommendations for future research are provided in this section.

First, a larger, randomly selected sample of students would provide more data from which to draw conclusions. A larger, randomly selected sample would also allow for more transferability of results to a greater population. Ideally, the sample would be taken from the population of students in the entire school district, rather than from a single school. Demographic information should also be collected from students in the sample, to ensure that the sample is representative of the population and to allow for a more informed analysis of students' understanding.

The sequence in which students take high school mathematics courses may have been a factor in the results of this study. Future research should study schools who follow the traditional sequence of Algebra 1, Geometry, then Algebra 2 as well as schools who follow the Geometry, Algebra 1, then Algebra 2 sequence. Studying the conceptual understanding of linear relationships for students from both course sequences may help determine whether course sequencing makes a difference in conceptual understanding of linear relationships for high school students.

This study found that students do not have a strong conceptual understanding of linear relationships if they are not able to move fluidly between representations of these relationships.

The specifics of which representations and which direction of moving between representations presented the largest struggle for students should be investigated further. The assessment created for this study did not contain an item to assess students' abilities to move between each type of representation in both directions (e.g., table to graph and graph to table). Additional items should be compiled to ensure that all representational movements are represented to get a more accurate picture of where students lack understanding.

Finally, student interviews would help the researcher gain insight into student thinking for items where their thought process is not immediately clear on paper. Through these interviews, the researchers may also gain information on the types of learning experiences the students have had in their mathematics courses related to linear relationships. A clear picture of what is happening in the mathematics courses could explain why students are more successful on some items than others.

Concluding Remarks

Linear relationships are a foundational topic in mathematics and students are introduced to different representations of linear relationships throughout middle and high school mathematics courses. The results of this study showed that students have a weak conceptual understanding of linear relationships and limited ability to make connections between representations of these relationships such as graphs, equations, tables, and written contexts. Going into this study, I thought students would be very comfortable moving between representations by the time they were in Algebra 2, but I found that even the Algebra 2 students did not show a strong overall conceptual understanding of linear relationships. Students are expected to transfer some of their understanding and representational fluency of linear relationships to their work with nonlinear relationships (e.g., quadratic and exponential

relationships). Without a strong understanding or ability to interpret and move fluidly between different representations of linear relationships, students will likely struggle with representational fluency in nonlinear relationships. Students need to be given more opportunities to engage in learning experiences where they are working with multiple representations of the same linear relationship to support their conceptual understanding of linear relationships.

Given that the school of study is in an under-performing, high-minority, high-poverty district, the stakes of the instructional decisions made for the students in this school are very high (NCTM, 1989; Berry et al., 2013; NCES, 2019). As was discussed in the review of the literature, mathematics education has the power to impact a student's academic, personal, and professional lives (Burdman, 2018; Knudson, 2019; NCTM, 2018; National Research Council, 2001), so it is imperative that each and every student is given access to a high-quality mathematics education. The results of this study show that the students in this school may not have been given access to a high-quality mathematics education in regards to developing a conceptual understanding of linear relationships, and there is significant room for improvement.

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Appendix A

CCSSM Standards used for Assessment Development

Construct	Standard Name	Standard Description	Item Numbers on Assessment
1	CCSS.MATH.CONTENT.7.RP.A.2	Recognize and represent proportional relationships between quantities.	#14
1	CCSS.MATH.CONTENT.7.RP.A.2.B	Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.	#14, #11, #7, #4, #5
2	CCSS.MATH.CONTENT.6.EE.B.7	Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p , q and x are all nonnegative rational numbers.	#6
2	CCSS.MATH.CONTENT.6.EE.C.9	Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.	#1, #2
2	CCSS.MATH.CONTENT.7.EE.B.4	Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.	#10
2	CCSS.MATH.CONTENT.8.F.B.4	Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.	#10, #3
3	CCSS.MATH.CONTENT.8.F.B.4	Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.	#12, #13
3	CCSS.MATH.CONTENT.8.F.A.2	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). <i>For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.</i>	#8, #9
3	CCSS.MATH.CONTENT.8.F.B.5	Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.	#15

Appendix B

IRB Approval Letters



Institutional Review Board

uwm.edu/irb
irbinfo@uwm.edu
414-662-3544

Date: March 15, 2022

To: DeAnn Huinker

Dept: Curriculum and Instruction

CC: Melissa Manley - Co-Inv (Full Access w/Notify)

IRB #: 22.215

Title: Conceptual Understanding of Linear Relationships at Various Grade Levels

The University of Wisconsin-Milwaukee Institutional Review Board has granted your protocol Exempt Status under Category 1 as governed by 45 CFR 46.104(d).

This exemption determination is valid for three years and will expire on **March 14, 2025**. Before the expiration date, you will receive an email explaining how to either keep the study open or close it. If the study is completed before the expiration date, you may notify the IRB by sending an email to irbinfo@uwm.edu.

Any proposed changes to the protocol must be reviewed by the IRB before implementation, unless the change is specifically necessary to eliminate apparent immediate hazards to the subjects.

It is your responsibility to:

- promptly report unanticipated problems to the IRB
- maintain proper documentation of study records
- ensure that all study staff receive appropriate training as outlined in the protocol
- adhere to the policies and guidelines set forth by the IRB, UWM, and the UW System, and to all applicable state and federal laws

Contact the IRB office if you have any further questions. Thank you for your cooperation and best wishes for a successful project.



Institutional Review Board

uwm.edu/irb
irbinfo@uwm.edu
414-662-3544

Date: November 10, 2022

To: DeAnn Huinker

Dept: Curriculum and Instruction

CC: Melissa Manley - Co-Investigator

IRB #: 22.215

Title: Conceptual Understanding of Linear Relationships at Various Grade Levels

The University of Wisconsin-Milwaukee Institutional Review Board has approved the following changes to your protocol:

- Adding online and verbal methods of obtaining parental consent.

Review level: Exempt

As a reminder, this study's IRB approval will expire on **March 14, 2025**. Before the expiration date, you will receive an email explaining how to either keep the study open or close it.

Any proposed changes to the protocol must be reviewed by the IRB before implementation, unless the change is specifically necessary to eliminate apparent immediate hazards to the subjects.

It is your responsibility to:

- promptly report unanticipated problems to the IRB
- maintain proper documentation of study records
- ensure that all study staff receive appropriate training as outlined in the protocol
- adhere to the policies and guidelines set forth by the IRB, UWM, and the UW System, and to all applicable state and federal laws

Contact the IRB office if you have any further questions. Thank you for your cooperation and best wishes for a successful project.

Appendix C

Research Approval from School District



Office of Communications and School Performance
Department of Research, Assessment, and Data
5225 W. Vliet St., Room 224
Milwaukee, WI 53208
(414) 475-8258 • mps.milwaukee.k12.wi.us

Date: 3/29/2022

Name: Melissa Manley
Organization: MPS, UWM
Email: melissamanley2@gmail.com

Application #: 2122-24

Project Start Date: 4/4/2022

Project End Date: 5/29/2023

Re: Application for Approval to Conduct a Research or Evaluation Project in MPS

Dear Melissa,

I am pleased to inform you that your research project 'Conceptual Understanding of Linear Relationships at Various Grade Levels' has been approved. It is recommended that you maintain a copy of this letter as proof that you have received district approval to conduct your study. Your application number and dates for your approval are indicated at the top of this letter for your convenience.

