Box Splines

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ABSTRACT

This report gives an introduction to multivariate cardinal spline theory. It is based on joint work with Carl de Boor and Sherman Riemenschneider and the particular topics discussed are: recurrence relations for box splines, approximation order, interpolation, convergence to functions of exponential type and subdivision algorithms.

AMS (MOS) Subject Classifications: 41A15, 41A63.

Key Words: box splines, recurrence relations, approximation, interpolation, subdivision algorithms.

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Introduction

This paper describes a generalization of univariate cardinal spline theory to several variables. Multivariate cardinal splines are defined as linear combinations of translates of box splines,

$$S_V := \text{span}\ \{B_V(\cdot - j)\ :\ j \in \mathbb{Z}^d\}.$$

The box spline $B_V : \mathbb{R}^d \mapsto \mathbb{R}$ is a natural generalization of the univariate cardinal B-spline which is apparent from its Fourier transform

$$\widehat{B}_V(x) = \prod_{v \in V} \frac{\sin(xv/2)}{xv/2}.$$

The freedom in the choice of the vectors V results in rich theory which has attracted considerable interest. Some multivariate results are rather unexpected from the univariate theory, others do not have a univariate analogue.

Box splines were introduced by de Boor and DeVore [BD] and their basic properties have been described in [BH82/83]. Subsequently, many different aspects of approximation by box splines have been studied. De Boor and I [BH83, BH86] have applied box spline techniques to analyze bivariate smooth piecewise polynomials on regular meshes. Further results along these lines, in particular on approximation order, have been obtained by Dahmen and Micchelli [DM83_2, DM84], Jia [J84_1], Barrar and Loeb [BL] and Chui [CL85_1, CW]. In a series of beautiful papers, Dahmen and Micchelli [DM85_2, DM86] have studied the combinatorial properties of box splines and established intriguing connections to diophantine equations and multivariate difference equations. De Boor, Riemenschneider and

I [BHR85_{1-3}, BHR86_{1-2}] have generalized several of Schoenberg's theorems on univariate cardinal interpolation [Sch73_1]. Extensions of these results and related questions are subject of current work by Jetter and Riemenschneider [JR] and Chui, Jetter and Ward [CJW]. Finally, subdivision algorithms for rendering of box spline surfaces have been independently developed by Böhm [Bö83], Cohen, Lyche and Riesenfeld [CLR83], Dahmen and Micchelli [DM83_3] and Prautsch [Pr]. The variety of results, obtained in a short period of four years, is remarkable. It reflects the rich mathematical structure of the theory and, sometimes, "competition" among the authors.

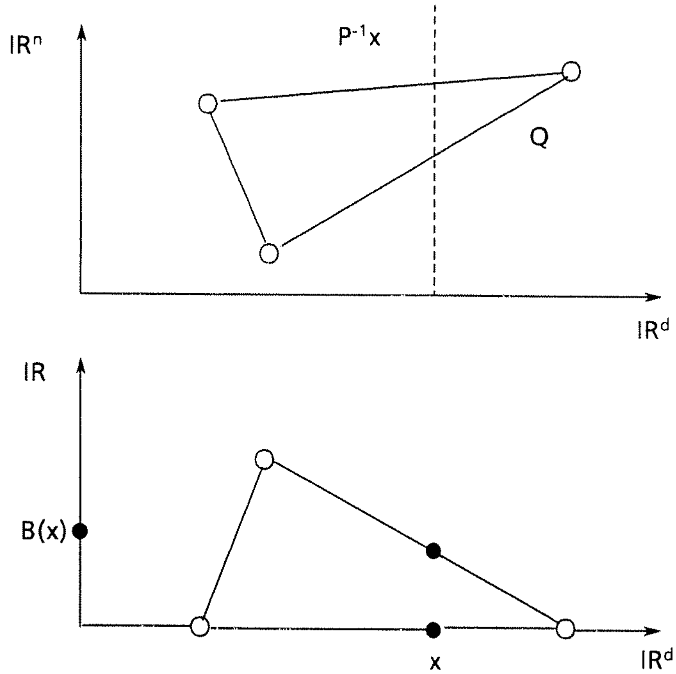
The material discussed in this survey is based on my joint work with Carl de Boor and Sherman Riemenschneider. Moreover, some of the recent work on subdivision of box spline surfaces is discussed. The selection of topics reflects my prejudiced view of the subject and I apologize for having omitted beautiful results which have been proved and which have yet to be proved [??].

Basic Properties

Generalizing the geometric interpretation of univariate B-splines due to Curry and Schoenberg [CS], de Boor [Bo] defined multivariate B-splines as volume densities of simplices. This idea led to the following more general definition [BH82]. The multivariate B-spline $B : \mathbb{R}^d \mapsto \mathbb{R}$, corresponding to a convex polyhedron $Q \subset \mathbb{R}^{d+n}$ with nonzero volume and a linear map $P : \mathbb{R}^{d+n} \mapsto \mathbb{R}^d$, is defined by (cf. Figure 1)

$$B(x) := \text{vol}_n\{Q \cap P^{-1}x\}, \tag{1}$$

i.e. $B(x)$ is the n-dimensional volume of the cross section of Q which is mapped by P onto x.



⟨ Figure 1 ⟩

The box spline is a special case of definition (1) with Q the centered unit cube. Multiplying (1) by a smooth test function φ and integrating over $\mathbb{R}^d$ yields the identity

$$\int_{\mathbb{R}^d} B(x)\varphi(x)dx = \int_Q \varphi(Py))dy.$$

This formula is the basis for the following analytical definition of box splines.

Definition 1 [BH82/83]. Let V be a multiset (i.e. a collection of not necessarily distinct objects) of vectors v in $\mathbb{R}^d$ with integer components. Denote by $\#V$ the number of vectors in V, counting multiplicities, and by $\langle V \rangle$ their linear span. The box spline B_V is the linear functional defined by

$$(B_V, \varphi) := \int_{[-1/2,1/2]^{\#V}} \varphi(\sum_{v \in V} t_v v) dt \quad \text{for} \ \ \varphi \in C_0(\mathbb{R}^d). \tag{2}$$

Unless stated otherwise, it is assumed that $\langle V\rangle = \mathbb{R}^d$. Then B_V can be identified with the function

$$B_V(x) = \mathrm{vol}_n\{t \in [-1/2, 1/2]^{\#V} : \sum_{v\in V} t_v v = x\}, \tag{3}$$

and $(B_V, \varphi) = \int B_V(x)\varphi(x)dx$.

