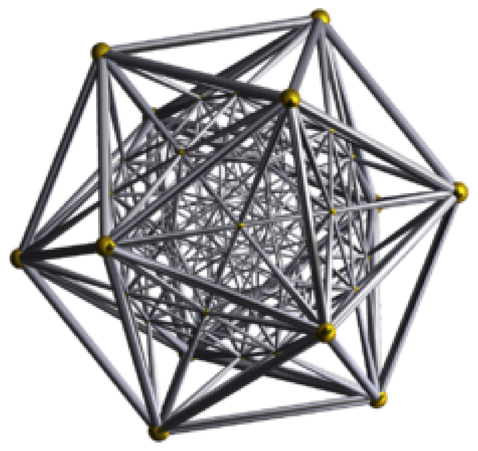


Algebras associated with the Hasse Graphs of Polytopes

Student Researchers: *Geoffrey Glover, Austin Holmes, & Tennie Jacobson*
Faculty Mentor: *Colleen Duffy*

The Power of AND

Motivation

There is a Hasse graph associated with each symmetry of every n-dimensional polytope, and there is an algebra associated with each Hasse graph. Each level of the graph represents the number of k-dimensional faces that remain fixed under a given automorphism (or symmetry) of the polytope. For each symmetry, we determine a polynomial $f(t)$ where the power of t represents the length of each path in the graph. The coefficient of t^0 is the number of points, the coefficient of t^1 is the number of paths of length 1, ..., and the coefficient of t^i is the number of unique paths of length i in the Hasse graph. Once we determine the polynomial associated with each symmetry, we can determine the structure of the algebra associated with the symmetry using the coefficients of the Hilbert series given by the generating function $H(t) = \frac{t-1}{1-tf(t)}$

Our goal is to determine the structure of all of the algebras associated with finite Coxeter groups (consisting of 4 families and 6 exceptional groups) by determining all Hasse graph polynomials $f(t)$. Duffy and past student research groups have accomplished finding the Hasse graph polynomials for the algebras associated with the A_n, B_n, D_n, $I_2(p)$ familes and H_3. We are working on the 600-Cell (H_4).

Future Directions

We are in the process of utilizing our programs to determine the containments of the fixed k-dimensional faces once a symmetry is applied to the 600-cell. Upon doing so, we will be able to construct a Hasse graph and count the directed paths of each length. This will essentially create the polynomial $f(t)$ that we can input into the generating function $H(t)$. Overall, we want to find the generating function for each symmetry conjugacy class.

We are also looking into the idea of implementing our programs for other exceptional Coxeter groups. Regardless of whether our programs will be used, we will continue working towards finding the Hasse graph polynomials for them.

Acknowledgments

- UWEC Mathematics Department
- UWEC Office of Research & Sustainability Programs
- Professor Colleen Duffy

University *of* Wisconsin
Eau Claire

Methodology

We determined the best way to represent the vertices of the 600-cell and created a program in Java to determine the sets of faces. From there, we wrote Maple programs which do the following:

- One creates a symmetry matrix for each symmetry by looking at where a given 3-D face (flag) gets mapped to. Another identifies a representative for each symmetry class.
- We have one program for each dimension to determine which faces are fixed by creating a matrix formed by the vertices for each face and multiplying that by a representative of each symmetry class.

Throughout this process, we have been using our programs on the icosahedron to compare and verify that our results are accurate. Our next step will involve using our programs to determine the containments of the fixed k-dimensional faces in order to create the Hasse graphs.

Example

Procedure

1. Apply an automorphism to the 600-cell.

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

   This symmetry will apply a reflection across the plane going through vertices 2, 3, 6 and 7

