

Distributionally Robust Gain Analysis for Systems Containing Complex Uncertainty¹

Sheila R. Ross

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
sross@cae.wisc.edu

B. Ross Barmish

Department of Electrical Engineering and Computer Science
Case Western Reserve University
brb8@po.cwru.edu

Abstract

The main result of this paper addresses the minimum and maximum expected values of various gain measures for the transfer function of a system which depends on a vector Δ of independent complex random gains. In the distributional robustness framework of this paper, the probability density function for Δ is not completely specified. It is assumed only that the distribution of each component Δ_i is non-increasing with respect to $|\Delta_i|$, radially symmetric and supported on the disc of radius r_i centered at zero in the complex plane. Under these conditions, the expected value of the magnitude-squared of the gain function at a fixed frequency $\omega \geq 0$ is seen to be maximized when each Δ_i is uniformly distributed over the disc of radius r_i and minimized when each Δ_i has the impulse distribution. The result is extended to show that an \mathcal{H}^2 measure of the gain is also maximized and minimized in the same way. These results apply to quotients of multilinear functions of Δ , which includes system transfer functions obtained using Mason's formula.

1 Introduction

The main result of this paper applies to systems which depend on a vector of independent complex random gains

$$\Delta \doteq (\Delta_1, \Delta_2, \dots, \Delta_n).$$

Consistent with considerations of unmodelled dynamics in the literature, the Δ_i can be viewed as realizations of rational function uncertainty at the frequency of interest. We first analyze the effect of this uncertainty on the overall system gain at a fixed frequency $\omega \geq 0$. In

the presence of uncertainty, the expected gain deviates from its nominal value. The main result addresses the maximum and minimum excursions that the expected fixed-frequency squared-gain may achieve, given a set of radial bounds r_i for the Δ_i . The result is then extended to maximize an \mathcal{H}^2 measure of the expected system gain over the given uncertainty bounds.

1.1 Range of Application

In this paper, we consider the class of uncertain systems with transfer functions of the form

$$H(s, \Delta) = \frac{N(s, \Delta)}{D(s, \Delta)}$$

where $N(s, \Delta)$ and $D(s, \Delta)$ are polynomials in s with coefficients that depend multilinearly on the vector of uncertain gains Δ . That is, each coefficient of N and D is a complex-valued linear function of each Δ_i . Polynomials with coefficients that are multilinear functions of uncertain parameters are analyzed in the context of stability for the case of real uncertain parameters in references such as [1] and [2]. In this regard, perhaps the Mapping Theorem is most notable; e.g., see [3].

Many uncertain systems have transfer functions which are a ratio of polynomials whose coefficients are multilinear functions of uncertain parameters. For example, in Figure 1.1.1, if $P(s)$ and $C(s)$ are ratios of polynomials in s , then the transfer function for the system is

$$H(s) = \frac{P(s) + \Delta_1}{1 + P(s)C(s)(1 + \Delta_2) + \Delta_1 C(s)(1 + \Delta_2)},$$

which is readily reduced to this multilinear form. In fact, any transfer function which can be computed using Mason's gain formula also has this form, as long as each gain Δ_i appears in only one branch. This fact is easily established: The gain for any forward path is either independent of Δ_i or a linear function of Δ_i . Then,

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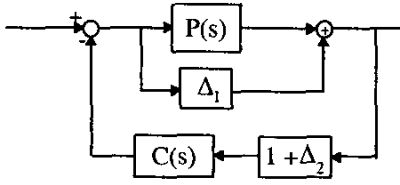


Figure 1.1.1: Possible application system

according to Mason's gain formula, the numerator of the transfer function must be affine linear in each Δ_i . Additionally, with Δ_i restricted to one branch, it cannot appear in two non-touching loops. With this in mind, it is straightforward to show that the denominator in Mason's gain formula must be affine linear in each Δ_i .

1.2 Distributional Robustness

The main result in this paper requires little information concerning the actual probability distribution of the gain vector Δ . We obtain the tightest possible bounds on the expected squared-gain with Δ having any distribution f from the prescribed class \mathcal{F} as defined in Section 1.3. In this sense, our result is said to be *distributionally robust*; see [4]-[6] for more details motivating this approach. In view of the importance of the robust performance of systems, this property of distributional robustness is of interest, as it allows a designer to evaluate performance tradeoffs from a probabilistic standpoint without introducing assumptions concerning the distribution of the uncertain gains.