The definition (2) is more convenient for computations than the geometric definition (3) since it does not require that the vectors V span $\mathbb{R}^d$. However, if this condition is not satisfied, then B_V is a linear functional with support contained in a hyperplane and identities involving B_V have to be interpreted in the sense of distributions.

The box spline has the following properties:

(i) B_V is positive,

(ii) supp $B_V = \{\sum_V t_v v : -1/2 \le t_v \le 1/2\}$,

(iii) B_V is a piecewise polynomial of degree $\le n := |V| - d$,

(iv) B_V is ϱ times continuously differentiable, where $\varrho := \min\{\#W : \langle V\backslash W\rangle \ne \mathbb{R}^d\} - 2$.

Properties (i) and (ii) follow directly from Definition 1. The last two assertions are less obvious and will be verified using identity (10) below.

If the vectors V form a basis for $\mathbb{R}^d$, then B_V is the normalized characteristic function of a parallelepiped spanned by the vectors V and centered at the origin, i.e.

$$B_V(x) = \begin{cases} 1/|\det V|, & \text{if } x = \sum t_v v \text{ with } -1/2 \le t_v \le 1/2; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

If V consists of the unit vectors e_ν with multiplicities m_ν, then B_V is the centered tensor product B-spline of degree $m_\nu - 1$ with respect to the ν-th variable and with equally spaced knots. In particular, the univariate B-spline

$$B_n(x) := (n+1)\ [-(n+1)/2, -(n-1)/2, \ldots, (n+1)/2]_t (t-x)_+^n$$

corresponds to $V = \{1, \ldots, 1\}$ with $\#V = n + 1$.

Setting $\varphi(x) = \exp(-iyx)$ in (2), one computes the **Fourier transform** of the box spline,

$$\widehat{B}_V(y) = \prod_{v \in V} \operatorname{sinc}(yv/2), \tag{5}$$

where $\operatorname{sinc}(t) := \sin(t)/t$. This formula stresses the analogy to univariate B-splines with equally spaced knots and is useful for deriving the basic recurrence relations for B_V.

Let D_w denote the derivative in the direction of the vector w, i.e. $D_w = \sum_\nu w_\nu \partial_\nu$ with ∂_ν the derivative with respect to the ν-th variable, and denote by δ_w the corresponding centered difference operator, i.e. $(\delta_w \varphi)(x) := \varphi(x + w/2) - \varphi(x - w/2)$. It follows from (5) that the **derivative** of a box spline is the difference of two box splines of lower degree, $D_v B = \delta_v B_{V \setminus v}$. More generally, if $W \subset V$,

$$D_W B_V = \delta_W B_{V \setminus W}, \tag{6}$$

where $D_W = \prod_{w \in W} D_w$ and δ_W is defined analogously. Setting $W = V$ in (6) yields a generalization of the Hermite Genocchi formula,

$$(B_V, D_V \varphi) = (\delta_V \varphi)(0), \tag{7}$$

i.e. the box spline is the Peano kernel for a product of centered difference operators. The **convolution** of two box splines yields a box spline of higher degree,

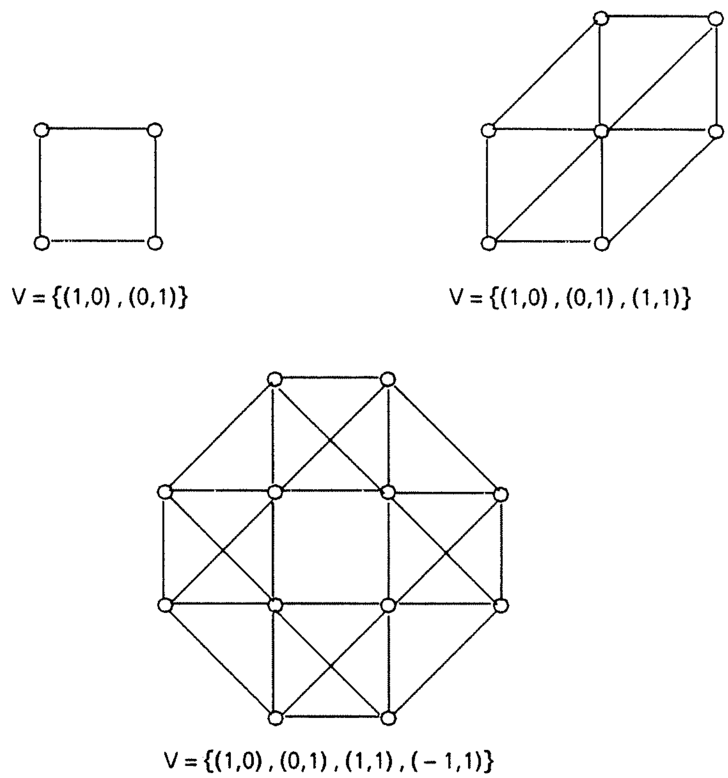
$$B_V * B_W = B_{V \cup W}. \tag{8}$$

A special case is the identity

$$B_{V \cup w}(x) = \int_{-1/2}^{1/2} B_V(x + tw)\,dt, \tag{9}$$

i.e. the box spline $B_{V \cup w}$ is obtained by averaging the box spline B_V in the direction w. This is illustrated in the following

Example 1. The box spline $B_{\{(1,0),(0,1)\}}$ is the characteristic function of the square $[-1/2, 1/2]^2$. Applying (9) with $w = (1,1)$, one obtains the piecewise linear "hat" function on the triangulation generated by the three vectors $(1,0)$, $(0,1)$ and $(1,1)$. Similarly, averaging the linear box spline in the direction of the vector $w = (-1,1)$ yields the piecewise quadratic element first studied by Zwart [Z] and independently derived by Powell [Po] and Sabin [PS]. Figure 2 shows the support of these box splines and the corresponding mesh.