Please submit a copy of your final report within 3 months after completion to Research, Assessment, and Data using the Application for Approval portal on our website.

I thank you for your interest in working with Milwaukee Public Schools and wish you success with this project. If you have any questions, please contact me at sandermc@milwaukee.k12.wi.us or phone at (414) 475-8570.

You may also find answers to your questions on our Research & Evaluation website:
<http://mps.milwaukee.k12.wi.us/en/District/Initiatives/Research-Development/Conducting-Research-in-MPS.htm>.

Sincerely,

A handwritten signature in black ink that reads "Marc Sanders".

Marc Sanders
Manager II
(414) 475-8570
sandermc@milwaukee.k12.wi.us

Start. Stay. Succeed.
Comienza. Quédate. Triunfa.

Appendix D
Teacher Script and Parent Letter to Explain Study

Hello,

As part of my doctoral dissertation research, I am completing a study to learn about how students' understanding of linear relationships differs between grade levels. To help me understand this, I am asking that students in grades 7, 8, 9, 10, and 11 all take the same test that I have created about linear relationships, and that includes you!

As a participant in this study, you would take the math test one day during your regular math class. On this test, you will only tell me your grade level and math course title, so I will not be able to tell whose test is whose. Your score on the test will not impact your math grade, and it will only be grouped with the scores from other people in your grade level to be compared with other grade levels. This data will be used to help me find differences in understanding between grade levels and ultimately earn my PhD.

Your participation in this study is optional, but I appreciate your consideration. If you are choosing to participate, please fill out the attached consent/assent form with your parent or guardian and return it to your math teacher.

Please let me know if you have any questions about this study.

Thank you,

Melissa Manley

mmmanley@uwm.edu

Appendix E

Student Assent Form



Minor Assent for Research Participation

IRB #: 22.215

IRB Approval Date: 03/15/2022

Study title	Conceptual Understanding of Linear Relationships at Various Grade Levels
Researcher	Melissa Manley - Doctoral candidate at the University of Wisconsin-Milwaukee

I'm inviting you to be in a research study. A research study is a way to learn new things. I am trying to learn more about how students at different grade levels understand linear relationships in math.

If you agree to be in this study, I will ask you to take a test about linear relationships during your regular math class. The test will take about one class period and you will complete it using a pencil and paper. The only personal information you will be asked for on the test is your grade level and the name of your math class, so no one will know which test is yours. The test will not be counted as a grade for any class.

Since this is a math test, you might feel anxious or nervous while taking it. If there is a question that you do not understand completely, I just ask that you try your best. If you don't get it right, that is ok. Remember, this test does not impact your grade and your results will not be connected to you in any way.

Being in this study might have some benefits for you. By taking this test, you might be able to see connections between things you have learned in math classes over the years. I also hope this study will help teachers improve their math instruction so it will help other students someday, too.

You don't have to be in this study. It is up to you. No one will be mad, no matter what you decide. If you say yes now, but then change your mind later, that's ok too. Just let me know.

When I am finished with this study, I will write a report about what I learned. This report won't have your name in it, or that you were in the study.

Signatures

If you decide you want to be in this study, write your name on the line below.

Name of Participant

Date

Appendix F

Parental Consent Form



Informed Consent for Research Participation

IRB #: 22.215

IRB Approval Date: 03/15/2022

Study title	Conceptual Understanding of Linear Relationships at Various Grade Levels
Researcher	Melissa Manley - Doctoral candidate at the University of Wisconsin-Milwaukee

Dear Parents/Guardians,

I am a teacher at Golda Meir and I am also working on my doctoral degree at UWM. I'm inviting your child to participate in my research study about how students understand linear relationships. The results of this study will be used to help us improve the math program at Golda.

What is the purpose of this study?

I want to see how students in different grade levels understand linear relationships and what strengths can be found at each grade level.

What will my child do?

Your child will take a test that I created. They will take it during their regular math class, and will have one class period to complete it. This test will not impact their grade at all, and they will not provide their name on the test, so it will be anonymous. Participation is completely voluntary. If you give consent for your child to participate but change your mind later, there are no negative consequences for your child to withdraw from the study.

Risks

Possible risks	How I'm minimizing these risks
Stress from taking a math test	- I am not asking for student names or anything other than their grade level and math class title, so the stress of wanting to do well on a math test should be taken away. - Your child's test is also only going to be analyzed as a part of their whole grade level's tests, not on an individual level.

Other Study Information

Possible benefits	Your child might start to see more connections between things they have learned in other grade levels as they take this test, which can also help them in future math classes.
Estimated number of participants	600
How long will it take?	One math class period (about 49 minutes)
Costs	None
Compensation	None
Future research	Data and results from this study may be shared with other researchers who are also interested in studying the same topic.
Conflicts of Interest	I am the researcher and am also a teacher at your child's school. This conflict is being managed by having the test being given to your child by their regular math teacher, so I am not giving any tests to any students.

Data Security

What identifying information will be collected and why?	I will only collect your child's grade level and math course title from them, which is necessary for me to compare test scores with students in other grade levels. No other identifying information will be collected.
How long will my child's data be kept?	Until May 2024
How is data kept secure?	• Data is anonymous since I will not be asking for your child's name or any other identifying information other than their grade level and course title. • I will store all completed tests in a locked filing cabinet in a locked office.

Who might see my data and why?

The researchers	To conduct the study and analyze the data
The IRB (Institutional Review Board) at UWM	To ensure we're following laws and ethical guidelines
The Office for Human Research Protections (OHRP) or other federal agencies	
Doctoral Committee Members at UWM	To make sure that I collected data accurately and determine whether I can receive my doctoral degree as a result of this study

Contact information:

For questions about the research, problems, or complaints	Melissa Manley	mmmanley@uwm.edu
For questions about your child's rights as a research participant, problems, or complaints	IRB (Institutional Review Board; provides ethics oversight)	414-662-3544 / irbinfo@uwm.edu

Signatures

If you have had all your questions answered and give permission for your child to participate in this study, sign on the lines below. Remember, your child's participation is completely voluntary, and you're free to remove them from the study at any time.

