Chord Transformations and Beethoven:

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Introduction:

In western standard tuning, frequency of notes are categorized into 12 pitch classes that are ordered and cyclical. These pitch classes are depicted in the *musical clock*. Line segments are notes played together and polygons are chords. There are a number of transformations to mathematically describe transitions between notes, phrases, and chords. These include transposition, inversion, and the transformations P, L and R.

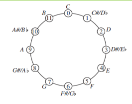
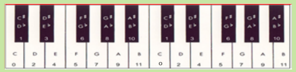
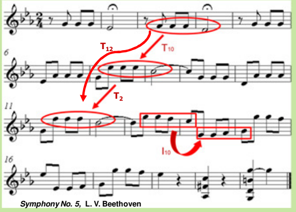


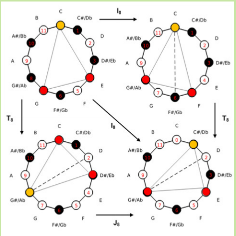
Figure 1: The musical clock.



Transposition & Inversion

- ***Transposition*:**
  1. Translation of note(s) upward or downward in pitch
  2. Defined by T_n: $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that
     $T_n(x) = x + n \pmod{12}$
  3. Measures 3-4 transposed to 7-8
     $T_{10}(\{5,5,5,2\}) = \{3,3,3,0\}$
  4. $(T_1(x))^n = T_n(x)$





- ***Inversion*:**
  1. Reflection of a set of notes about an axis
  2. Defined by I_n: $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that
     $I_n(x) = -x + n \pmod{12}$
  3. Measures 13-14 inverted to 14-15
     $I_{10}(\{7,7,5,3\}) = \{3,3,5,7\}$
  4. $I_n(x)^{2m} = I_0(x)$

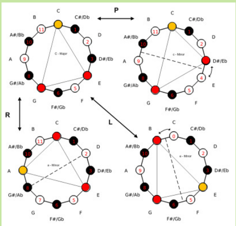
- ***Composition*:**

  $T_m \circ T_n = T_{m+n} \bmod 12$

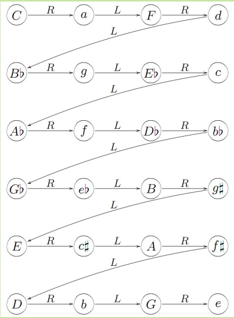
  $T_m \circ I_n = I_{m+n} \bmod 12$

  $I_m \circ T_n = I_{m-n} \bmod 12$

  $I_m \circ I_n = T_{m-n} \bmod 12$



Function	Axis of Inversion Spanned by
P	$\frac{y_1+y_3}{2}, \frac{y_1+y_3}{2}+6$
L	$\frac{y_2+y_3}{2}, \frac{y_2+y_3}{2}+6$
R	$\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}+6$



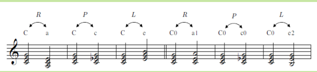
PLR Group:

- ***P Transformation*:**
  1. Maps major to **pure** minor (same leading tone)
  2. Lowers harmonic 3rd (the 2nd note in triad) by one semitone
  3. Defined by P: $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that
     $P\langle y_1, y_2, y_3\rangle = I_{y_1+y_3}\langle y_1, y_2, y_3\rangle$
  4. Map C-major to c-minor
     $P\langle 0, 4, 7\rangle = I_7\langle 0, 4, 7\rangle = \langle 7, 3, 0\rangle = \langle 0, 3, 7\rangle$

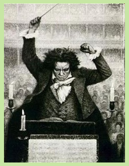
- ***L Transformation*:**
  1. **Leading tone exchange** lowers leading tone by one semitone and rearranges order
  2. Defined by L: $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that
     $L\langle y_1, y_2, y_3\rangle = I_{y_2+y_3}\langle y_1, y_2, y_3\rangle$
  3. Map C-major to e-minor
     $L\langle 0, 4, 7\rangle = I_{11}\langle 0, 4, 7\rangle = \langle 11, 7, 4\rangle = \langle 4, 7, 11\rangle$

- ***R Transformation*:**
  1. Maps major to **relative** minor (same sharps/flats but different leading tone)
  2. Defined by R: $\mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that
     $R\langle y_1, y_2, y_3\rangle = I_{y_1+y_2}\langle y_1, y_2, y_3\rangle$
  3. Map C-major to a-minor
     $R\langle 0, 4, 7\rangle = I_4\langle 0, 4, 7\rangle = \langle 4, 0, 9\rangle = \langle 9, 0, 4\rangle$

- ***Properties of the PLR group*:**
  1. The set of all major and minor chords form a nonabelian group of order 24 under compositions of the operations P, L, and R, and is isomorphic to the dihedral group of order 24.
  2. The entire group of 24 major and minor chords can be formed by starting at any chord and alternately applying L and R (see chart to left).
  3. **$P = R(LR)^3$**
  4. Thus, L and R generate the PLR group and their compositions define the entire group.



