LOW DIMENSIONAL A_∞ ALGEBRAS

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1. WHAT IS THE TENSOR COALGEBRA?

A vector space V over a field $\mathbb{K}$ is said to be $\mathbb{Z}_2$-*graded* if it is equipped with a fixed decomposition $V = V_0 \oplus V_1$. The tensor algebra $T(V)$ is the $\mathbb{Z}_2$-graded algebra $T(V) = \bigoplus_{k=0}^{\infty} V^k$, where V^k is the k-th tensor product of V with itself.
The coalgebra structure on $T(V)$ is the map $\Delta : T(V) \to T(V) \otimes T(V)$ given by

$$\Delta(v_1 \cdots v_n) = \sum_{k=0}^{n} v_1 \cdots v_k \otimes v_{k+1} \cdots v_n.$$

An automorphism of the tensor coalgebra is a bijective map $g : T(V) \to T(V)$ such that

$$(g \otimes g) \circ \Delta = \Delta \circ g.$$

A coderivation of the tensor algebra is a map $\delta : T(V) \to T(V)$ such that

$$(\delta \otimes 1 + 1 \otimes \delta) \circ \Delta = \Delta \circ \delta.$$

The $\mathbb{Z}_2$-graded Lie bracket of two coderivations δ and μ is given by

$$[\delta, \mu] = \delta \circ \mu + (-1)^{\delta\mu+1} \mu \circ \delta$$

2. WHAT IS AN A_∞ ALGEBRA?

If $\alpha \in \operatorname{Hom}(V^k, V)$, then it determines a coderivation of $T(V)$ by

$$\alpha(v_1 \cdots v_n) = \sum_{i=0}^{n-k} (-1)^{(v_1+\cdots+v_k)\alpha} v_1 \cdots v_i \cdot \alpha(v_{i+1} \cdots v_{i+k}) \cdot v_{i+k+1} \cdots v_n.$$

Using the formula above we are able to identify the Lie algebra $\operatorname{Coder}(T(V))$ of coderivations with

$$C(V) = \operatorname{Hom}(T(V), V) = \bigoplus_{k=0}^{\infty} \operatorname{Hom}(V^k, V) = \bigoplus_{k=0}^{\infty} C^k(V).$$

As a consequence, any coderivation δ has a decomposition as a power series $\delta = \sum_{k=0}^{\infty} \delta_k$. An odd coderivation $d = d_1 + d_2 + \cdots$ such that $[d, d] = 0$ is called an A_∞ algebra structure on V.

Define the coboundary operator on an A_∞ structure d by $D(\varphi) = [d, \varphi]$ for any $\varphi \in C(V)$. Then $D^2 = 0$, so we can define the cohomology $H(d)$ by

$$H(D) = \ker(D) / \operatorname{Im}(D).$$

When d consists of a single term, we can refine this cohomology to define $H^n(d)$ for all natural numbers n.

3. COALGEBRA AUTOMORPHISMS

A coalgebra automorphism g acts on coderivation α by $g^*(\alpha) = g^{-1} \circ \alpha \circ g$. To check that $g^*(\alpha)$ is a coderivation, we compute

$$\Delta(g^*(\alpha)) = \Delta(g^{-1}\alpha g) = (g^{-1} \otimes g^{-1})(\alpha \otimes 1 + 1 \otimes \alpha)(g \otimes g)\Delta = (g^*(\alpha) \otimes 1 + 1 \otimes g^*(\alpha))\Delta.$$

In fact, g^* is a Lie algebra automorphism, because

$$g^*([\alpha, \mu]) = g^{-1}(\alpha\mu + (-1)^{\alpha\mu+1}\mu\alpha)g = g^{-1}\alpha g g^{-1}\mu g + (-1)^{\alpha\mu+1} g^{-1}\mu g g^{-1}\alpha g = [g^*(\alpha), g^*(\mu)].$$

If α is an even coderivation, then $\exp(\alpha)$, if well-defined, is a coalgebra automorphism, since

$$\begin{aligned}\Delta \exp(\alpha) &= \sum_{n=0}^{\infty} \frac{1}{n!} \Delta \alpha^n = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha \otimes 1 + 1 \otimes \alpha)^n \Delta. \\ &= \sum_{k+l=n} \frac{1}{k!l!} (\alpha^k \otimes \alpha^l)\Delta = (\exp(\alpha) \otimes \exp(\alpha))\Delta.\end{aligned}$$

If $g = \exp(\alpha)$ then $g^* = \exp(-\operatorname{ad}_\alpha)$, where $\operatorname{ad}_\alpha(\varphi) = [\alpha, \varphi]$. This means that

$$g^*(\delta) = \delta + [\delta, \alpha] + \tfrac{1}{2}[[\delta, \alpha], \alpha] + \cdots .$$

4. THE MODULI SPACE OF A_∞ ALGEBRAS

An invertible even linear map $\lambda : V \to V$ extends in a natural way to a coalgebra automorphism of $T(V)$. Moreover, if $\alpha_k \in C^k(V)$ for $k > 1$, then $\exp(\alpha)$ is always defined. An arbitrary coalgebra morphism g can be written in the form $g = \lambda \exp(\alpha_2) \exp(\alpha_3) \cdots$, where $\lambda \in \mathbb{GL}(V)$ and $\alpha_k \in C^k(V)$. Then

$$g^* = \Big(\prod_{k=\infty}^{2} \exp(-\operatorname{ad}_{\alpha_k}) \Big) \lambda^*.$$

The important fact about the above formula is that it is computable!