Name of Child (print)

Name of Parent, Guardian or Legally Authorized Representative (print)

Appendix G

Test Items, Constructs, Potential Distractors, and Source

Item Number	Construct	Justification for Use on Test & What I am Hoping to Learn from this Item	Distractors/Potential Misconceptions	Source & Performance Data
1	2	This item presents students with a table of values showing the relationship between the number of cards sold and profit made and asks them to choose the correct equation that represents the relationship. The relationship is directly proportional. Hopefully, students see that 0 cards yields a \$0 profit and realize it is a directly proportional relationship. I chose to include this item to see if students would be able to identify the equation of a simple, proportional relationship from given values which would show that they are able to fluently move between a tabular and algebraic representation of a linear relationship.	Answer: A Distractors/Potential Misconceptions: B - This equation shows the rate of change being 2. A student who chose this option may have thought the variables were reversed, as $n=2p$ would have been a valid equation. C - A student who chose this equation likely looked at the Monday values and noticed that the profit was 2 "less" than 4, so $n-2$ made sense for Monday. If they had looked at the other days, they would have seen that the $n-2$ rule did not work for any other day. D - This distractor may have been random, as I do not see an immediate misconception represented by $p=n+1$ E - Similar to C, a student may have looked at the Monday values and saw that the profit was equal to 6-n, so they chose equation E. This rule does not work for any of the other days, so they should not have chosen this answer choice.	Adapted (clarifying variables added into question text and answer choices re-ordered) from 2007 NAEP Grade 8 Assessment Question ID: 2007-8M7 #15 M145001 *A: 54% B: 17% C: 13% D: 6% E: 9% Omitted: 1%
2	2	This item asks students to select the correct graph that represents the symbolic equation that is given. This requires students to understand the relationship between the symbolic and graphical representations of the same linear relationship. I hope to learn whether students are able to visually see this relationship through the use of this item.	Answer: A Distractors/Potential Misconceptions: B - This line has the correct y-intercept, but shows a slope of positive 2 instead of negative 2. The student may not understand what a negative slope looks like visually on a graph. C - This line shows a y-intercept of 2 instead of 1, but an x-intercept of 1. It does show the correct slope of -2, so the student may have understood that the slope was negative and what that should look like on the graph, but either mixed up the x- and y-axis or didn't know what to do with the 1 from the equation. D - This line has the correct y-intercept of (0,1), but shows a slope of positive 1/2. This student likely struggles with translating the rate of change in the equation to the visual representation of slope on the graph. E - This graph has a y-intercept of (0,-2) and a slope of positive 2, with an x-intercept at (1,0). This student may have chosen this graph because there was a point on each axis corresponding with the numbers in the equation if they did not know what the numbers in the equation meant for the linear relationship.	2007 NAEP Grade 8 Assessment Question ID: 2007-8M11 #11 M106701 *A: 25% B: 15% C: 15% D: 31% E: 13% Omitted: 1%
3	2	This item asks students to identify an equation that represents the same relationship shown in a table. This relationship has a nonzero y-intercept, so it is not directly proportional. I chose this item to push students further than item 1 and to see if they are able to move fluidly between the tabular and algebraic representation of a linear relationship that is not directly proportional.	Answer: C Distractors/Potential Misconceptions: A - The equation $y = x+2$ is a correct equation if the student only looked at the set of values in the first row of the table. If they looked at any other pair of values, it would not work. The student likely did not look past the first line before choosing their answer. B - Again, the equation $y = 3x$ is a correct equation if the student only looked at the first row of the table, but it does not work for any other values in the table. D - The equation $y = 4x - 1$ is also a correct equation if the student only looked at the first row of the table, but it does not work for any other values in the table. E - The equation $y = 3x - 1$ is true for the second row of values, but is not true for any other row, so a student may have looked at only that row when choosing an answer.	2013 NAEP Grade 8 NAEP Test Question ID: 2013-8M6 #12 M141501 A: 156% B: 7% *C: 67% D: 5% E: 4% Omitted: 1%

4	1	This item requires students to fill in missing values in a table representing hours walked (n) and miles walked (d). Students were told that the person walks at a constant speed, which should signal to them that this is a proportional relationship. I chose to include this item to see whether students are able to find the unit rate and use it to complete the table's missing inputs and outputs.	Answers: 3.2 2.5 16 Potential Misconceptions: Students may subtract 6.4 from 8 to get 1.6 as the unit rate, not realizing that the hours column is not increasing by 1 for each row. This would make their answers be 4.8, 3 or 4, 9.6. Students may also correctly calculate the unit rate of 3.2 by dividing 6.4 by 2 so they can figure out the two miles walked blanks, but they may struggle to figure out the time for 8 miles walked, because that requires them to go "backwards"	Adapted (part A) from Illustrative Mathematics Task: Walk-a-thon 2 https://tasks.illustrativemathematics.org/content-standards/7/RP/A/2/tasks/1526
5	1	This item explicitly asks students to state the walking rate in miles per hour. This is the unit rate, and I chose to include this item to see whether students were able to connect the idea of a unit rate to the "walking rate" in context.	Answer: 3.2 miles per hour Potential Misconceptions: Students may have subtracted 6.4 from 8 to get 1.6 as the unit rate, not realizing that the hours column is not increasing by 1 for each row.	Adapted (part B) from Illustrative Mathematics Task: Walk-a-thon 2 https://tasks.illustrativemathematics.org/content-standards/7/RP/A/2/tasks/1526
6	2	In this item, students are asked to move from the tabular representation of the relationship to an algebraic representation by writing an equation to relate the number of hours walked (n) with the distance walked (d). I chose to include this item to determine whether students would be able to move from a tabular representation to algebraic representation on their own without having multiple answer choices to choose from.	Answer: $d = 3.2n$ Potential Misconceptions: Students may write $d = n/3.2$ instead of using multiplication, because they divided to find the unit rate in the first place. Students may write $n = 3.2d$ because they mixed the variables up in their heads. Students may write $n = d/3.2$, which is technically a correct representation of the relationship, but is not what the question specifically asked them to do, which is to write an equation for the distance, d.	Adapted (part C) from Illustrative Mathematics Task: Walk-a-thon 2 https://tasks.illustrativemathematics.org/content-standards/7/RP/A/2/tasks/1526
7	1	Students are asked to calculate the unit rate for two proportional, linear relationships. One is presented in a table and the other is present with an equation. I chose to include this to determine whether students could successfully calculate a unit rate from multiple representations of directly proportional relationships.	Answer: Company P: \$0.12 per kwh Company M: \$0.15 per kwh Potential Misconceptions: Students may not know that the unit rate is the rate of change or the slope in the equation for Company M, so they may not answer that part of the problem. Students may subtract values from the table instead of dividing to get a unit rate.	Adapted from MC ² PARCC Practice Test Items Grade 8 Mathematics #31, p. 85 https://mc2.nmsu.edu/teachers/paarc/spring_2018/8thMC2PracticeTestItemPacket_Spring2018-Revision.pdf
8	3	Students are asked to choose the least expensive company and explain their choice. This forces students to interpret the unit rate for two linear relationships, one being represented in a table and the other represented with an equation. I chose to include this item to see how students would interpret the unit rate aspect of the relationships when presented with a real world situation.	Answer: Company P, because their cost per kwh is less than that of Company M. Potential Misconceptions: Students may choose Company M if they incorrectly calculated the unit rate for Company P in item #7. Students may also choose the wrong company if they do not make the connection between unit rates/cost per kwh and the overall cost to use a company.	Adapted from MC ² PARCC Practice Test Items Grade 8 Mathematics #31, p. 85 https://mc2.nmsu.edu/teachers/paarc/spring_2018/8thMC2PracticeTestItemPacket_Spring2018-Revision.pdf