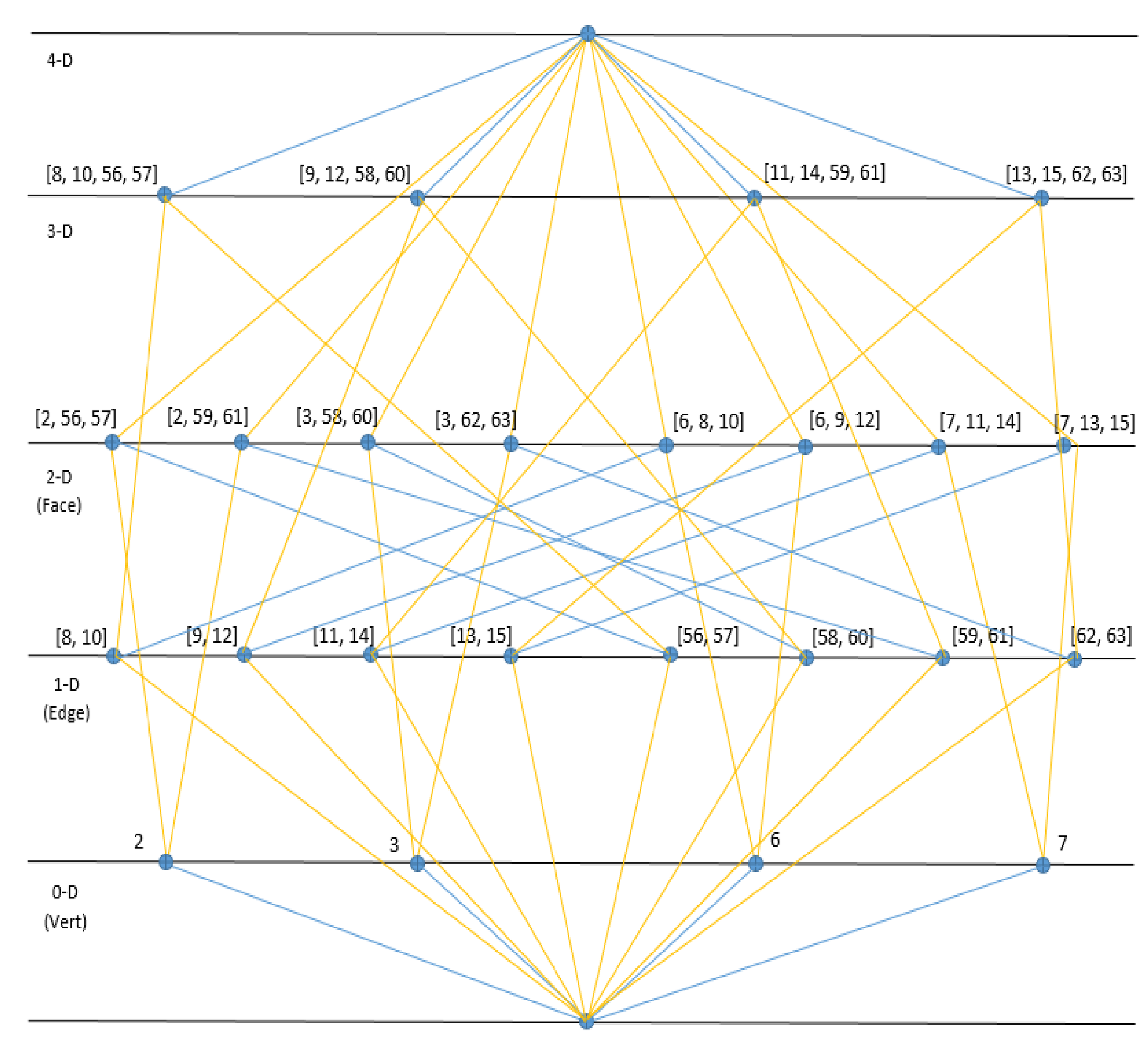
2. Count the number of faces fixed under the automorphism. Running it through our programs, we get the faces shown in the diagram.

3. Using the fixed faces, construct a Hasse Graph, where the level of the graph corresponds to the dimension of the fixed face.

4. Connect the points in the Hasse graph to show the containments of the fixed faces.

5. Count the (signed) directed paths of each length, and create the polynomial $f(t)$ such that the coefficient of t^i is the number of directed paths of length i.

Example: reflection across plane



$$f(t) = 26 - 16t - 32t^2 + 16t^3 + 8t^4 - t^5$$

References

[1] Duffy, Colleen. Graded Traces and Irreducible Representations of $Aut(A(\Gamma))$ Acting on $grA(\Gamma)$ and $grA(\Gamma)^{-1}$. Rutger's University.

[2] Chan, Schneider, Franko. Algebra Associated with the Hasse Graph of the Hypercube. University of Wisconsin - Eau Claire. 2012.

[3] Duffy, Holmes, Lemons, Riedl. Algebras associated with the Hasse graphs of polytopes. University of Wisconsin - Eau Claire. 2016.

[4] Images from Wikipedia and The University of Wisconsin - Eau Claire websites.