⟨ Figure 2 ⟩

From (4) and (9) it is clear that B is a piecewise polynomial of degree $\leq \#V - d$ and it is also not difficult to derive its exact smoothness. An alternative derivation of properties

(iii) and (iv) is based on the following

Lemma 1. Denote by V_* the set of all subsets W of V with $\langle V \backslash W \rangle \neq \mathbb{R}^d$. Then, for any $y \in \mathbb{R}^d$, there exist constants a_W so that

$$D_y^r = \sum_{W \subset V,\ \#W = r,\ W \notin V_*} a_W D_W \quad + \sum_{W \subset V,\ \#W \leq r,\ W \in V_*} a_W D_y^{r-\#W} D_W. \tag{10}$$

This Lemma is proved by induction on r. For the induction step, (10) is differentiated in the direction y. If $W \subset V$ and $W \notin V_*$, the vectors in $V \backslash W$ span $\mathbb{R}^d$. Therefore,

$$D_y D_W = \sum_{w \in V \backslash W} a_w D_w D_W = \sum a_w D_{w \cup W},$$

which yields (10) with r replaced by $r+1$.

Proof of (iii). If $r = \#V - d + 1 = n + 1$, the first sum on the right hand side of (10) is empty. Therefore, $D_y^r B_V$ is a linear combination of shifts of derivatives of the box splines $B_{V \backslash W}$ with $W \in V_*$. The support of these box splines (as linear functionals) is contained in hyperplanes which implies that all $(n+1)$ order derivatives of B_V vanish on the complement of these hyperplanes.

Proof of (iv). If $r = \varrho + 1$, it follows from the definition of ϱ that the second sum on the right hand side of (10) is empty. In this case, $D_y^r B_V$ is a linear combination of shifts of the box splines $B_{V \backslash W}$ with $W \notin V_*$. Since for such W, $\langle V \backslash W \rangle = \mathbb{R}^d$, these box splines are bounded functions and it follows that all derivatives of B_V of order $\varrho + 1$ are bounded.

Recurrence relation [BH82/83]. If the box splines $B_{V \backslash v}$ are continuous at $x = \sum_V t_v v$, then

$$(\#V - d) B_V(x) = \sum_V \big((1/2 + t_v) B_{V \backslash v}(x + v/2) + (1/2 - t_v) B_{V \backslash v}(x - v/2) \big). \tag{11}$$

A technical difficulty in the implementation of the recurrence relation is that (11) is in general not valid on the set where some of the box splines $B_{V\backslash v}$ are not continuous. The reason for this is that no consistent assignment at the discontinuities can be made which replaces the univariate notion of "limit from the right (or left)". Therefore, one should start the recurrence relation at the level of continuous (usually linear) box splines.

Proof of (11). Denote by ω_ν the ν-th coordinate function, i.e. $\omega_\nu(x) := x_\nu$. By (6),

$$\sum_\nu \omega_\nu \partial_\nu B_V(x) = D_x B_V(x) = \sum_V t_v \big(B_{V\backslash v}(x+v/2) - B_{V\backslash v}(x-v/2)\big).$$

Therefore, the right hand side of (11) equals

$$\sum_\nu \omega_\nu \partial_\nu B_V + \frac{1}{2}\sum_V \big(B_{V\backslash v}(\cdot + v/2) + B_{V\backslash v}(\cdot - v/2)\big)$$

and the Fourier transform of this expression is

$$\sum_\nu (i\partial_\nu)(i\omega_\nu \widehat{B}_V) + \sum_V \cos(\cdot\, v/2)\widehat{B}_{V\backslash v}.$$

By direct computation one checks that this sum equals $(\#V - d)\widehat{B}_V$, the Fourier transform of the left hand side of (11).

Definition 2. Multivariate cardinal splines, denoted by S_V, are by definition linear combinations of translates of box splines, i.e.

$$S_V := \text{span}\ \{B_V(\cdot - j) : j \in \mathbb{Z}^d\}$$

where $\mathbb{Z}$ denotes the integers.

As for univariate splines with equally spaced knots, algorithms for computing with box spline series are particularly simple. For example, a box spline series is differentiated by differencing its coefficients. With δ^- denoting the backward difference, i.e. $\delta_w^- a_j := a_j - a_{j-w}$, one sees from (6) that

$$D_W\Big(\sum_j a_j B_V(\cdot - j)\Big) = \sum_j \delta_W^- a_j \; B_{V\setminus W}(\cdot - j + \xi_W) \tag{12}$$

where $\xi_W := \frac{1}{2}\sum_{w\in W} w$. This identity plays a crucial role in deriving many of the properties of box splines.

Theorem 1 [DM83$_2$, J84$_2$]. The box splines $B_V(\cdot - j)$, $j \in \mathbb{Z}^d$, form a basis of S_V if and only if

$$|\det W| = 1 \quad \text{for any basis} \quad W \subset V. \tag{13}$$

Proof. Assume that $|\det W| > 1$ for some basis $W \subset V$. Then, the vectors W generate a proper sublattice Λ of $\mathbb{Z}^d$,

$$\Lambda = \{\sum_W j_w w : j \in \mathbb{Z}^d\}.$$

By (4) we have

$$\sum_{j\in\Lambda} B_W(\cdot - j - k) = 1/|\det W| \tag{14}$$

for any k in the factor group $\mathbb{Z}^d/\Lambda$. From (8) one sees that (14) remains valid with W replaced by V which shows that the translates of the box splines are linearly dependent.