1.3 Probabilistic Performance

Deterministic robust control techniques may be employed to find the maximum and minimum values of various measured related to the system gain for given uncertainty bounds. This would allow a system designer to set uncertainty limits in order to guarantee that the system gain will stay within a desired range. However, this deterministic approach may be unnecessarily conservative. In many applications, some small risk of excessive gain may be acceptable. This fact motivates consideration of robust performance from a probabilistic point of view, as in [2]-[13]. Distributional robustness analysis provides information concerning the risk of violating a performance criterion without knowing the probability density functions for the uncertain parameters. This information may allow a designer to increase the uncertainty limit beyond the level provided through deterministic analysis. Increasing the amount of uncertainty allowed is important from an applications standpoint; increased parameter tolerance can improve design feasibility, manufacturing cost and other aspects of system design. Often, the risk of performance violation is small even when the magnitude of the uncertainty far exceeds the de-

terministic stability margin. In addition, the computation of a deterministic robustness bound is often NP-hard, as the complexity of the algorithm increases exponentially with the number of uncertain parameters in many cases; e.g., see [14]-[18]. However, distributional robustness analysis avoids many computational difficulties by using Monte Carlo methods to estimate risk of robust performance violation; e.g., see [5].

1.4 Admissible Probability Distributions

In this paper, consistent with the paradigm introduced in [6], it is assumed that each independent gain Δ_i has an associated probability density function f_i , which is unknown except for the fact that it has the following properties: it is non-increasing with respect to $|\Delta_i|$, radially symmetric and supported on the disc of radius r_i centered at zero in the complex plane. The joint probability density function for the random vector Δ is thus

$$f(\Delta) \doteq f_1(\Delta_1)f_2(\Delta_2)\cdots f_n(\Delta_n)$$

and we let \mathcal{F} denote the class of joint probability density functions obtained this way. For further explanation of the paradigm underlying this type of probabilistic setup, see [4]-[6] and [13]. These assumptions on the marginal distributions, motivated by manufacturing considerations, are justified in the sense that small-magnitude values of Δ_i are typically more probable than large-magnitude Δ . The radial symmetry property implies that all phase errors are equally likely.

1.5 Additional Notation

In the sequel, we let Δ^J denote the vector of independent random gains with joint probability distribution $f \in \mathcal{F}$ with associated radial bounds r_i on the gains Δ_i . In addition, we let u denote the joint distribution which occurs when each Δ_i is uniformly distributed over the disc of radius r_i centered at zero in the complex plane. We take δ to be the joint impulse distribution for Δ . That is, each component Δ_i of Δ has the Dirac delta function centered at $\Delta_i = 0$ as its probability density function. Finally, we let $\mathcal{D}(r)$ denote the disc of radius r centered at zero in the complex plane.

2 Main Result and Proof

We now present a theorem concerning minimization and maximization of the expected value of the magnitude-squared of the system gain at a fixed frequency. Theorem 2.5 extends this result to maximize and minimize an \mathcal{H}^2 measure of the system gain.

2.1 Theorem

Let $N(s, \Delta)$ and $D(s, \Delta)$ be polynomials in s with coefficients that depend multilinearly on Δ ,

with $D(j\omega, \Delta) \neq 0$ for all $\Delta_i \in \mathcal{D}(r_i)$ and all $\omega \geq 0$. Then with $H(s, \Delta) \doteq \frac{N(s, \Delta)}{D(s, \Delta)}$,

$$\max_{f \in \mathcal{F}} E \left[|H(j\omega, \Delta^f)|^2 \right] = E \left[|H(j\omega, \Delta^u)|^2 \right]$$

and

$$\min_{f \in \mathcal{F}} E \left[|H(j\omega, \Delta^f)|^2 \right] = E \left[|H(j\omega, \Delta^\delta)|^2 \right]$$

holds for all $\omega \geq 0$.

The following lemma is used to prove Theorem 2.1.