9	3	In this item, students are given two linear relationships, one represented with a table and the other with an equation. They are given four statements relating the two functions and are asked to choose the statement that is incorrect. I chose this item because it forces students to interpret different aspects of the given relationships such as slope and y-intercept and compare these aspects of the two relationships to each other.	Answer: C Distractors: A - The rate of change for function 1 (1.5) is greater than the rate of change for function 2 (4/3). A student may have missed the "incorrect" aspect of the question and chose this as a correct answer. They also may not have known whether 1.5 or 4/3 was greater since one is a decimal and one is a fraction. Some students may have seen the 4 as the numerator and thought the fraction was greater than 1.5. B - The y-intercepts are equal for the two functions. A student may have missed the "incorrect" aspect of the question and chose this as a correct answer. D - The line of the graph for function 2 is less steep than the line of the graph for function 1. A student who chose this answer choice may not have known that "steepness" is directly connected to the rate of change. They also may have missed the "incorrect" aspect of the question and chose this as a correct answer.	Adapted from PARCC Released Test Items 2017 #22 Item Number: M20868P https://www.aps.edu/assessment/educator-resources/released-items/math/parcc-grade-8-math-item-set-2017
10	2	This item asks students to choose an expression that represents the given verbal description of a linear relationship. This item was included on the test to see if students are able to fluidly move from a verbal representation of a linear relationship to an algebraic representation.	Answer: E Distractors/Potential Misconceptions: A - A student who chose this answer likely did not understand what was going on in the story. They may have seen the word "plus" in the verbal description and decided to add all values from the description together. B - A student who chose this answer may not have understood the situation at hand. C - A student who chose this answer seems to have mixed up the hourly rate and the constant fee. D - A student who chose this answer seems to know that something needed to be multiplied and something else needed to be added, but they are adding the hours and multiplying the two costs, so that is incorrect.	1996 Grade 8 NAEP Assessment Question ID: 1996-8M3 #9 M050701 A: 15% B: 8% C: 11% D: 8% *E: 58%
11	1	This item is a very simple unit rate problem. The students are given a proportional relationship verbally, and are asked to find the unit rate. I chose this item to see if students understand the concept of a unit rate in a proportional relationship.	Answer: B Distractors/Potential Misconceptions: A - Someone who chose the answer 45 likely guessed. Mathematically, they could have gotten to 45 by dividing 720 by 8, then dividing that again by 2, but that seems like an unlikely path to take. C - 728 may have been chosen by combining the 720 and 8 either by addition or simply choosing the answer choice that had all nonzero digits in it. This student likely does not understand the concept of a unit rate. D - A student who picked this answer choice probably multiplied 720 by 8 to get to 5760. They do not fully understand the concept of a unit rate.	SpringBoard Course 1, Unit 4 Practice (2014) #21

12	3	This item asks students to interpret the meaning of values from a given equation in context of the real world situation presented. This particular problem addresses the y-intercept, or initial cost in the situation. I chose to include this item to see how students do at interpreting the meaning of numerical values in an algebraic representation of a linear relationship.	Answer: B Distractors/Potential Misconceptions: A - This answer switched the meaning of slope and y-intercept, so a student who chose this answer choice also probably had them switched in their heads, where they associated the rate of change with the cost of making zero bowls. C - A student who chose this answer realized that 50 had to do with the cost of each bowl, but they thought that it decreased the total cost by \$50, when it actually increased it since the 50 is positive in the equation. D - This answer also switched the meaning of slope and y-intercept, so a student who chose this answer associated the y-intercept with the incremental change in cost based on the number of bowls made. They likely switched the meaning of the two values in their head. E - A student who chose this answer simply added the numerical values from the equation together and decided that was the cost for each bowl. This student has a very limited understanding of interpreting the algebraic representation of linear equations.	2017 Grade 8 NAEP Assessment Question ID: 2017-8M9 #11 M355601 A: 12% *B: 36% C: 17% D: 16% E: 16% Omitted: 1%
13	3	This item asks students to interpret the meaning of values from a given equation in context of the real world situation presented. This particular problem addresses the slope, or rate of change in the situation. I chose to include this item to see how students do at interpreting the meaning of numerical values in an algebraic representation of a linear relationship.	Answer: D Distractors/Potential Misconceptions: A - This answer switched the meaning of slope and y-intercept, so a student who chose this answer choice thought that the y-intercept represented the rate of change. B - A student who chose this answer correctly identified that the slope value from the equation represents the filling rate, but they didn't interpret the negative sign in the equation correctly as emptying instead of filling. C - A student who chose this answer likely divided 800 by 50 to arrive at 16. This student has very limited understanding of interpreting the algebraic representation of a linear relationship. E - A student who chose this answer recognized that water was emptying from the minus sign, but they switched the slope and y-intercept meaning and thought that the y-intercept was the rate of change.	2011 Grade 8 NAEP Assessment Question ID: 2011-8M12 #15 M154101 A: 11% B: 21% C: 13% *D: 48% E: 6% Omitted: 1%
14	1	This item presents students with verbal information about two people driving for given times, and they are asked to tell who was driving faster, then explain how they know this. This item is a unit rate problem, but does not explicitly ask students to find the unit rate. I chose to include this item to see whether students understand when a unit rate is useful in solving a problem involving proportional relationships.	Answer: Julius, because he was driving 65 mph while Alisha was only driving 62 mph. Potential Misconceptions: Students may not calculate the unit rates and may just choose based on the numbers they see, then try to justify their thinking. Students may multiply the values together and get 3,968 for Alisha and 7,865 for Julius so they would still choose Julius, but not because of the unit rate.	Self-created
15	3	This item gives students a graphical representation of two runners. They are asked to fill in a table with the verbal interpretation of different aspects of the graphical representations. I chose to include this item because it touches on multiple aspects of interpreting a graphical representation of linear relationships and will show whether students are able to make those connections and interpretations between a graph and a real-world situation.	Answers: Slopes: Runner A was running faster than runner B because the slope of Runner A's line is more steep OR Runner A is running about 8 mph and Runner B is running about 3.5 mph Point of Intersection: At about 11:45am, both runners were a little less than 5 miles from the finish line. This is the point where Runner A passed Runner B. x-intercepts: Runner A finished the race at 12:15pm and Runner B finished the race at 1:00pm Distractor/Incorrect Response: Students may not know the vocabulary of "slopes" "point of intersection"	2009 Grade 12 NAEP Assessment Question ID: 2009-12M2 #9 M1809CL,M180901,M180902,M180903 Incorrect: 34% Minimal: 13% Partial: 24% Satisfactory: 13% Complete: 1% Omitted: 13% Off Task: 2%

Appendix H

Linear Relationships Assessment

Linear Relationships Assessment

This assessment is based on linear relationships. Your score will not count toward your math grade for class, and your answers will not be able to be connected with you individually. Results from this test will be looked at in groups based on grade levels, so your responses will be anonymous. Please try your best and show your work.