For the proof of the converse assume that V satisfies the condition (13). If $\#V = d$, the supports of the box splines $B_V(\cdot - j)$ are essentially disjoint and their linear independence is obvious. Therefore, by induction on the number of vectors in V, one may assume that the Theorem holds for all V' with $\#V' < \#V$. Let $W \subset V$ be a basis. Since $|\det W| = 1$, the vectors W can be mapped onto the unit vectors $e_1, \ldots, e_d$ by an integer matrix A with

$|\det A| = 1$ and it is clear that AV also satisfies condition (13). Therefore, by a change of variables, one can assume without loss of generality that V contains the unit vectors. For the induction step one shows that

$$s := \sum_j a_j B_V(\cdot - j) = 0$$

implies that $a = 0$ as follows. By (12),

$$0 = \partial_1 s = \sum_j \delta_{e_1}^- a_j \; B_{V\backslash e_1}(\cdot - j + e_1/2). \tag{15}$$

There are two cases:

(a) If $\langle V\backslash e_1\rangle = \mathbb{R}^d$, the inductive assumption implies that

$$\delta_{e_1}^- a = 0. \tag{16}$$

(b) If $\langle V\backslash e_1\rangle \neq \mathbb{R}^d$, then $\langle V\backslash e_1\rangle = \{0\} \times \mathbb{R}^{d-1}$. In this case, the same conclusion as in (a) holds; but the argument is slightly more complicated. The sum in (15) has to be interpreted as a functional with support in the hyperplanes $\{j_1 - 1/2\} \times \mathbf{R}^{d-1}$ with $j_1 \in \mathbb{Z}$, i.e.

$$0 = \sum_{j_2,\ldots,j_d\in\mathbb{Z}} \delta_{e_1}^- a_j \int_{[-1/2,1/2]^{\#V-1}} \varphi((j_1 - 1/2, j_2, \ldots, j_d) + \sum_{V\backslash e_1} t_v v) dt.$$

Since all vectors in $V\backslash e_1$ are of the form $(0, v')$ with $v' \in \mathbb{R}^{d-1}$, the integral equals

$$\int_{\mathbb{R}^{d-1}} B_{V'}(x)\varphi(j_1 - 1/2, (j_2, \ldots, j_d) + x) dx$$

where $\{0\} \times V' = V\backslash e_1$. Since V' satisfies condition (13), (16) follows also in this case from the inductive assumption.

Clearly, (16) holds for any unit vector which implies that a is constant, and hence equal to zero.

Identity (14) of the above proof implies in particular that the box splines form a **partition of unity**, i.e.

$$\sum_j B_V(\cdot - j) = 1. \tag{17}$$

This identity is the simplest example of explicit box spline representations of polynomials which are studied in more detail in the next section.

Approximation order

Denote by σ_h the scaling operator, i.e. $(\sigma_h f)(x) := f(x/h)$ and denote by $S_V^h := \sigma_h S_V$ the "scaled" cardinal spline space. The approximation order from S_V is defined as

$$\max\{r : \operatorname{dist}(f, S_V^h) = O(h^r) \text{ for all smooth } f\}.$$

Strang and Fix [FS, SF] have developed a general theory for approximation by integer translates of compactly supported functions. In particular they have given several equivalent characterizations of the approximation order. Their theory can be applied to the spaces S_V, and I think this will lead to progress on the difficult problem of computing the approximation order for translates of several box splines. However, for cardinal splines, a direct approach, based on the special properties of box splines, is simpler. The key result is

Theorem 2 [BH82/83]. Denote by π_m the polynomials of total degree $\leq m$ and set $\pi := \bigcup_m \pi_m$. With V_* defined in Lemma 1,

$$\pi \cap S_V = \bigcap_{W \in V_*} \ker D_W. \tag{18}$$

Proof. Let

$$p := \sum_j a_j B_V(\cdot - j) \in \pi \cap S_V.$$

By (12),

$$D_W p = \sum_j \delta_W^- a_j \; B_{V \setminus W}(\cdot - j + \xi_W).$$

For $W \in \Lambda$, the box splines $B_{V \setminus W}$ have support on a set of measure zero which implies that the polynomial $D_W p$ vanishes identically, i.e. lies in the kernel of D_W.

For the proof of the converse, note first that

$$K := \bigcap_{W \in V_*} \ker D_W \subset \pi_{\#V-d}.$$

This follows from Lemma 1. If $r > \#V - d$, the first sum on the right hand side of (10) is empty and therefore, all derivatives D_y^r with $r > \#V - d$ vanish on functions in K.

To complete the proof, one shows by induction on r that

$$\pi_r \cap K \subset S_V.$$

For the induction step, it is sufficient to prove that

$$p \in \pi_r \cap K \;\text{ implies }\; q := p - \sum_j p(j) B_V(\cdot - j) \in \pi_{r-1} \cap K. \tag{19}$$

If $W \in V_*$, it follows from (7) that $\delta_W^- p = (B_W, D_W p(\cdot - \xi_W)) = 0$ and, by (12), $D_W q = 0$. This shows that $q \in K$. By (10),

$$(D_y)^r q = \sum_{W \subset V,\ \#W = r,\ W \notin V_*} a_W \left(D_W p - \sum_j (\delta_W^- p)(j)\ B_{V \setminus W}(\cdot - j + \xi_W)\right).$$

Since p is a polynomial of degree $\leq r$, $D_W p = \delta_W^- p$, and it follows from (17) that $D_y^r q = 0$.

Corollary 1. The maximal degree k for which $\pi_k \subset S_V$ equals $\varrho + 1$ (where ϱ is defined in (iv)).

Proof. By definition of ϱ there exists W with $\#W = \varrho + 2$ and $\langle V \setminus W \rangle \neq \mathbb{R}^d$. Therefore, by the Theorem, there are polynomials of degree $\varrho + 2$ which are not contained in S_V. On the other hand, $\#W > \varrho + 1$ for $W \in V_*$, and it follows that $\pi_{\varrho+1} \subset S_V$.

