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A GENERALIZATION OF LOVASZ'S SANDWICH THEOREM

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ABSTRACT

For a graph with $G(V, E)$ with n vertices, let $\omega_k(G)$ be the size of the largest induced subgraph that can be covered with k cliques. Define a $k-$multicoloring of G to be a coloring of its vertices such that each vertex is assigned a set of k colors, and two adjacent vertices are assigned disjoint sets of colors. Let $\chi_k(G)$ be the minimum number of colors needed for a valid $k-$multicoloring of the graph G.

We present a polynomial-time computable function $\vartheta_k(G)$ which satisfies the following inequality:

$$\omega_k(G) \leq \vartheta_k(G) \leq \chi_k(G).$$

Thus $\vartheta_k(G)$ is sandwiched between two NP-hard parameters, namely $\omega_k(G)$ and $\chi_k(G)$. This generalizes Lovasz's **Sandwich Theorem** [Lov86], which demonstrates a polynomial-time computable function $\vartheta(G)$ satisfying the following inequality:

$$\omega(G) \leq \vartheta(G) \leq \chi(G),$$

where $\omega(G)$ is the size of the largest clique of G, and $\chi(G)$ is the chromatic number of G.

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1. Introduction

The term "sandwich theorem" was coined by Lovasz to describe a theorem that asserts the existence of a polynomially computable function that is bounded between two NP-Hard graph parameters [Lov86]. Lovasz presents a sandwich theorem that exhibits a polynomially computable function $\vartheta(G)$ that is bounded between the chromatic number $\chi(G)$, and the clique number $\omega(G)$ [Lov79]. That is,

$$\omega(G) \leq \vartheta(G) \leq \chi(G).$$

A sandwich theorem provides polynomially computable bounds on the graph parameters involved, as well as a polynomial time algorithm to compute these parameters for classes on which they agree. Indeed, Lovasz's sandwich theorem leads to a polynomial algorithm to compute the chromatic number (and the clique number) for perfect graphs *.

In his monograph [Lov86], Lovasz states that it would be interesting to find other sandwich theorems. We present here a new sandwich theorem which can be viewed as a generalization of Lovasz's theorem. The parameters involved in our theorem are $\omega_k(G)$, which is the size of the largest induced subgraph that can be covered with k cliques, and $\chi_k(G)$, whose definition follows. The *k-chromatic number* of a graph G, $\chi_k(G)$, is the minimum number of colors needed to color the vertices of a graph such that each vertex is assigned a set of k colors and every two adjacent vertices are assigned disjoint sets of colors. Note that $\chi_1(G) = \chi(G)$ and $\omega_1(G) = \omega(G)$. Moreover, $\chi_k(G) \geq k\omega_1(G) \geq \omega_k(G)$. We define a function $\vartheta_k(G)$ and show that it is polynomial time computable via the Ellipsoid Method †, and that

$$\omega_k(G) \leq \vartheta_k(G) \leq \chi_k(G), \quad \text{for } 1 \leq k \leq n.$$

Thus, our sandwich theorem provides a polynomially computable bound ϑ_k for ω_k and for χ_k. The function $\vartheta_k(G)$ is related to the spectra of a family of matrices associated with the graph G. Spectral properties of graphs are related to several graph parameters [CDS79]. For example, the largest eigenvalue of a graph is at least as large as its chromatic number [Wil67]. We show here that for every positive integer k, the sum of the k largest eigenvalues of a graph G is at least as large as $\omega_k(G)$. This result, interesting in its own right, plays a vital role in the proof of our sandwich theorem.

Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{1, 2, ..., n\}$. We define the *adjacency matrix* $A(G) = (a_{ij})$ of G as follows: $a_{ij} = 1$ if either $i = j$, or

* *A graph is perfect if for every induced subgraph H, $\chi(H) = \omega(H)$.*

† *see for example [Lov86].*

i and j are adjacent. Otherwise $a_{ij} = 0$. We note that our definition of the adjacency matrix differs from the common definition since $A(G)$ is defined to have 1's along the main diagonal. Consequently, the eigenvalues of the adjacency matrix as defined here are shifted by 1 from the eigenvalues of the adjacency matrix defined with zeros along the diagonal. We use the term "spectrum of a graph" to denote the spectrum of its adjacency matrix as defined above. All the matrices in this paper are real and symmetric, and hence their spectra consist of real numbers. Thus, we may assume that $\lambda_1 \geq \lambda_2 \geq ...\lambda_n$. This is a standing assumption in this paper.