What grade are you in? (You must circle one):

7 th	8 th	9 th	10 th	11 th
-----------------	-----------------	-----------------	------------------	------------------

What math class are you currently in? (You must circle one):

7 th grade math	8 th grade math	Algebra 1
Geometry	Algebra 2	Precalculus

Test Directions:

- For multiple choice questions, circle the best answer choice.
- For short answer questions, please write your answer clearly where the question is asked.
- Workspace has been provided to the right of each question if you need it.

1. Angela makes and sells special-occasion greeting cards. The table below shows the relationship between the number of cards sold (n) and her profit (p). Based on the data in the table, which of the following equations shows how the number of cards sold (n) and profit (p) (in dollars) are related?

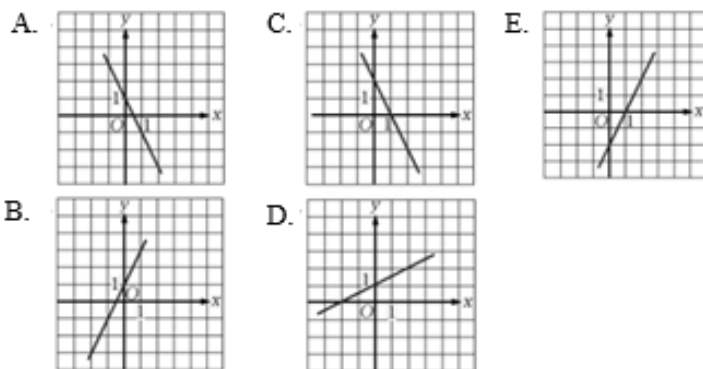
	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Number Sold, n	4	0	5	2	3	6
Profit, p	\$2.00	\$0.00	\$2.50	\$1.00	\$1.50	\$3.00

Workspace (if needed):

- A. $p = 0.5n$
 B. $p = 2n$
 C. $p = n - 2$
 D. $p = n + 1$
 E. $p = 6 - n$

2. Which of the following is the graph of the line with equation $y = -2x + 1$?

Workspace (if needed):



3. The table below lists the coordinates of several points on a line.

x	y
1	3
2	5
3	7
4	9

Workspace (if needed):

Which of the following is an equation of the line?

- A. $y = x + 2$
 B. $y = 3x$
 C. $y = 2x + 1$
 D. $y = 4x - 1$
 E. $y = 3x - 1$

Use the following information for questions 4 through 6.

Julianna participated in a walk-a-thon to raise money for cancer research. She recorded the total distance she walked at several different points in time, but a few of the entries got smudged and can no longer be read. The times and distances that can still be read are listed in the table below.

4. Assume Julianna walked at a constant speed. Complete the table with the missing miles walked and time in hours.

Time in Hours (n)	Miles walked (d)
1	
2	6.4
	8
5	

Workspace (if needed):

5. What was Julianna's walking rate in miles per hour?
6. Write an equation for the distance d , in miles, that Julianna walked in n hours.

Use the following information for questions 7 and 8.

Two utility companies sell electricity in units of kilowatt-hours. The cost of electricity for company P is shown in the table. The cost of electricity for company M can be found by using the equation shown, where y represents the total cost in dollars for x kilowatt-hours of electricity.

Electricity Costs		
Company P		Company M
Number of Kilowatt-hours	Total Cost (dollars)	$y = 0.15x$
1,250	150.00	
1,650	198.00	

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

Unit Rate: _____

Company M

Workspace:

Unit Rate: _____

8. Which is the *least* expensive company? Explain your answer.

9. Two linear functions are shown.

Function 1

Input (x)	Output (y)
-3	-2.5
-1	0.5
2	5
4	8

Function 2

$$y = \frac{4}{3}x + 2$$

Workspace (if needed):

Which statement about the functions is incorrect?

- A. The rate of change for function 1 is greater than the rate of change for function 2.
- B. The y-intercept of the line for function 1 is equal to the y-intercept of the line for function 2.
- C. The y-intercept of the line for function 1 is less than the y-intercept of the line for function 2.
- D. The line shown for the graph of function 2 is less steep than the line shown for the graph of function 1.

10. A plumber charges customers \$48 for each hour worked plus an additional \$9 for travel. If h represents the number of hours worked, which of the following expressions could be used to calculate the plumber's total charge in dollars?

Workspace (if needed):

- A. $48 + 9 + h$
- B. $48 \times 9 \times h$
- C. $48 + (9 \times h)$
- D. $(48 \times 9) + h$
- E. $(48 \times h) + 9$

11. A factory can produce 720 wagons in an 8-hour day.
What is the unit rate of wagons made per hour?

A. 45
B. 90
C. 728
D. 5760

Workspace (if needed):

12. Jessica makes clay bowls.

The following equation relates the number of clay bowls, n , she makes and the total amount of money, d , in dollars, it costs for her to make them.

$$d = 375 + 50n$$

Based on the equation, which of the following statements is true?

- A. Without making any clay bowls, Jessica's total cost is \$50.
B. Without making any clay bowls, Jessica's total cost is \$375.
C. Each clay bowl Jessica makes decreases her total cost by \$50.
D. Each clay bowl Jessica makes increases her total cost by \$375.
E. Each clay bowl Jessica makes costs \$425.

Workspace (if needed):

13. The number of gallons of water, y , in a tank after x hours may be modeled by the linear equation $y = 800 - 50x$. Which of the following statements about the tank is true?

- A. It is filling at a rate of 800 gallons per hour.
- B. It is filling at a rate of 50 gallons per hour
- C. It is emptying at a rate of 16 gallons per hour.
- D. It is emptying at a rate of 50 gallons per hour.
- E. It is emptying at a rate of 800 gallons per hour.

Workspace (if needed):

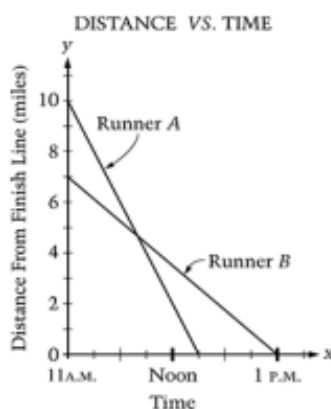
14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

How do you know?

Workspace (if needed):

15. The graph below shows distance *versus* time for a race between runners *A* and *B*. The race is already in progress, and the graph shows only the portion of the race that occurred after 11 A.M.



The given table lists several characteristics of the graph. Interpret these characteristics in terms of what happened during this portion of the race. Include times and distances to support your interpretation. (A sample interpretation of the y -intercepts is given in the table.)