If $\sum_j p(j) B_V(\cdot - j) = 0$, then $\{p(j) : B_V(x - j) \neq 0\}$ must change sign for any x. In particular, p cannot be a polynomial. It follows that the map

$$p \mapsto Ap := \sum_j p(j) B_V(\cdot - j)$$

is one to one on π_m. Moreover, if $\pi_m \subset S_V$, it follows from (19) that

$$p \mapsto p - Ap \ :\ \pi_m \mapsto \pi_{m-1}. \tag{20}$$

This implies that the range of $A_{|\pi_m}$ is contained in π_m, and therefore A is bijective on π_m. Using these facts, a quasi-interpolant can be defined by

$$f \mapsto Qf := \sum \lambda f(\cdot + j)\ B_V(\cdot - j)$$

where λ denotes a norm preserving extension of the functional $p \mapsto (A^{-1}p)(0)$ which is defined on π_m. Since A is translation invariant, $(A^{-1}p(\cdot + j))(0) = (A^{-1}p)(j)$ for $p \in \pi_m$.

Therefore, $Qp = A(A^{-1}p) = p$, i.e. Q reproduces polynomials of degree $\leq m$. By standard arguments, it follows that

$$\|\sigma_h Q \sigma_{1/h} f - f\|_p = O(h^{m+1})$$

for any smooth function $f \in L_p(\mathrm{I\!R}^d)$. In conjunction with Corollary 1 this proves one direction of

Theorem 3 [BH82/83]. The approximation order of S_V is $\varrho + 2$.

To show that the order $\varrho + 2$ is best possible, let p be a polynomial of degree $\varrho + 2$ with $D_W p \neq 0$ for some W with $\langle V \backslash W \rangle \neq \mathrm{I\!R}^d$. Assume that there exists a sequence of splines $s_h \in S_V^h$ with

$$\|p - s_h\|_{p,\Omega} = o(h^{\varrho+2})$$

where Ω is a bounded set. Let $W = W' \cup w$. By Markov's inequality,

$$\|D_{W'}p - D_{W'}s_h\|_{p,\Omega} = o(h). \tag{21}$$

Since the support of $D_w D_{W'} s_h$ is contained in hyperplanes, $D_{W'} s_h$ is piecewise constant along any line $\{x + tw : t \in \mathrm{I\!R}\}$ while $D_{W'}p$ is linear. Thus (21) cannot hold.

Chui and Diamond [CD] gave an explicit formula for the functional λ in the definition of the quasi-interpolant Q. This representation is based on the observation that the operator A can be inverted explicitly on π_m,

$$A^{-1} = (id - (id - A))^{-1} = \sum_{k=0}^{m} (id - A)^k, \tag{22}$$

where $id\, p := p$. This identity holds since by (20) the operator $(id - A)$ is degree reducing which implies that $(id - A)^{m+1} = 0$ on π_m. Therefore, the functional λ can be defined by

$$\lambda f := \sum_{k=0}^{m} ((id - A)^k f)(0). \tag{23}$$

From the definition of A one sees that the right hand side of (23) is a finite linear combination of the function values $\{f(k-j) : k \in \mathbb{Z}^d\}$ with the corresponding weights given in terms of B_V.

Convergence to Functions of Exponential Type

Schoenberg [Sch73$_2$] showed that a sequence of univariate cardinal splines s_n of degree $\leq n$ converges to a function $f \in L_2(\mathbb{R})$ as the degree n tends to infinity if and only if f is an entire function of exponential type π. Moreover, cardinal interpolation is an optimal approximation process. The analysis of the convergence of a sequence of multivariate cardinal splines $s_n \in S_{V_n}$, $|V_n| \to \infty$, is more difficult. The results depend on the particular sequence V_n. A natural choice is to let the multiplicities of the vectors in a fixed multiset V tend to infinity. For this case, the analogue of Schoenberg's result is

Theorem 4 [BHR86$_2$]. Let nV denote the multiset consisting of n copies of V. Define the set

$$\Omega_V := \{x : |\widehat{B}_V(x + 2\pi j)| < |\widehat{B}_V(x)|,\ j \neq 0\}.$$

Then, for $f \in L_2(\mathbb{R}^d)$, there exists a sequence of cardinal splines

$$s_n = \sum_j a_{n,j} B_{nV}(\cdot - j)$$

with coefficients $a_n \in \ell_2(\mathbb{Z}^d)$ which converges to f in $L_2(\mathbb{R}^d)$ if and only if the support of the Fourier transform of f is contained in $\bar{\Omega}$.

This Theorem is valid, more generally, for B_V replaced by any compactly supported function. This becomes clear from the proof given below which essentially only uses the decay of the Fourier transform of B_V, i.e. that $|\widehat{B}_V(x)| = O((1+|x|)^{-1})$.

Lemma 2. The set Ω is a fundamental domain, i.e.

$$\bar{\Omega} \cap (\Omega + 2\pi j) = \emptyset, \ j \neq 0,$$

$$\text{measure}\left(\mathbb{R}^d \setminus \cup_j (\Omega + 2\pi j)\right) = 0.$$

Proof. To prove the first assertion, let $x = \lim_{\nu \to \infty} x_\nu$ with $x_\nu \in \Omega$. Then, the assumption $x - 2\pi j \in \Omega$ with $j \neq 0$ leads to the contradiction

$$1 > |\widehat{B}_V((x-2\pi j)+2\pi j)/\widehat{B}_V(x-2\pi j)| = \lim_{\nu \to \infty} |\widehat{B}_V(x_\nu)/\widehat{B}_V(x_\nu - 2\pi j)| \geq 1.$$

For the second assertion, consider the set

$$\{f_j(x) := |\hat{B}_V(x+2\pi j)|^2 \ : \ j \in \mathbb{Z}^d\}.$$

If this set has a unique maximum, say for $j = j_*$, then $x \in \Omega + 2\pi j_*$. Thus the complement of $\cup_j(\Omega + 2\pi j)$ is contained in

$$\{x : f_j(x) = f_k(x), \text{ some } j \neq k\}.$$

Since the zero set of a nontrivial real analytic function is of measure zero, it remains to show that $f_j - f_k$ cannot vanish identically. But, $f_j = f_k$ implies that f is periodic in the direction $j - k$ which contradicts the decay of $\widehat{B}_V$.