2.2 Lemma

Let x^t be a complex-valued random variable with probability distribution f which is uniform over $\mathcal{D}(t)$, with x^δ having the impulse distribution centered at $x = 0$. Then, for any complex scalars a , b , c , and d such that $cx + d \neq 0$ for all $x \in \mathcal{D}(t)$,

$$\max_{0 \leq t \leq r} E \left[\left| \frac{ax^t + b}{cx^t + d} \right|^2 \right] = E \left[\left| \frac{ax^r + b}{cx^r + d} \right|^2 \right]$$

and

$$\min_{0 \leq t \leq r} E \left[\left| \frac{ax^t + b}{cx^t + d} \right|^2 \right] = E \left[\left| \frac{ax^\delta + b}{cx^\delta + d} \right|^2 \right].$$

2.3 Proof of Lemma 2.2

The expected value above is first written as

$$\begin{aligned} E \left[\left| \frac{ax^t + b}{cx^t + d} \right|^2 \right] &= \frac{1}{\pi t^2} \int_0^t \int_0^{2\pi} \left| \frac{a\rho e^{j\theta} + b}{c\rho e^{j\theta} + d} \right|^2 \rho \, d\rho \, d\theta \\ &= \frac{2}{t^2} \int_0^t \rho \, g(\rho) \, d\rho \end{aligned}$$

where

$$g(\rho) \doteq \frac{|a|^2 \rho^2 - 2 \cos(\angle ad - \angle bc) \rho^2 |abc| / |d| + |b|^2}{|d|^2 - |c|^2 \rho^2}.$$

Noting that

$$\frac{dg}{d\rho} = 2 \frac{\rho |ad - bc|^2}{(|d|^2 - |c|^2 \rho^2)^2} \geq 0$$

so that $g(\rho)$ is non-decreasing, the inequality $\rho g(\rho) \leq \rho g(t)$ holds for $\rho \leq t$. Therefore,

$$\int_0^t \rho g(\rho) \, d\rho \leq \frac{1}{2} t^2 g(t).$$

The derivative,

$$\frac{d}{dt} \frac{1}{t^2} \int_0^t \rho g(\rho) \, d\rho = \frac{g(t)}{t} - \frac{2}{t^3} \int_0^t \rho g(\rho) \, d\rho$$

is thus non-negative for $t > 0$. Therefore,

$$\max_{0 \leq t \leq r} E \left[\left| \frac{ax^t + b}{cx^t + d} \right|^2 \right] = E \left[\left| \frac{ax^r + b}{cx^r + d} \right|^2 \right]$$

and

$$\begin{aligned} \min_{0 \leq t \leq r} E \left[\left| \frac{ax^t + b}{cx^t + d} \right|^2 \right] &= \lim_{t \rightarrow 0^+} \frac{1}{t^2} \int_0^t \rho g(\rho) \, d\rho \\ &= E \left[\left| \frac{ax^\delta + b}{cx^\delta + d} \right|^2 \right] \\ &= \frac{|b|^2}{|d|^2}. \end{aligned}$$

2.4 Proof of Theorem 2.1

Given a vector $t = (t_1, t_2, \dots, t_n)$ with components $0 \leq t_i \leq r_i$, let Δ^t be the random vector with each component Δ_i^t having the uniform distribution $u^{t_i}(\Delta_i)$ over the disc of radius t_i centered at zero in the complex plane. We claim that the expected value is maximized with each $t_i = r_i$, and note that a similar proof can be given to show that the expected value is minimized with $t_i = 0$.

Indeed, according to a result in [13],

$$\max_{f \in \mathcal{F}} E \left[|H(j\omega, \Delta^f)|^2 \right] = \max_{t \in [0, r_1] \times \dots \times [0, r_n]} E \left[|H(j\omega, \Delta^t)|^2 \right].$$

Thus, we seek truncation radii (t_1, t_2, \dots, t_n) which maximize the expected value. Proceeding by contradiction, suppose that for some t_k , we have $t_k < r_k$; without loss of generality, say $k = n$. Noting that the integrand associated with the expectation operator is non-negative and the Δ_i are independent, we write

$$\begin{aligned} E \left[|H(j\omega, \Delta^t)|^2 \right] &= \\ &\int_{\Delta_1 \in \mathcal{D}(t_1)} u^{t_1}(\Delta_1) d\Delta_1 \cdots \int_{\Delta_n \in \mathcal{D}(t_n)} |H(j\omega, \Delta)|^2 u^{t_n}(\Delta_n) d\Delta_n. \end{aligned}$$