We say that d and d' are *equivalent* A_∞ algebra structures if there is a coalgebra automorphism g of the tensor coalgebra such that $g^*(d') = d$, and write $d' \sim d$.

Theorem 1 *Suppose that*

$$d = d_k + d_{k+1} + \cdots$$

$$d' = d'_l + d'_{l+1} + \cdots .$$

Then $k = l$ and there is a linear automorphism λ of $T(V)$ such that $d' = \lambda^*(d)$.

Because of this theorem, we know the first step in classifying the A_∞ algebras is to classify all nonequivalent A_∞ algebras consisting of a single term $d_k \in C^k$.

5. EXTENDING A CODIFFERENTIAL

Let $d = d_k + d_{k+1} + \cdots$ be a codifferential. Then

$$0 = [d, d] = \sum_{n=k}^{\infty} [d_k, d_n] + [d_{k+1}, d_{n-1}] + \cdots [d_n, d_k].$$

Define $D(\varphi) = [d_k, \varphi]$. Then the above equation yields an infinite sequence of equations

$$D(d_n) = -\tfrac{1}{2} \sum_{l=k+1}^{n-1} [d_l, dn + k - l].$$

This means we can solve for d_n in terms of the operator D and $d_{k+1}, \ldots, d_{n-1}$. In particular, if r is the first coefficient larger than k such that $d_r \neq 0$, then $D(d_r) = 0$. Suppose that $d_r = D(\alpha)$. Then if $g = \exp(-\alpha)$, we have

$$g^*(d) = d_k + d_r - [d_k, \alpha] + \mathrm{ho},$$

where ho stands for terms of higher order than r. Since $d_r = [d, \alpha]$, the terms of order r cancel, and we are left with an equivalent codifferential with the second term of higher order than r.

Theorem 2 *Suppose $H^n(d_k) = 0$ whenever $n > k$. Then any codifferential d with leading term d_k is equivalent to d_k.*

6. APPLYING THE THEORY

if $V = \langle e_1, \ldots, e_n \rangle$, then a k-multi-index I is an ordered k-tuple of integers between 1 and n. If e_I denotes the element $e_{i_1} \otimes \cdots e_{i_n} \in V^k$, then define $\varphi^I_i : V^k \to V$ by $\varphi^I_i(e_J) = \delta^I_J e_i$. Then the φ^I_i form a basis for the coderivations of $T(V)$.

Example 1 *As an example, suppose that V is a 1-dimensional odd space. Then for each k, there is only 1 multi-index $I_k = (1, \ldots, 1)$. Denote $\psi_k = \varphi_1^{I_{2k}}$ and $\phi_k = \varphi_1^{I_{2k+1}}$. Then ψ_k is odd and ϕ_k is even. We have the following formulas for the bracket.*

$$\begin{aligned}[] [\psi_k, \psi_l] &= 0 \\ [\psi_k, \phi_l] &= (2k-1)\psi_{k+l} \\ [\phi_k, \phi_l] &= 2(k-l)\phi_{k+l}.\end{aligned}$$

As a consequence, if $d = a_1\psi_1 + a_2\psi_2 + \cdots$, then d is a codifferential. Moreover, if $d_k = a_k\psi_k$ is the leading term in d, then the above formulas show that $H^n(d_k) = 0$ for $n > 2k$, so by Theorem 2, $d \sim d_k$. This means that the moduli space of A_∞ algebras on a 1-dimensional space is a 1 parameter family $d_k = \psi_k$, indexed by the natural numbers. When we study the deformations on the moduli space, we obtain the following picture, which completely characterized the deformations of the moduli space.

$$\cdots \rightsquigarrow d_3 \rightsquigarrow d_2 \rightsquigarrow d_1.$$

References

[1] D. Bodin, C. DeCleene, W. Hager, C. Otto, M. Penkava, M. Phillipson, R. Steinbach, and E. Weber. The moduli space of 1|1-dimensional complex associative algebras. preprint, 2009.

[2] J. Cox and M. Penkava. The classification of 1-dimensional A_∞ algebras and their applications. preprint, 2009.

[3] A. Fialowski and M. Penkava. Deformation theory of infinity algebras. *Journal of Algebra*, 255(1):59–88, 2002. math.RT/0101097.

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