Characteristic of Graph	Interpretation in Terms of the Race
y -intercepts	At 11 am Runner <i>A</i> is 10 miles from the finish line and Runner <i>B</i> is 7 miles from the finish line.
Slopes	
Point of Intersection	
x -intercepts	

Appendix I

Rubric Used to Score Linear Relationships Assessment

		Score	
Item Number	0	0.5	1
#1	Student did not choose the correct answer (A)	N/A	Student chose the correct answer (A)
#2	Student did not choose the correct answer (A)	N/A	Student chose the correct answer (A)
#3	Student did not choose the correct answer (C)	N/A	Student chose the correct answer (C)
#4	Student entered 0 of 3 missing values into the table correctly	Student entered 1-2 of 3 missing values into the table correctly	Student entered 3 of 3 missing values into the table correctly (3.2, 2.5, 16)
#5	Student had completely incorrect walking rate and no evidence of thought process	Student had an incorrect walking rate, but showed evidence of a thought process.	Student gave the correct walking rate (3.2 miles per hour)
#6	Student did not provide an equation	Student provided an accurate verbal representation of the relationship, but did not provide it in equation form with the given variables.	Student provided the correct equation ($d = 3.2n$)
#7	Student did not correctly identify the unit rate for either company P or company M.	Student correctly identified the unit rate for ONE company, either P (0.12 per kwh) or M (0.15 per kwh)	Student correctly identified the unit rates for both companies (P: 0.12 per kwh and M: 0.15 per kwh)

#8	Student did not identify the correct company as being least expensive AND did not provide an explanation.	Student either correctly identified the least expensive company (P) but gave no explanation, OR identified the wrong company but provided an explanation that supported their choice based on their work in #7	Student correctly identified the least expensive company AND gave an explanation that supported their choice. (Company P, connection to rate of change or cost per kwh being lower than the other company)
#9	Student did not choose the correct answer (B)	N/A	Student chose the correct answer (B)
#10	Student did not choose the correct answer (E)	N/A	Student chose the correct answer (E)
#11	Student did not choose the correct answer (B)	N/A	Student chose the correct answer (B)
#12	Student did not choose the correct answer (B)	N/A	Student chose the correct answer (B)
#13	Student did not choose the correct answer (D)	N/A	Student chose the correct answer (D)
#14	Student provided the wrong person's name, or student did not answer the item.	Student provided the correct name (Julius), but did not provide a mathematical explanation.	Student provided the correct name (Julius) and provided a mathematical explanation including the unit rate in mph. Julius, because he was driving 65 mph while Alisha was only driving 62 mph.

#15	Student provided 0 of 3 interpretations of the characteristics of the graph correctly	Student provided 1-2 of 3 interpretations of characteristics of the graph correctly	Student provided all 3 interpretations of characteristics of the graph correctly Slopes: Runner A was running faster than runner B because the slope of Runner A's line is more steep OR Runner A is running about 8 mph and Runner B is running about 3.5 mph Point of Intersection: At about 11:45am, both runners were a little less than 5 miles from the finish line. This is the point where Runner A passed Runner B. x-intercepts: Runner A finished the race at 12:15pm and Runner B finished the race at 1:00pm
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Appendix J

Student Work Samples for Open Response Items

Item 4 – Correct Responses

Student 7a6

Time in Hours (n)	Miles walked (d)
1	3.2
2	6.4
2.5	8
5	16

Workspace (if needed):

$$6.4 / 2 = 3.2$$

$$8 / 3.2 = 2.5$$

$$5 \cdot 3.2 = 16$$

Student 7a8

Time in Hours (n)	Miles walked (d)
1	3.2
2	6.4
2.5	8
5	16

Workspace (if needed):

$$\frac{2}{6.4} = \frac{x}{8}$$

$$2x = 6.4$$

$$x = 3.2$$

$$\frac{5}{2.5} = \frac{25}{8}$$

$$40 = 25x$$

$$16 = x$$

$$16 = 6.4x$$

$$x = 2.5$$

Student 7b12

Time in Hours (n)	Miles walked (d)
1 $\times 3.2$	3.2
2 $\times 3.2$	6.4
2.5 $\times 3.2$	8
5	16

Workspace (if needed):

$$\begin{array}{r} 3.2 \\ 2 \overline{) 6.4} \end{array}$$

Student 9c29

Time in Hours (n)	Miles walked (d)
1	3.2
2	6.4
2.5	8
5	16

Workspace (if needed):

$$\frac{2}{6.4} = \frac{x}{8} \rightarrow \frac{16}{6.4} = \frac{6.4 \cancel{x}}{\cancel{6.4}}$$

$$x = 2.5$$

$$\frac{1}{x} = \frac{2}{6.4} \rightarrow \frac{6.4}{2} = \frac{2x}{2}$$

$$x = 3.2$$

$$\frac{5}{x} = \frac{2}{6.4} \rightarrow \frac{32}{2} = \frac{2x}{2}$$

$$x = 16$$

Student 10e15

Time in Hours (n)	Miles walked (d)
1	3.2
2	6.4
2.5	8
5	16

$$2 \overline{) 6.4}$$

$$2 \overline{) 3.2}$$

$$2 \overline{) 1.6}$$

B

$$\begin{array}{r} 6.4 \\ - 3.2 \\ \hline 3.2 \end{array}$$

Workspace (if needed):

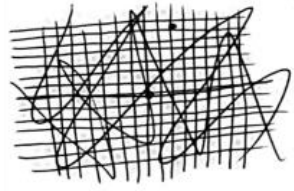
$$1.6 \overline{) 3.2}$$

$$16 \overline{) 32}$$

$$\begin{array}{r} 1 \ 3.2 \\ + \ 2 \ 6.4 \\ + \ 3 \ 9.6 \\ + \ 4 \ 12.8 \\ + \ 5 \ 16 \end{array}$$

Student 10e2

Time in Hours (n)	Miles walked (d)
1	3.2
2	6.4
2.5	8
5	16



3.2 mph

Workspace (if needed):

$$\frac{2}{6.4} \neq \frac{1}{x}$$

$$\frac{2x}{2} = \frac{6.4}{2} = 3.2$$

$$\frac{2}{6.4} \neq \frac{x}{8}$$

$$\frac{16}{6.4} = \frac{6.4x}{6.4} = 2.5$$

Item 4 – Common Errors

Student 8b14

Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)
1	1.6
2	6.4
4.8	8
5	9.6

Workspace (if needed):

$$\begin{aligned}
 8 - 6.4 &= 1.6 \\
 \cancel{6.4} + 1.6 &= 8 \\
 8 - 6.4 &= 1.6 \\
 8 + 1.6 &= 9.6
 \end{aligned}$$

Student 9c15

Time in Hours (<i>n</i>)	Miles walked (<i>d</i>)
1	4.8
2	6.4
3	8
5	9.6

$$\begin{array}{r}
 5 \cancel{16} 14 \\
 - 1.6 \\
 \hline
 4.8
 \end{array}$$

Workspace (if needed):

$$\begin{array}{r}
 1.6 \\
 + 1.6 \\
 \hline
 3.2
 \end{array}$$

Student 10d7

Time in Hours (n)	Miles walked (d)
1	4.8
2	6.4
3	8
5	9.6

Workspace (if needed):

$$8 - 6.4 = 1.6$$

$$6.4 - 1.6 = 4.8$$

$$1 + 1 = 2 \quad 1 + 2 = 3$$

$$2 + 3 = 5 \quad 5 - 2 = 3$$

Student 11e11

Time in Hours (n)	Miles walked (d)
1	4.8
2	6.4
?	8
5	9.6

1.6

Workspace (if needed):

$$\begin{array}{r} 8.0 \\ - 6.4 \\ \hline 1.6 \end{array} \quad \begin{array}{r} 56.4 \\ - 1.6 \\ \hline 4.8 \end{array}$$

$$\begin{array}{r} 8.0 \\ + 1.6 \\ \hline 9.6 \end{array}$$

Item 7 & 8 – Correct Responses

Student 7a19

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\frac{1250}{1250 \div 150 = X}$$

$$\frac{1250}{1250} \times \frac{150}{150} = \frac{150}{1}$$

$$X = 0.12$$

Unit Rate: 0.12

Company M

Workspace:

$$Y = 0.15 \times 5$$

$$0.15 \times 5 = 0.75$$

Unit Rate: 0.15

8. Which is the *least* expensive company? Explain your answer.

Company P is less expensive because the cost on each kilowatt is 0.12.