Beethoven's 9th Symphony:



Examples of these transformations are prevalent throughout the works of L. V. Beethoven and especially noteworthy in his *Symphony No. 9* measures 143-172. We will examine these transformations in the context of Beethoven's 9th Symphony.

- ***Procedure*:**

  We reduced notes into pitch classes (accounting for necessary transpositions due to instrument tunings) to find the overall chord structure of each measure.

  Each measure in this excerpt has at most 3 pitch classes, making chordal reduction much clearer.

  Chord reductions:
  - measure 143: $\langle 0, 4, 7\rangle$ — C-major
  - measure 144: $\langle 0, 9, 4\rangle = \langle 9, 0, 4\rangle$ — a-minor
  - measure 145: $\langle 0, 9, 5\rangle = \langle 5, 9, 0\rangle$ — F-major
  - measure 146: $\langle 2, 9, 5\rangle = \langle 2, 5, 9\rangle$ — d-minor

  Mathematical transformations:
  - measure 143 to 144:
    $R\langle 0, 4, 7\rangle = I_4\langle 0, 4, 7\rangle = \langle 4, 0, 9\rangle = \langle 9, 0, 4\rangle$
  - measure 144 to 145:
    $L\langle 4, 0, 9\rangle = I_9\langle 4, 0, 9\rangle = \langle 5, 9, 0\rangle$
  - measure 145 to 146:
    $R\langle 5, 9, 0\rangle = I_2\langle 5, 9, 0\rangle = \langle 9, 5, 2\rangle = \langle 2, 5, 9\rangle$

Measure	Notes Played	Chord Name	Transformation to Next Chord
143	C, G, E	C-major	R
144	C, A, E	a-minor	L
145	C, A, F	F-major	R
146	D, A, F	d-minor	T_0
147	D, A, F	d-minor	T_0
148-150	Rests		
151	D, A, F	d-minor	L
152	D, B♭, F	B♭-major	R
153	D, B♭, G	g-minor	L
154	E♭, B♭, G	E♭-major	T_0
155	E♭, B♭, G	E♭-major	T_0
156-158	Rests		
159	E♭, B♭, G	E♭-major	R
160	E♭, C, G	c-minor	L
161	E♭, C, A♭	A♭-major	R
162	F, C, A♭	f-minor	L
163	F, D♭, A♭	D♭-major	R
164	F, D♭, B♭	b♭-minor	L
165	G♭, D♭, B♭	G♭-major	R
166	G♭, E♭, B♭	e♭-minor	L
167	G♭/F, E♭, C♭/B	C♭/B-major	R
168	A♭/G#, E♭/D#, C♭/B	a♭/g#-minor	L
169	A♭/G#, F♭/E, C♭/B	F♭/E-major	R
170	G#, E, C#	c#-minor	L
171	A, E, C#	A-major	R
172	A	N/A	T_1
173	A#	N/A	T_0
174	A#	N/A	T_1
175	B	N/A	T_0
176	B	N/A	N/A

- ***Analysis*:**

  Through simplification of the notes into the chord structure of each measure, we can see that measures 142-172 follow the chord progression that maps the entire set of chords by using successive R and L.

  Though much of music utilizes chord progressions which can be mapped with L and R, such an extensive chord progression as this is fairly unique in music.

Current Music:

Much of current music follows a standard progression of chords, namely the I-vi-IV-V progression.
- I: Major 1st : major chord where leading tone is root tone of key
- vi: minor 6th : minor chord where leading tone is 6 diatonic notes above root of key
- IV: Major 4th : major chord where leading tone is 4 diatonic notes above root of key
- V: Major 5th : major chord where leading tone is 5 diatonic notes above root of key

This chord progression can be produced using the transformations R, P, and T, for example:

$$\text{C-maj} \xrightarrow{R} \text{A-min} \xrightarrow{P} \text{F-maj} \xrightarrow{T_2} \text{G-maj} \xrightarrow{T_5} \text{C-maj}$$

Since the progression is cyclic, it can be started at any one of the chords. Its arbitrary starting point and arbitrary key choice makes this chord progression the harmonic foundation for a huge percentage of popular music.