Proof of Theorem 4. For

$$x \in D := \{x \in \mathrm{I\!R}^d \ : \ \widehat{B}_V(x) \neq 0\} \supset \Omega,$$

we define

$$a_j(x) := \widehat{B}_V(x + 2\pi j)/\widehat{B}_V(x)$$

and introduce the trigonometric polynomial

$$P_n(x) := \sum_j B_{nV}(j) \exp{(ijx)} = \sum_j \widehat{B}_{nV}(x + 2\pi j) = \widehat{B}_{nV}(x) \sum_j (a_j(x))^n, \tag{24}$$

where the last equality follows from (8) and holds, at least, on D. For any $j \neq 0$ and $x \in \Omega$,

$$|a_j(x)| \leq 1 - \epsilon(j, x) \tag{25.1}$$

for some positive $\epsilon(j,x)$, while, because of the decay of the Fourier transform of B_V, there exists a positive constant C such that for all but finitely many j,

$$|a_j(x)| \leq 1/(1 + C|j|). \tag{25.2}$$

Consequently, for $x \in \Omega$,

$$P_n(x)/\widehat{B}_{nV}(x) = \sum_j (a_j(x))^n \to 1, \ n \to \infty, \tag{26}$$

and the convergence is uniform on compact subsets Ω_1 of Ω. This shows in particular that, for large enough n, P_n does not vanish on such Ω_1.

(a) Assume that $f \in L_2$ and that $\hat{f}$ vanishes a.e. outside $\bar{\Omega}$. Denote by χ the characteristic function of a compact subset Ω_1 of Ω. Since Ω is a fundamental domain, we can expand $\hat{f}\chi/P_n$ in a Fourier series,

$$(\hat{f}\chi/P_n)(x) =: \sum_j c_{n,j} \exp{(ijx)}, \ x \in \Omega, \tag{27}$$

with coefficients $c_n \in \ell_2(\mathbb{Z}^d)$. This implies that

$$s_n := \sum_j c_{n,j} B_{nV}(\cdot - j) \ \in \ L_2. \tag{28}$$

Since $\hat{f}$ vanishes a.e. outside $\bar{\Omega}$ and, by Lemma 2, $\bar{\Omega}\backslash\Omega$ has measure zero,

$$\|\hat{f} - \hat{s}_n\|_2^2 = \|\hat{f} - \hat{s}_n\|_{2,\Omega}^2 + \sum_{j\neq 0} \|\hat{s}_n(\cdot + 2\pi j)\|_{2,\Omega}^2.$$

The first term is estimated by

$$\|\hat{f} - \hat{s}_n\|_{2,\Omega} \leq \|\hat{f} - \chi\hat{f}\|_{2,\Omega} + \|\chi\hat{f} - \chi\hat{f}\widehat{B}_{nV}/P_n\|_{2,\Omega},$$

where the first norm on the right hand side is small if Ω_1 is chosen close to Ω, while, for fixed Ω_1, the second norm is small by (26) if n is sufficiently large. The j-th term in the sum is the square of

$$\|\widehat{B}_{nV}(\cdot + 2\pi j)(\hat{f}\chi/P_n)\|_{2,\Omega} = \|(a_j)^n \widehat{B}_{nV}(\hat{f}\chi/P_n)\|_{2,\Omega}$$

$$\leq (\|a_j\|_{\infty,\Omega_1})^n \|\widehat{B}_{nV}/P_n\|_{\infty,\Omega_1} \|\hat{f}\|_{2,\Omega_1}.$$

By (25) this implies that the sum is small for large n.

(b) Assume that $s_n = \sum_j c_{n,j} B_{nV}(\cdot - j)$ converges to f in L_2. Since

$$\hat{s}_n(x + 2\pi j) = (a_j(x))^n \hat{s}_n(x), \ x \in D,$$

we see from (25) that for $j \neq 0$

$$\|\hat{s}_n\|_{2,\Omega_1 + 2\pi j} \leq (\|a_j\|_{\infty,\Omega_1})^n \ \|\hat{s}_n\|_2 \to 0$$

for any compact subset Ω_1 of Ω. Since $\mathbb{R}^d \backslash \cup_j (\Omega + 2\pi j)$ has measure zero, it follows that, as an element of L_2, $\hat{f}$ vanishes outside $\bar{\Omega}$.

In this generality, little more can be said about the structure of the set Ω_V and the construction of optimal approximations s_n. However, if the polynomial P_n is strictly positive, then cardinal interpolation is an optimal approximation process.

Corollary 2 [BHR85$_{1-2}$]. If $P_n = \sum_j B_{nV}(j)\exp(ij\cdot)$ is strictly positive, then for any bounded function f, there exists a unique bounded spline $I_n f \in S_{nV}$ with

$$I_n f(k) = f(k),\ k \in \mathbb{Z}^d.$$

If $f \in L_2(\mathbb{R}^d)$ and $\operatorname{supp}\hat{f} \subset \Omega_V$, then

$$\|f - I_n f\|_2 \mapsto 0,\ n \to \infty.$$

Proof. The existence and uniqueness of a cardinal interpolant follows from standard theory for Töplitz matrices. The convergence of $s_n := I_n f$ to f can be established as in the proof of Theorem 4. Since Ω_V is a fundamental domain and $\operatorname{supp}\hat{f} \subset \Omega_V$,

$$\hat{f}(x) = \sum_j f(j)\exp(ijx),\ x \in \Omega_V, \tag{29}$$

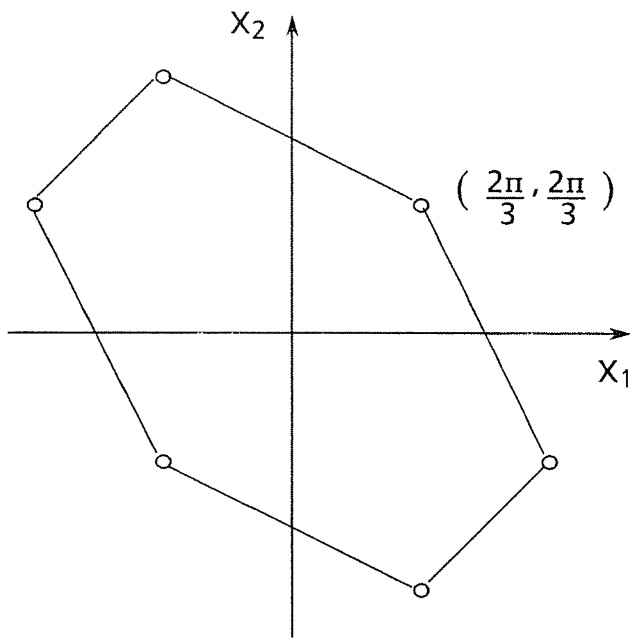
i.e. the Fourier coefficients of $\hat{f}$ are the function values of f at $\mathbb{Z}^d$. The right hand side of (29) defines a periodic extension $\hat{f}_p$ of $\hat{f}$. Using that $P_n(x) = \sum_j \widehat{B}_{nV}(x + 2\pi j)$, one veryfies that