Student 7b11

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$150 \div 1,250 = \$0.12$$

Company M

Workspace:

Unit Rate: \$0.12 per kilowatt-hour

Unit Rate: \$0.15 per kilowatt-hour

8. Which is the *least* expensive company? Explain your answer.

company p is least expensive, because it charges
less per kilowatt-hour.

Student 9c14

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$0.12x$$

Unit Rate: \$0.12 \$/kh

Company M

Workspace:

$$0.15x$$

$$\begin{array}{r} 1250 \\ \times 0.15 \\ \hline 6250 \\ 12500 \\ \hline 18750 \end{array}$$

Unit Rate: \$0.15 \$/kh

8. Which is the *least* expensive company? Explain your answer.

Company P is the least expensive. The found unit rate of 0.12 \$/kh is cheaper than the given unit rate for company M of 0.15 \$/kh by \$0.03, so company P is the least expensive & company M costed \$187.50 for 1250 kilowatt-hours where company P costed \$150

Student 9c28

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{array}{r} 1250 \\ + 1650 \\ \hline 2900 \end{array}$$

$$\frac{348}{2900} = 0.12$$

Unit Rate: 0.12

Company M

Workspace:

Unit Rate: 0.15

8. Which is the *least* expensive company? Explain your answer.

Company P because the unit rate is smaller
and if graphed has a less steep slope.

Student 9c35

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{array}{r} 1,250 \\ \times 50 \\ \hline \end{array}$$

$$\begin{array}{r} 198 \\ - 150 \\ \hline 48 \end{array}$$

$$\begin{array}{r} 48 \\ \times 400 \\ \hline \end{array}$$

Unit Rate: .12

Company M

Workspace:

Unit Rate: .15

8. Which is the *least* expensive company? Explain your answer.

Company P is the least expensive. It's 3 cents cheaper than Company M.

Student 10d9

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{aligned} 150 - 198 &= -48 = y \\ 1250 - 1650 &= -400 = x \\ \frac{-48}{-400} &= 0.12 \end{aligned}$$

Unit Rate: \$0.12

Company M

Workspace:

$$y = \boxed{0.15}x$$

↑
Rate

Unit Rate: \$0.15

8. Which is the *least* expensive company? Explain your answer.

Company P, because for each kilowatt-hours
you'd used, it would be less than
Company M.

Student 10e2

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{aligned} 1250/150 &= 8.\overline{3} \\ 1650/198 &= 8.\overline{3} \\ y &= 8.\overline{3} \\ 150/1250 &= 0.12 \\ 198/1650 &= 0.12 \end{aligned}$$

Unit Rate: ~~\$8.30~~ \$0.12

Company M

Workspace:

$$y = 0.15x$$

Unit Rate: \$0.15

8. Which is the *least* expensive company? Explain your answer.

Company P is less expensive because they charge \$0.12 per kilowatt-hour while Company M charges \$0.15 per kilowatt-hour

Student 10e4

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\frac{198}{1650} = 0.12$$

$$y = 0.12x$$

Unit Rate: 0.12 per hour

Company M

Workspace:

$$y = 0.15x$$

$$y = 0.15(1650) = 247.5$$

$$\frac{247.5}{1650} = 0.15$$

Unit Rate: 0.15 per hour

8. Which is the *least* expensive company? Explain your answer.

Company P is least expensive because for 1650 hours it is 198 compared to company M's 247.5 for 1650 hours.

Student 10e10

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{array}{l} \Delta x = 400 \\ \Delta y = 48 \end{array} \quad \frac{48}{400}$$

Unit Rate: .12 \$

Company M

Workspace:

Unit Rate: .15 \$

8. Which is the *least* expensive company? Explain your answer.

Company P, they have lower cost per kilowatt hours
meaning they will charge less.

Item 7 & 8 – Common Error

Student 9c29

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$1,250 \div 150 = 8.\overline{3}$$

$$1,650 \div 198 = 8.\overline{3}$$

Unit Rate: $8.\overline{3}$ per kilowatt-hour

Company M

Workspace:

$$y = 0.15x$$

Unit Rate: 0.15 per kilowatt-hour

8. Which is the *least* expensive company? Explain your answer.

The least expensive company is Company M because
they only make 15 cents per kilowatt-hour while
Company P makes \$8.33 per kilowatt-hour.

$$0.15 < 8.3$$

Student 10e1

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$1,256 / 150 = 8.33$$

.83

$$y = 8.33x$$

Unit Rate: ~~1.83~~ ~~8.33~~

Company M

Workspace:

Unit Rate: .15

8. Which is the *least* expensive company? Explain your answer.

Company M. While M is 15¢ per hour, Company P is over \$8.

Student 10e18

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$1.250h = 150d$$

$8.\overline{3}$ unit rate

Unit Rate: $8.\overline{3}$

Company M

Workspace:

$$y = 0.15x$$

Unit Rate: $0.15x$

8. Which is the *least* expensive company? Explain your answer.

company M is the least expensive,
their unit rate is 0.15 units cheaper
than company P.

Student 7a21

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

$$\begin{array}{r} 1,650 \\ 198.50 \overline{)1,650} \\ \hline \end{array}$$

$\$8.3$ per hour connection

Unit Rate: $\$8.3$ per hour

Company M

Workspace:

$$y = 0.15$$

Unit Rate: _____

8. Which is the *least* expensive company? Explain your answer.

Student 8b11

7. Use the information provided to find the unit rate, in dollars per kilowatt-hour, for each company.

Company P

Workspace:

Unit Rate:

$8.\bar{3}x$

Company M

Workspace:

Unit Rate:

$0.15x$

8. Which is the *least* expensive company? Explain your answer.

Company M because 0.15 is less than $8.\bar{3}$.

Item 14: Correct Responses

Student 7a19

14. Alisha drove 496 miles in 8 hours.

Julius drove 715 miles in 11 hours.

Who was driving faster?

Julius

How do you know?

Because I divided both
of the people's miles and hours
to get my answer. 65 is
greater than 62 so therefore
Julius was driving faster.

Workspace (if needed):

$$496 \div 8 = 62$$

$$715 \div 11 = 65$$

Student 7b7

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Julius

How do you know?

I know because I found
the unit rate then
multiplied it by 11 to
see how many miles Alisha
drove in 11 hours, it was only
682 miles compared to
Julius's 715

Workspace (if needed):

Student 7b8

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Julius

How do you know?