2. A Generalization of Wilf's Inequality

Wilf [Wil67] showed that the largest eigenvalue of the adjacency matrix of a graph (with ones along the diagonal), is at least as large as the chromatic number of the graph, which in turn is at least as large as the size of the largest clique in the graph. We generalize this result. We show that the sum of the k largest eigenvalues of the adjacency matrix is at least as large as the size of the largest induced subgraph of G that can be covered with k cliques. Our proof is based upon the well known Cauchy Inequality [HG51], (also known as the Interlacing Theorem).

The Interlacing Theorem : Let A be a Hermitian matrix and let $A^{(n-1)}$ be the principal submatrix of A with rows and columns $1, 2, \ldots, n-1$. Then

$$\lambda_s(A) \geq \lambda_s(A^{(n-1)}) \geq \lambda_{s+1}(A).$$

for $1 \leq s \leq n-1$.

Lemma 1 : Let $A = (a_{ij})$ be a real, symmetric $n \times n$ matrix. Then

$$\sum_{i=1}^{k} a_{ii} \leq \sum_{i=1}^{k} \lambda_i(A), \ \forall\, k \leq n.$$

Proof : Let $A^{(k)}$ be the principal submatrix of A with rows and columns $1, 2, \ldots, k$. Denote its eigenvalues by $\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}$. By the interlacing theorem we have

$$\lambda_1^{(k+1)} \geq \lambda_1^{(k)} \geq \lambda_2^{(k+1)} \geq \ldots \geq \lambda_k^{(k)} \geq \lambda_{k+1}^{(k+1)}.$$

Using the fact that the sum of the eigenvalues of a matrix equals its trace, we get:

$$\begin{aligned}\sum_{i=1}^{k} a_{ii} &= \sum_{i=1}^{k} \lambda_i^{(k)} \\ &\leq \sum_{i=1}^{k} \lambda_i^{(k+1)} \\ &\ \ \vdots \\ &\leq \sum_{i=1}^{k} \lambda_i^{(n)} \\ &= \sum_{i=1}^{k} \lambda_i(A). \quad \blacksquare\end{aligned}$$

Note that since simultaneous permutation of the rows and columns leave the spectrum unchanged, $\sum_{i=1}^{k} \lambda_i(A)$ is at least as large as the sum of the k largest diagonal elements of A. We also note that Lemma 1 follows from the following theorem of Ky Fan.

Lemma 2 : [Fan49] Let A be a Hermitian matrix. Then $\sum_{i=1}^{k} \lambda_i(A)$ is the maximum of $\sum_{j=1}^{k}(Ax_j, x_j)$, when k orthonormal vectors $x_j \quad (1 \leq j \leq k)$ vary in the space.

Lemma 1 follows from Lemma 2 when we let x_j be the vector with all components 0 except for the j-th component which is 1. We next state an elementary result in linear algebra.

Lemma 3 : [ND77] For a real, symmetric matrix A there exists a non-singular, orthogonal matrix P, such that $P^T AP = \text{diag}(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$.

Theorem 1 : Let $G(V, E)$ be a graph with adjacency matrix A. Then

$$\sum_{i=1}^{k} \lambda_i(A) \geq \omega_k(G), \quad \forall\, k \leq n,$$

where $\lambda_i(A)$ is the i^{th} largest eigenvalue of A and $\omega_k(G)$ is the size of the largest induced subgraph of G that can be covered with k cliques.

Proof : Let H be the largest induced subgraph of G that can be covered with k cliques, $C_1', C_2', \ldots, C_k'$. The cliques $C_1', C_2', \ldots, C_k'$ are not necessarily vertex disjoint. By reducing the size of some of the cliques we obtain the collection of vertex disjoint cliques $C_1, C_2, \ldots, C_k$ of sizes $c_1 \geq c_2 \geq \ldots \geq c_k$. It follows that $\omega_k(G) = \sum_{i=1}^{k} c_i$. With no loss of generality we assume that the vertex set of H is $\{1, ..., h\}$. Let A_H be the adjacency matrix of the induced subgraph H. Clearly $A_H = A^{(h)}$. Let A' be the adjacency matrix of a graph G' composed of k disjoint cliques of sizes $c_1, c_2, \ldots, c_k$. Then,