$$\widehat{I_n f} = \hat{f}_p \widehat{B}_{nV} / P_n.$$

This shows that s_n is given by (27, 28) with χ the characteristic function of the set Ω_V and the arguments in the proof of Theorem 4 apply.

Clearly, the linear independence of the translates of the box splines is a necessary condition for the well posedness of cardinal interpolation. In the bivariate case, this condition is also sufficient [BHR85$_1$]. However, this is in general not true as the counterexample in [BHR85$_3$] shows. In the univariate case, the well posedness of cardinal interpolation is a consequence of the total positivity of the matrix $\{B_n(j-k)\}$ (B_n is the univariate cardinal B-spline) which implies that the minimum of the characteristic polynomial is attained at π. For the multivariate problem, a comparable result could only be obtained in a very special case.

Theorem 5 [BHR86$_1$]. For $V = \{(1,0),(1,1),(0,1)\}$, the polynomial P_n attains its minimum at $(2\pi/3, 2\pi/3)$ mod $2\pi\mathbb{Z}^2$ for all n. The fundamental domain Ω_V is the convex hull of the points $\pm(2\pi/3, 2\pi/3)$, $\pm(4\pi/3, -2\pi/3)$ and $\pm(2\pi/3, -4\pi/3)$ (cf. Figure 3).



⟨ Figure 3 ⟩

Since the proof of the above result requires rather technical estimates it is not given here.

Subdivision Algorithms

Standard methods for graphic display of spline curves and surfaces are based on subdivision techniques. The basic example is Chaikin's algorithm [Ch] for approximating a quadratic spline curve

$$t \mapsto s(t) = \sum_j a_j B_2(t-j)$$

with coefficients $a_j \in \mathbb{R}^2$ by a sequence of polygons. From the "control polygon" p which, by definition, connects the vectors a_j, a refined polygon p' with vertices a'_j is generated by the rule

(I) $b_{2j+\nu} := (a_j + a_{j+\nu})/2,\ \nu = 0,1;$

(II) $a'_j := (b_j + b_{j+1})/2.$

Repeating this process, the polygons p, p', p'', ..., converge quadratically to the spline curve s. Typically, a few steps of the algorithm are sufficient to approximate the spline curve within the resolution of standard displays. Similar algorithms exist for rendering of splines of arbitrary degree as well as for other purposes (cf. [CLR80]). Subdivision algorithms for box splines are very similar to the above example which, after all, is a special case of the multivariate theory.

Definition 3. A box spline surface $\{s\}$ is a two dimensional surface in $\mathbb{R}^3$ which can be parameterized in the form

$$x \mapsto s(x) = \sum_j a_j B_V(x-j),\ x \in \mathbb{R}^2, \tag{30}$$

with coefficients $a_j \in \mathbb{R}^3$. It is assumed that the parameterization is regular, i.e. that $|\partial_1 s(x) \times \partial_2 s(x)| \neq 0$ for all $x \in \mathbb{R}^2$.

Two parameterizations s and $\tilde{s}$ are equivalent (i.e. represent the same surface) if there exists a smooth, 1-1 map $\psi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with $s \circ \psi = \tilde{s}$. For example, $\psi(x) := \lambda x + \xi$ yields an equivalent parameterization if $\lambda \neq 0$.

One can assume without (too much) loss of generality that V contains the vectors $U := \{(1,0),(1,1),(0,1)\}$ (up to symmetry, this merely excludes tensor product B-splines). Then, the piecewise linear surface $\{p\}$ which is parameterized by

$$x \mapsto p(x) = \sum_j a_j B_U(x-j), \ x \in \mathbb{R}^2, \tag{31}$$

is called the "control polygon" for the surface $\{s\}$. Since

$$B_U(j) = \begin{cases} 1, & \text{if } j = 0; \\ 0, & \text{if } j \in \mathbb{Z}^2 \backslash 0, \end{cases} \tag{32}$$

the function $p : \mathbb{R}^2 \mapsto \mathbb{R}^3$ interpolates the vectors a_j at the lattice points $j \in \mathbb{Z}^2$. Therefore, the surface $\{p\}$ is the union of the triangles with vertices a_j, $a_{j+(1,0)}$, $a_{j+(1,1)}$ and the triangles with vertices a_j, $a_{j+(0,1)}$, $a_{j+(1,1)}$.

Algorithm (cf. [Bö83, CLR83, DM83$_3$, Pr]). For a piecewise linear surface $\{p\}$, a refined piecewise linear surface $\{p'\}$, which is parameterized by

$$x \mapsto p'(x) = \sum_j a'_j B_U(x-j), \ x \in \mathbb{R}^2,$$

is defined in two steps:

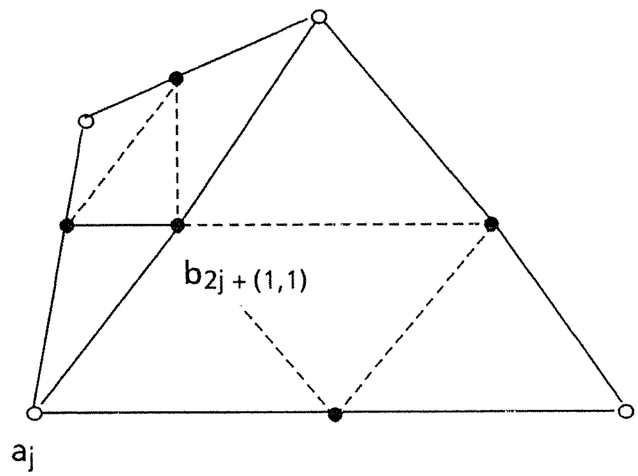
(I) $b_{U,2j+(\nu,\mu)} := (a_j + a_{j+(\nu,\mu)})/2, \quad (\nu,\mu) = (0,0),(1,0),(0,1),(1,1);$

(II) $W := U;$

for $v \in V \backslash U$ do: $\quad b_{W \cup v, j} := (b_{W,j} + b_{W,j+v})/2;$

$a'_j := b_{V,j}$.