I know because Julius is
driving at 65 mph and Alisha
is driving at 62 mph
which makes Julius faster.

Workspace (if needed):

$$496/8 = 62$$

$$715/11 = 65$$

Student 8d4

14. Alisha drove 496 miles in 8 hours.

Julius drove 715 miles in 11 hours.

Who was driving faster?

Julius

How do you know?

$496/8 < 715/11$
Divide the miles driven by the
number of hours and you get the
MPH. Julius' MPH was greater than
Alisha's.

Workspace (if needed):

$$\begin{array}{r} 496 \\ 8 \overline{) } \\ 715 \\ 11 \overline{) } \end{array}$$

Student 9c9

14. Alisha drove 496 miles in 8 hours.

Julius drove 715 miles in 11 hours.

Who was driving faster?

Julius

How do you know?

I thought it would be
multiplying the amount of
hours to the total amount
of miles they drove altogether,
but then I realized it would
make more sense for how
much they drove per hour.
Julius drove 65 miles per
hour, meanwhile Alisha
drove 62 miles per hour

Workspace (if needed):

$$\begin{array}{r} 74 \\ 496 \\ 8 \overline{) } \\ 3,968 \end{array} \quad \text{Alisha}$$

$$\begin{array}{r} 715 \\ \times 11 \\ 715 \\ 7150 \\ + \\ 7865 \end{array} \quad \text{Julius}$$

$$496 \div 8 = 62 \text{ HOURS}$$

↳ Alisha

$$715 \div 11 = 65 \text{ HOURS}$$

↳ Julius

Student 11e11

14. Alisha drove 496 miles in 8 hours.

Julius drove 715 miles in 11 hours.

$$\frac{496}{8} = 62$$

Who was driving faster?

Julius

$$\frac{715}{11} = 65$$

How do you know?

I know this because I
divided each persons number
of miles with their number
of hours and it gave me
the number of miles they
drove per hour. Julius
has a higher # of miles
so therefore he drove
faster than Alisha.

Workspace (if needed):

Item 14: Common Error – Choosing Lower MPH as Faster Person

Student 8b8

14. Alisha drove 496 miles in 8 hours. = 62
 Julius drove 715 miles in 11 hours. = 65

Who was driving faster?

Alisha

How do you know?

Because when you divide
the numbers Alisha miles
per hour is faster by
3 62, 65.

Workspace (if needed):

$$\begin{array}{r} 62 \\ 8 \overline{)496} \\ \underline{-480} \\ 16 \end{array} \quad \begin{array}{r} 65 \\ 11 \overline{)715} \\ \underline{-660} \\ 55 \\ \underline{-55} \\ 0 \end{array}$$

Student 8b12

14. Alisha drove 496 miles in 8 hours.
 Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

Because I think I did the
math right

Workspace (if needed):

$$\begin{array}{r} 63 \\ 11 \overline{)715} \\ \underline{-660} \\ 55 \\ \underline{-55} \\ 0 \end{array} \quad \begin{array}{r} 62 \\ 8 \overline{)496} \\ \underline{-480} \\ 16 \\ \underline{-16} \\ 0 \end{array}$$

Student 11e1

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

Alisha drove 62 mph (496 ÷ 8)
Julius drove 65 mph (715 ÷ 11)

Workspace (if needed):

Student 11e25

14. Alisha drove 496 miles in 8 hours. 62
Julius drove 715 miles in 11 hours. 65

Who was driving faster?

Alisha

How do you know?

dividing the miles by
the hours

Workspace (if needed):

Item 14: Common Error 2 – Choosing Based on Amount of Time

Student 7a18

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

Alisha only drove 8 hours
and 496 miles which is way
less than Julius. Which she
had 715 miles in 11 hours.
Julius took way longer.

Workspace (if needed):

Student 8b15

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

She drove more in 8 hours.

Workspace (if needed):

Student 8b19

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

I know because she
496 in 8 hours which
is less than 11 hours.

Workspace (if needed):

Student 9c18

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

because she drove faster
in a little and fewer amount
of time

Workspace (if needed):

496 = 8
715 = 11
73

Student 9c19

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

Alisha drove half the
speed Julius road in only
8 hours.

Workspace (if needed):

Student 10d19

14. Alisha drove 496 miles in 8 hours.
Julius drove 715 miles in 11 hours.

Who was driving faster?

Alisha

How do you know?

because she got to alot
of miles in less hours

Workspace (if needed):

Item 15: Correct Responses

Student 7b9

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner a completes the race at a pace of 3.9 mph Runner b completes the race at a pace of 2.5 mph.
Point of Intersection	at 11:49 am Runner a and b are both 4.5 miles from the finish
x-intercepts	at 12:15 PM Runner a makes it to the finish line. at 1pm Runner b makes it to the finish line

Student 7d2

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A ran $\frac{2}{15}$ of a mile every minute, and runner B ran $\frac{1}{15}$ of a mile every minute
Point of Intersection	The runners were the same distance from the finish
x-intercepts	Runner A finished at 12:15 and Runner B finished at 1:00

Student 9c28

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A runs at a constant speed of 8mph towards the finish line and Runner B runs at a constant speed of 3.5 mph towards the finish line.
Point of Intersection	At 11:45 am Runner A and B meet 4.75 miles from the finish line.
x-intercepts	At 12:15 pm Runner A reaches the finish line and at 1 pm Runner B reaches the finish line.

Student 9c30

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A ran at a constant speed of 8 MPH while Runner B ran at a constant speed of 3.5 MPH
Point of Intersection	Just before 11:45, is when Runner A and Runner B caught up to each other.
x-intercepts	At 12:15, Runner A had completed the race and crossed the finish line. At 1 P.M., Runner B also finished and crossed the finish line

Student 10e7

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	RUNNER A IS RUNNING FASTER THAN RUNNER B BECAUSE THEIR SLOPE IS MUCH SHARPER THAN RUNNER B'S. A = HIGHER RISE/LOWER RUN VS. B = LOWER RISE/HIGHER RUN, MAKING B TAKE LONGER
Point of Intersection	SINCE RUNNER A STARTED FARTHER (3 MILES) FROM THE FINISH LINE THAN RUNNER B, THEIR INTERSECTION IS THE POINT AT WHICH THEIR INCREASED SPEED OVERTAKES THE DISTANCE THAT B HAS ALREADY RUN BUT SLOWER
x-intercepts	THE X INTERCEPTS REPRESENT THE POINTS AT WHICH THE RUNNERS FINISHED THEIR RACE, BECAUSE THEY HAD NO DISTANCE FROM THE FINISH LINE

Student 10e2

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A ⁴ has a steeper slope, so they were running faster than Runner B
Point of Intersection	They met each other at 11:45, 4.5 miles from the finish line
x-intercepts	runner B reached the finish line at 1pm and runner A reached the finish line at 12:15pm.

Student 11e2

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 am Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	Runner A was running faster than Runner B.
Point of Intersection	At 11:45, Runner A and Runner B were both about 5 miles from the finish line.
x-intercepts	At 12:15 Runner A finished the race and Runner B finished at 1:00 pm.