The integrand can be written as

$$|H(j\omega, \Delta)|^2 = \left| \frac{a\Delta_n + b}{c\Delta_n + d} \right|^2$$

where a , b , c , and d are complex-valued functions of $\Delta_1, \Delta_2, \dots, \Delta_{n-1}$ and ω . We now write

$$\int_{\Delta_n \in \mathcal{D}(t_n)} |H(j\omega, \Delta)|^2 u^{t_n}(\Delta_n) d\Delta_n = E \left[\left| \frac{a\Delta_n^{t_n} + b}{c\Delta_n^{t_n} + d} \right|^2 \right]$$

and apply Lemma 2.2 to obtain

$$\begin{aligned} E \left[\left| \frac{a\Delta_n^{t_n} + b}{c\Delta_n^{t_n} + d} \right|^2 \right] &\leq E \left[\left| \frac{a\Delta_n^{r_n} + b}{c\Delta_n^{r_n} + d} \right|^2 \right] \\ &= \int_{\Delta_n \in \mathcal{D}(r_n)} |H(j\omega, \Delta)|^2 u^{r_n}(\Delta_n) d\Delta_n. \end{aligned}$$

It now follows that

$$E \left[|H(j\omega, \Delta^t)|^2 \right] \leq E \left[|H(j\omega, \Delta^{t^*})|^2 \right]$$

where $t^* = (t_1, t_2, \dots, t_{n-1}, r_n)$. Thus, to maximize the expected value, t_n must equal r_n . Arguing in a similar manner, it follows with each t_i equalling r_i , we obtain u as the joint distribution maximizing the expected value. A similar proof can be given to establish that the joint impulse distribution minimizes the expected value.

Theorem 2.1 is now used to show that an \mathcal{H}^2 measure of the system gain is maximized with the joint uniform distribution and minimized with the joint impulse distribution.

2.5 Theorem

Let $N(s, \Delta)$ and $D(s, \Delta)$ be polynomials in s with coefficients that depend multilinearly on Δ , with $D(j\omega, \Delta) \neq 0$ for all $\Delta_i \in \mathcal{D}(r_i)$ and all $\omega \geq 0$. Then with $H(s, \Delta) \doteq \frac{N(s, \Delta)}{D(s, \Delta)}$,

$$\max_{f \in \mathcal{F}} E \left[\int_{\omega=0}^{\infty} |H(j\omega, \Delta^f)|^2 d\omega \right] = E \left[\int_{\omega=0}^{\infty} |H(j\omega, \Delta^u)|^2 d\omega \right]$$

and

$$\min_{f \in \mathcal{F}} E \left[\int_{\omega=0}^{\infty} |H(j\omega, \Delta^f)|^2 d\omega \right] = E \left[\int_{\omega=0}^{\infty} |H(j\omega, \Delta^\delta)|^2 d\omega \right].$$

2.6 Proof of Theorem 2.5

Since the integrand is non-negative, we may reverse the order of integration associated with the composition of expectation and integration above to obtain

$$\begin{aligned} \sup_{f \in \mathcal{F}} E \left[\int_0^{\infty} |H(j\omega, \Delta^f)|^2 d\omega \right] &= \sup_{f \in \mathcal{F}} \int_0^{\infty} E \left[|H(j\omega, \Delta^f)|^2 \right] d\omega \\ &\leq \int_0^{\infty} \sup_{f \in \mathcal{F}} E \left[|H(j\omega, \Delta^f)|^2 \right] d\omega. \end{aligned}$$

In view of the requirement of Theorem 2.1, for each $\omega \geq 0$, we have

$$E \left[|H(j\omega, \Delta^f)|^2 \right] \leq E \left[|H(j\omega, \Delta^u)|^2 \right]$$

which implies that

$$\int_0^{\infty} E \left[|H(j\omega, \Delta^f)|^2 \right] d\omega \leq \int_0^{\infty} E \left[|H(j\omega, \Delta^u)|^2 \right] d\omega.$$

Hence, the supremum above is achieved by the uniform distribution. A similar argument shows that the joint impulse distribution minimizes the expected value of the same \mathcal{H}^2 measure of the system gain.