Geometrically, the vectors $b_{U,2j+(\nu,\mu)}$, $(\nu,\mu) \neq (0,0)$, defined in step (I) of the Algorithm, are the midpoints of the edges of the triangles which make up the piecewise linear surface $\{p\}$ (cf. Figure 4). With $s_W := \sum_j b_{W,j} B_W(\cdot - j)$, it is therefore clear that $\{p\} = \{s_U\}$.



⟨ Figure 4 ⟩

Theorem 6 (cf. [Bö83, CLR83, DM83$_3$, Pr]). Assume that s and p are given by (30) and (31) respectively. Then, the sequence of piecewise linear surfaces $\{p\}$, $\{p'\}$, $\{p''\}$, ..., generated by the Algorithm converges to the box spline surface $\{s\}$.

The **proof** is divided into three steps.

(a) The parameterizations $x \mapsto s(x) = \sum_j a_j B_V(x-j)$ and $x \mapsto s'(x) := \sum_j a'_j B_V(x-j)$ are equivalent.

This is clearly true for $V = U$ since then $\{s\} = \{p\}$ and step (I) of the algorithm leaves the piecewise linear surface $\{p\}$ unchanged. In this case, it follows from (32) and the definition of $b_{U,j}$ that

$$\sum a_j B_W(\cdot - j) = \sum_j b_{W,j} B_W(2\cdot - j - \xi_{W\setminus U}) = s_W(2\cdot - \xi_{W\setminus U}) \tag{33}$$

holds for $W = U$. To show that (33) remains valid with W replaced by $W \cup w$, note that by (9),

$$\begin{aligned}\int_{-1/2}^{1/2} B_W(2(x+tw) - j - \eta)dt &= \int_{-1}^{1} B_W(2x + tw - j - \eta)dt/2 \\ &= \Big(B_{W\cup w}(2x - j - \eta') + B_{W\cup w}(2x - j + w - \eta')\Big)/2\end{aligned}$$

where $\eta := \xi_{W\setminus U}$ and $\eta' := \eta + w/2 = \xi_{W\cup w\setminus U}$. Using this identity in (33), one obtains

$$\begin{aligned}\sum_j a_j B_{W\cup w}(\cdot - j) \;&=\; \int_{-1/2}^{1/2} \Big(\sum_j a_j B_W(\cdot - j + tw)\Big)dt \\ &= \sum_j b_{W,j}\left(B_{W\cup w}(2\cdot - j - \eta') + B_{W\cup w}(2\cdot - j + w - \eta')\right)/2 \\ &= \sum_j \left(b_{W,j} + b_{W,j+w}\right)/2 \; B_{W\cup w}(2\cdot - j - \eta') \\ &= \sum_j b_{W\cup w,j} B_{W\cup w}(2\cdot - j - \eta')\end{aligned}$$

which proves (33) for any $W \subset V$. Since by step (II) of the Algorithm $\{s_V\} = \{s'\}$, the identity (33) implies that $\{s\} = \{s'\}$.

(b) The Algorithm reduces the distance of neighboring coefficients by a factor 2, i.e.

$$\sup_{j\in\mathbb{Z}^2, w\in U} |a'_j - a'_{j+w}| \le \frac{1}{2} \sup_{j\in\mathbb{Z}^2, w\in U} |a_j - a_{j+w}|. \tag{34}$$

From the geometric interpretation of step (I) of the Algorithm it is clear that (34) is valid for linear box splines, i.e. for $V = U$. It is then easily seen that the averaging process in step (II) of the Algorithm does not increase the right hand side of (34).

(c) Denote by $\{p^{(n)}\}$ with $p^{(n)} = \sum_j a_j^{(n)} B_U(\cdot - j)$ the piecewise linear surface generated by n applications of the Algorithm. By (a), $\{s\}$ can be parameterized by

$$x \mapsto s^{(n)}(x) := \sum_j a_j^{(n)} B_V(x - j)$$

for all n. Using (17),

$$\begin{aligned} s^{(n)}(x) - p^{(n)}(x) &= \sum_j a_j^{(n)} \big(B_V(x-j) - B_U(x-j)\big) \\ &= \sum_j \big(a_j^{(n)} - a_k^{(n)}\big)\big(B_V(x-j) - B_U(x-j)\big). \end{aligned} \tag{35}$$

From (b) one sees that

$$|a_j^{(n)} - a_k^{(n)}| \;\le\; c\, 2^{-n} \sup_{\ell \in \mathbb{Z}^2, w \in U} |a_\ell - a_{\ell+w}|, \quad \text{if} \quad \|j - k\|_\infty \le c.$$

Since the summands on the right hand side of (35) are nonzero only if j is in the support of $B_V(x - \cdot)$, it follows that

$$|s^{(n)}(x) - p^{(n)}(x)| = O(2^{-n})$$

if k is chosen as the nearest lattice point to x.

A more subtle argument [D] shows that, under appropriate assumptions, the convergence of the sequence $\{p\}$, $\{p'\}$, ..., is quadratic. Hence, as for tensor products, the subdivision algorithm yields a fast method for the approximate evaluation of box spline surfaces.

Example 3. Let $V = U \cup U$ which corresponds to a C^2 quartic box spline. Then step (II) of the Algorithm can be rewritten as

$$a'_j = 1/8 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} b_{U,j}$$

i.e. a' is a convex combination of neighboring coefficients b_j.

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