$$A' = \begin{pmatrix} J_{c_1} & 0 & \ldots & 0 \\ 0 & J_{c_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_{c_k} \end{pmatrix}$$

where J_{c_i} is a square matrix of all ones of order c_i. Since the cliques $C_1, C_2, \ldots, C_k$ are disjoint, we can label the vertices of G such that

$$A_H = \begin{pmatrix} J_{c_1} & A_{12} & \ldots & A_{1k} \\ A_{21} & J_{c_2} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \ldots & J_{c_k} \end{pmatrix}.$$

The spectrum of J_{c_i} is $\{c_i, 0, \ldots, 0\}$. By Lemma 3, there exist matrices P_i of order c_i such that $P_i^T J_{c_i} P_i = \mathrm{diag}(c_i, 0, \ldots, 0)$ for $i = 1, \ldots, k$. Let

$$P = \begin{pmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P_k \end{pmatrix}.$$

It follows that the diagonal elements of $P^T A_H P$ are the same as the diagonal elements of $P^T A' P$, which are the eigenvalues of A'. Since $\lambda_i(A') = \lambda_1(J_{c_i}) = c_i$ for $i = 1, \ldots, k$, we have:

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i(A_H) &= \sum_{i=1}^{k} \lambda_i(P^T A_H P) \\ &\geq \text{sum of the } k \text{ largest diagonal elements of } P^T A_H P \\ &= \text{sum of the } k \text{ largest diagonal elements of } P^T A' P = \sum_{i=1}^{k} c_i = \omega_k(G). \end{aligned}$$

Finally, since A_H is a principal submatrix of A, it follows from the Interlacing Theorem that $\lambda_i(A) \geq \lambda_i(A_H)$ for $i = 1, \ldots, k$. Thus

$$\sum_{i=1}^{k} \lambda_i(A) \geq \omega_k(G), \quad \forall\, k \leq n. \quad \blacksquare$$

3. The Sandwich Theorem

We refer to the next Theorem as "Lovasz's Sandwich Theorem".

Theorem 2 [Lov86] : For any graph G, there exists a polynomial-time computable function $\vartheta(G)$, which satisfies the following inequality:

$$\omega(G) \leq \vartheta(G) \leq \chi(G).$$

The function $\vartheta(G)$ is defined as the minimum of the largest eigenvalues of matrices related to the adjacency matrix of G. More precisely, let $\mathcal{A}(G)$ be the set of all real symmetric $n \times n$ matrices $A = (a_{ij})$ for which $a_{ij} = 1$ if $i = j$ or if i and j are adjacent in G and whose largest eigenvalue satisfies $\lambda_1 \leq n$. The elements of A corresponding to non-adjacent positions are allowed to be arbitrary real numbers provided that the matrix remains symmetric. We say that a position q in row p is a *fixed position* if $b_{pq} = 1$ for every matrix $B = (b_{ij})$ in $\mathcal{A}(G)$. Thus, q is a fixed position in row p either if $p = q$ or if $(v_p, v_q) \in E(G)$. The function $\vartheta(G)$ is defined as

$$\vartheta(G) = min\{\lambda_1(A) : A \in \mathcal{A}(G)\}.$$

The parameter $\vartheta(G)$ was first introduced in [Lov79] where it was used to estimate the Shannon Capacity of C_5. Later it was shown to be polynomial time computable via the Ellipsoid Method [GLS81].
We define here a set of functions $\vartheta_k(G)$ which generalize $\vartheta(G)$.

$$\vartheta_k(G) = min\Big\{\sum_{i=1}^{k} \lambda_i(A) : A \in \mathcal{A}(G)\Big\}, \quad \text{for } 1 \leq k \leq n.$$

Lemma 4 : Let $B \in \mathcal{A}(\mathcal{G})$. Then $\|B\|_\infty \leq n^2$.