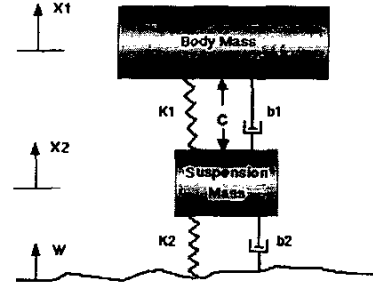


Figure 3.1: Model of bus suspension system [19]

3 Numerical Example

The maximum expected magnitude-squared gain is now computed at several different levels of uncertainty for the bus suspension system of [19]; see Figure 3.1.

The output of the system, $x_1 - x_2$, represents vertical bus displacement and the input, w , represents disturbance due to road irregularities. The model parameters are: bus body mass $m_1 = 2500$ kg, suspension system mass $m_2 = 320$ kg, suspension system spring constant $k_1 = 80,000$ N/m, spring constant of wheel and tire $k_2 = 500,000$ N/m, suspension system damping constant $b_1 = 350$ Ns/m and damping constant of wheel and tire $b_2 = 15,020$ Ns/m. With a PID control, $C(s)$, designed to improve the performance of the suspension system, the block diagram for the system is pictured in Figure 3.2. The corresponding transfer

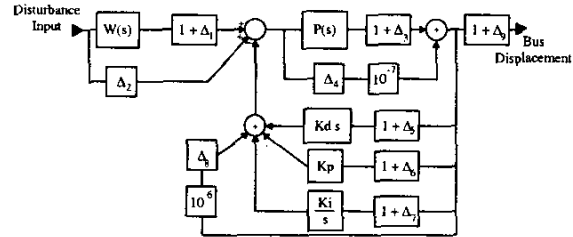


Figure 3.2: Model of bus suspension system

functions are

$$W(s) = 10^7 \frac{-3.76s^4 - 125s^3}{2820s^2 + 15020s + 500000};$$

$$P(s) = 10^{-7} \frac{2820s^2 + 15020s + 500000}{0.08s^4 + 3.85s^3 + 148.1s^2 + 137.7s + 4000}$$

and the controller gains are

$$K_d = 416,050; \quad K_p = 1,664,200; \quad K_i = 1,248,150.$$

To illustrate the application of our results, nine uncertain complex gains representing unmodelled dynamics have been inserted into the system as shown in Figure 3.2. We now apply Theorem 2.1 to compute the

maximum expected value of the magnitude-squared gain at a fixed frequency of $\omega = 2$ Hz (simulating a bumpy road as disturbance input). The system gain at this frequency is as follows:

$$\left| \frac{x_1 - x_2}{w} \right|^2 = \frac{[W(j4\pi)(1 + \Delta_1) + \Delta_2]\tilde{P}(j4\pi, \Delta)[1 + \Delta_9]}{1 + \tilde{P}(j4\pi, \Delta)C(j4\pi, \Delta)}$$

where

$$\tilde{P}(s, \Delta) \doteq P(s)(1 + \Delta_3) + 10^{-7}\Delta_4$$

and

$$C(s, \Delta) \doteq sK_d(1 + \Delta_5) + K_p(1 + \Delta_6) + \frac{K_i}{s}(1 + \Delta_7) + 10^6\Delta_8.$$

The maximum expected magnitude-squared gain is computed for several different radii of uncertainty. To facilitate computation, the uncertain gains were assigned the same radial bound r . Scaling factors were added to Δ_4 and Δ_8 to illustrate the effect of the uncertainty. The radius r was varied from 0 to 0.5. The upper bound on the radius ensures that the denominator of the transfer function will not vanish. At each radius, the expected magnitude-squared gain is estimated using the uniform distribution for the Δ_i and Monte Carlo integration with over 1 million samples. Figure 3.3 relates the uncertainty radius to the expected gain. We see that the nominal value of the gain-squared is about 0.075, but it increases rapidly for $r > 0.4$.