Proof : Let $B = (b_{ij}) \in \mathcal{A}(G)$. By Lemma 3 we can write $B = P^T D P$, where $P = (p_{ij})$ is an orthogonal matrix and $D = diag(\lambda_1(B), \ldots, \lambda_n(B))$. Hence $b_{ij} = \sum_{k=1}^{n} \lambda_k(B) p_{ik} p_{jk}$. Using the fact that $\sum_{i=1}^{n} \lambda_i(B) = Trace(B) = n$, we get $\lambda_1(B) \geq \lambda_i(B) = n - \sum_{j \neq i} \lambda_j(B) \geq n - (n-1)\lambda_1(B)$ for all i. Since trivially $\lambda_1(B) \leq n$, we have $|\lambda_i(B)| \leq n^2$. Hence

$$|b_{ij}| = \left| \sum_{k=1}^{n} \lambda_k(B) p_{ik} p_{jk} \right| \leq n^2 \sum_{k=1}^{n} |p_{ik}||p_{jk}|$$

$$\leq n^2 \left(\sum_{k=1}^{n} |p_{ik}|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |p_{jk}|^2 \right)^{\frac{1}{2}} = n^2 . \blacksquare$$

It follows that there exist a matrix $B \in \mathcal{A}(\mathcal{G})$ such that $\vartheta_k(G) = \sum_{i=1}^{k} \lambda_i(B)$. Note that $\vartheta_1(G) = \vartheta(G)$. We also note that Lemma 2 provides a min-max characterization for the functions ϑ_k. We use these functions to generalize Theorem 2 as follows:

Theorem 3 : For any graph G, there exist polynomial-time computable functions $\vartheta_k(G)$ which satisfy the following inequality:

$$\omega_k(G) \leq \vartheta_k(G) \leq \chi_k(G), \quad \text{for } 1 \leq k \leq n.$$

Theorem 3 generalizes Lovasz's result since for $k = 1, \omega_k(G) = \omega(G)$ and $\chi_k(G) = \chi(G)$, and in this case the function $\vartheta_1(G) = \vartheta(G)$.

The proof of Theorem 3 follows from Theorems 4, 6 and 7 below. In Theorem 4 we show that $\vartheta_k(G)$ is a tighter upper bound for $\omega(G)$ than the one provided by Theorem 1.

Theorem 4 : For any graph G, $\omega_k(G) \leq \vartheta_k(G)$, for $1 \leq k \leq n$.

Proof: We claim that Theorem 1 can be refines as follows. For every matrix $B \in \mathcal{A}(G)$,

$$\sum_{i=1}^{k} \lambda_i(B) \geq \omega_k(G), \quad \text{for } 1 \leq k \leq n.$$

The claim can be verified by following the steps in the proof of Theorem 1 replacing A by B, and $A^{(h)}$ by $B^{(h)}$. The proof of the theorem now follows. $\blacksquare$

Let $\alpha_k(G)$ denote the size of the largest subgraph of G that can be covered by k independent sets in G. Then $\alpha_k(G)$ is also the size of the largest induced k-partite subgraph in G or equivalently, the largest induced k-colorable subgraph of G. The problem of computing $\alpha_k(G)$ has been recently considered for special classes of graphs [MN87, SL87, YL88]. We note that $\omega_k(G) = \alpha_k(\overline{G})$ where $\overline{G}$ is the complement of G. The next Corollary provides a polynomially computable upper bound for $\alpha_k(G)$.

Corollary 5 : $\alpha_k(G) \leq \vartheta_k(\overline{G})$, for $1 \leq k \leq n$.

Theorem 6 below establishes the relation between ϑ_k and χ_k. Before we prove Theorem 5, we show that $k-$multicoloring a graph is equivalent to a regular coloring of a modified graph. Define the *Composition* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ to be the graph $G(V, E) = G_1[G_2]$ such that $V = V_1 \times V_2$ and $((x_i, y_i), (x_j, y_j)) \in E$ if either $x_i = x_j$ and $(y_i, y_j) \in E_2$, or $(x_i, x_j) \in E_1$. Given a graph G, let $G^{(k)} = G[K_k]$, where K_k is the complete graph on k vertices. A $k-$multicoloring of G can be converted into a coloring of $G^{(k)}$ by assigning the set of k distinct colors given to vertex v_i to the k vertices in G_i. Since this process is reversible $\chi_k(G) = \chi(G^{(k)})$.

Theorem 6 : For any graph G, $\vartheta_k(G) \leq \chi_k(G)$, for $1 \leq k \leq n$.

proof: The graph $G^{(k)}$ can be considered as the graph obtained by replacing each vertex v_i in G by a graph $G_i = K_k$, and then adding the appropriate edges. Each of the k vertices in G_i are similar in the sense that they are adjacent to the same set of vertices outside of G_i, besides being adjacent to each other. This fact is reflected in the structure of every matrix B in $\mathcal{A}(G^{(k)})$ in that two rows corresponding to similar vertices have the same set of fixed positions. This implies that if we replace a row B corresponding to vertex in G_i by a convex combination of rows corresponding to vertices in G_i, the new matrix is still in $\mathcal{A}(G^{(k)})$.

It follows from the definition of ϑ that $\vartheta(G^{(k)}) = \lambda_1(B')$ for some matrix $B' \in \mathcal{A}(G^{(k)})$ of order nk. Moreover, by Theorem 2, $\lambda_1(B') \leq \chi(G^{(k)})$. It is known that $\lambda_1(B)$ is a convex function of the entries in B [Fan49]. This fact, together with the discussion above imply that there exists a matrix $B" \in \mathcal{A}(G^{(k)})$ such that $\lambda_1(B") = \lambda_1(B') = \vartheta(G^{(k)})$ and the rows and columns corresponding to the similar (and mutually adjacent) vertices within each G_i are identical. Consequently, $B"$ can be divided into n sets of k rows (columns) such that each of the rows (columns) in a set is identical. Since $B"$ is symmetric, we can simultanousely permute its rows and columns (or relabel the vertices of $G^{(k)}$) such that the resulting matrix B has the following property. B can be divided into n^2 blocks where each block is of size $k \times k$ and all the entries within a block are equal.

Now consider the $n \times n$ symmetric matrix A obtained from B by replacing each $k \times k$ block in B by any entry from that block (the entries are all equal). Clearly, $B \in \mathcal{A}(G)$.

It is obvious that if

$$\mathbf{e} = (x_1, \ldots, x_n)$$

is an eigenvector of A, then

$$\mathbf{e}' = (x_{11}, \ldots, x_{1k}, x_{21}, \ldots, x_{2k}, \ldots, x_{n1}, \ldots, x_{nk})$$

is an eigenvector of B where

$$x_{ij} = x_i, \quad j = 1, \ldots, k, \quad i = 1, \ldots, n.$$

Consequently, if $\lambda_1(A), \ldots, \lambda_n(A)$ are the n eigenvalues of A, and $\lambda_1(B), \ldots, \lambda_{nk}(B)$ are the nk eigenvalues of B, then they are related as follows:

$$\lambda_{k(i-1)+j}(B) = k\lambda_i(A), \quad j = 1, \ldots, k.$$

In particular, $\lambda_1(B) = k\lambda_1(A)$. Now,

$$\vartheta_k(G) \leq \sum_{i=1}^{k} \lambda_i(A) \leq k\lambda_1(A) = \lambda_1(B) \leq \chi(G^{(k)}) = \chi_k(G). \quad \blacksquare$$

Theorem 7 : For every graph G, $\vartheta_k(G)$ can be computed in polynomial time for arbitrary k.

Proof : The proof that $\vartheta_k(G)$ can be computed in polynomial time is very similar to Lovasz's proof that $\vartheta(G)$ can be computed in polynomial time [Lov86]. It was proved by Ky Fan [Fan49] that for all $n \times n$ Hermitian matrices A, the functions

$$\phi_k(A) = \sum_{i=1}^{k} \lambda_i(A), \quad k = 1, \ldots, n,$$

are convex. For a fixed k and a rational matrix $B \in \mathcal{A}(\mathcal{G})$, the convex function $\phi_k(B)$ can be minimized over the affine subspace $\mathcal{A}$ using the ellipsoid algorithm, provided we can find a ball or cube about which we know *a priori* that it contains the minimum [Lov86]. Lemma 4 guarantees the existence of such a ball. Since $\phi_k(A)$ can be computed in polynomial time for each rational $A \in \mathcal{A}$, $\vartheta_k(G)$ can be computed in polynomial time. $\blacksquare$