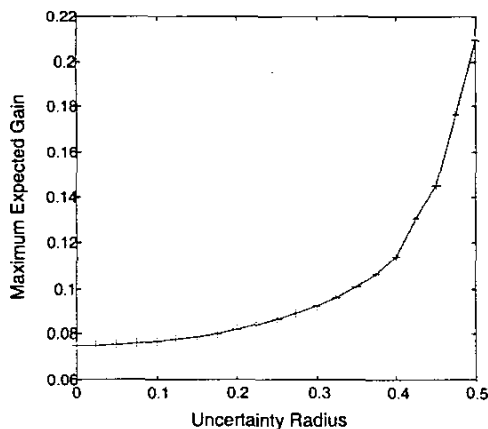


Figure 3.3: Maximum expected gain vs. uncertainty radius

4 Concluding Remarks

4.1 Probability of Instability

As motivation for further research, it is of interest to note that the *probability of instability* of a system in

the class considered here is not necessarily maximized with uniform or impulse distributions for Δ_i . For example, more assumptions are required in order to conclude that

$$\text{Prob}\{H(s, \Delta^f) \text{ is unstable}\} \leq \text{Prob}\{H(s, \Delta^u) \text{ is unstable}\}$$

holds for all $f \in \mathcal{F}$. To establish this fact, we provide a simple counterexample: Consider the system pictured in Figure 4.1, which contains a single uncertain gain. Now with a very large uncertainty bound $r = 10^5$

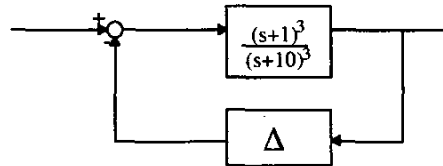


Figure 4.1: System for counterexample

for Δ , we claim the probability of instability is not maximized with the uniform or impulse distribution. To this end, let u^t denote a truncated uniform distribution which is obtained as the uniform distribution over the disk of radius t centered at zero in the complex plane, where $t \leq r$. Now, we compute the probability of instability for distributions having various truncation radii t . This is shown in Figure 4.2, which contains a plot of the probability of instability versus truncation radius t . The graph shows that probability of instability is not an increasing function of t . From this graph, the truncation which maximizes the probability of instability seems to be around $t = 60$. Since $r > 60$, neither the uniform distribution over the disc of radius r nor the impulse distribution maximizes the probability of instability for this system.

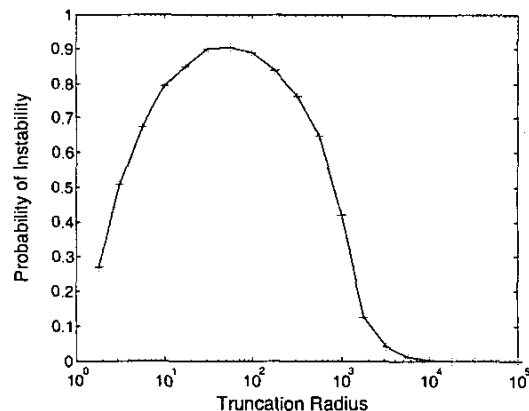


Figure 4.2: Probability of instability vs. truncation radius

4.2 Areas For Further Research

We have just demonstrated that neither the uniform distribution nor the impulse distribution is the worst-case distribution with regard to a particular performance criterion, stability in this case. This motivates the search for the optimal distributions which will maximize/minimize the criterion of interest. This topic is discussed in [5], [13] and [20].

Counterexamples of the sort above also motivate further research towards finding a distributionally robust estimate of the probability of instability. The systems studied here have a transfer function denominator polynomial $D(s, \Delta)$ which has multilinear functions of Δ as coefficients. This type of uncertain polynomial is studied in [1] and in [2], which provides an estimate of the probability of instability of the polynomial for the case of real uncertain parameters. For real Δ , with an appropriately chosen function φ , it is found that

$$\max_{f \in \mathcal{F}} \text{Prob}\{D(s, \Delta^f) \text{ is unstable}\} \leq E\{\varphi(\Delta^u)\}.$$

It is possible that these results could be extended to the case of complex uncertainty, to obtain an estimate of the probability of instability of the systems considered here.

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