4. Discussion

For perfect graphs the function $\vartheta(G)$ coincides with $\omega(G)$ and with $\chi(G)$. This fact is the essential ingredient in the polynomial-time algorithm to compute $\omega(G)$ and $\chi(G)$ for perfect graphs [GLS81]. Unfortunately, we cannot use the polynomially computable function $\vartheta_k(G)$ in the same manner to compute its bounding parameters for perfect graphs because $\chi_k(G)$ and $\omega_k(G)$ are not equal on perfect graphs. For example, for the complete graph $G = K_p$, which is a perfect graph, $\chi_k(G) = k\omega_k(G) = kp$. In fact, in this case the class $\mathcal{A}(G)$ consists only of the adjacency matrix $A(G)$, and hence $\omega_k(G) = \vartheta_k(G) = p$. These facts leave us with two open problems.

One open problem is to classify graphs for which $\omega_k = \chi_k$. For example for C_{2p}, the simple cycle of length $2p$, we have $\chi_k(C_{2p}) = \omega_k(C_{2p}) = kp, \quad k \leq p$. Another example where equality holds is for trees that have a matching of size k, in which case $\omega_k = \chi_k = 2k$.

The other open problem is to identify classes of graphs for which the functions $\vartheta_k \quad k \geq 1$, can be used to compute either one of its bounding parameters in polynomial time. For example, the function $\chi_k(G)$ is polynomial time computable for perfect graphs since $\chi_k(G) = \chi(G^{(k)}) = k\chi(G) = k\vartheta_1(G)$. In contrast, when k is arbitrary, computing $\omega_k(G)$ for perfect graphs is an NP-Hard problem. This follows from the fact that it is NP-Hard to compute $\omega_k(G)$ (for k arbitrary) for the class of chordal graphs [Yan87]. We believe that ϑ_k provides a tighter bound for ω_k than for χ_k. We have already seen that although $\chi_k(K_p) \neq \omega_k(K_P)$, it is the case that $\vartheta_k(K_p) = \omega_k(K_P)$. It will be interesting to identify classes of graphs for which $\omega_k(G) = \vartheta_k(G)$ since for these graphs this equality will provide a polynomial algorithm to compute ω_k.

Finally, we propose one last open problem. We know that ϑ_1 is an integral valued function for perfect graphs. Is ϑ_k integral valued on perfect graphs for $k > 1$?

References :

[CDS79] D. M. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs: Theory and Application, Academic Press, New York, 1979.

[Fan49] Fan, K., On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations I, *Proc. Natl. Acad. Sci. USA*, **35**, (1949), 652-655.

[GLS81] M. Grotshcel, L. Lovasz, and A. Schrijver, The Ellipsoid Method and its Consequences in Combinatorial Optimization, *Combinatorica*, **1**, (1981), 169-197.

[HG51] H. L. Hamburger and M¿ E. Grimsbaw, Linear Transformations in n-

Dimensional Vector Space, Cambridge University Press, 1951.

[Lo79] L. Lovasz, On the Shannon Capacity of a Graph, *IEE Transactions on Information Theory*, **IT-25**, no. 1, (1979), 1-7.

[Lov86] Lovasz L., An Algorithmic Theory of Numbers, Graphs and Convexity, CBMS 50, SIAM 1986, Philadelphia, Pennsylvania.

[MN87] R. Manber and G. Narasimhan, Algorithms for the Maximum Induced Bipartite Subgraph Problem on Interval and Circular-arc Graphs, *CS-TR 696-87*, University of Wisconsin - Madison, (1987).

[ND77] B. Noble, and J. Daniel, Applied Linear Algebra, 2nd edition, 1977, Prentice Hall, Inc., NJ 07632.

[SL87] Sarrafzadeh, M., and Lee, D.T., A New Approach to Topological Via Minimization, Tech. Report, Center for Integrated Microelectronic Systems, NorthWestern Univ., November 1987.

[Wil67] Wilf, H. S., The Eigenvalues of a Graph and its Chromatic Number, *J. of London Math. Soc.*, **42**, (1967), 330-332.

[Yan87] M. Yannakakis, and F. Gavril, The Maximum $k-$Colorable Subgraph Problem for Chordal Graphs, *Inform. Proc. Letters*, **24**, (1987).

[YL88] S. Yeh and A. LaPaugh, Algorithms for Finding a Maximum Bipartite Subgraph for Special Classes of Graphs, *CS-TR 149-88*, Princeton University (1988).