On the Dimension of Group Boundaries

by

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ABSTRACT

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The goal of this dissertation is to find connections between the small-scale dimension (i.e. covering dimension and linearly controlled dimension) of group boundaries and the large scale dimension (i.e. asymptotic dimension and macroscopic dimension) of the group. We first show that generalized group boundaries must have finite covering dimension by using finite large-scale dimension of the space. We then restrict our attention to CAT(0) group boundaries and develop metrics on the boundary that allow us to study the linearly controlled dimension. We then obtain results relating the linearly controlled dimension of CAT(0) boundaries to the large scale dimension of the CAT(0) space.

Table of Contents

List of Figures

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Chapter 1

Overview

One of the goals of Geometric Group Theory is to better understand algebraic properties of finitely generated groups by studying the geometry of metric spaces that the group acts on in a nice way. Two types of groups that are often studied are CAT(0) and δ-hyperbolic groups, which act geometrically on CAT(0) and δ-hyperbolic spaces. These geodesic metric spaces may be defined completely in terms of thinness of triangles and thus have very interesting geometry. In particular, analyzing the large-scale geometry of these metric spaces has proven to be very fruitful and enlightening. Much of this large-scale geometry is captured in the space at infinity, or boundary, of these spaces.

Hyperbolic and CAT(0) group boundaries may both be defined in terms of equivalence classes of geodesic rays, where two rays are equivalent if they fellow travel (for more details, see Section 3.2.1). However, for groups that are neither hyperbolic nor CAT(0), this definition of a group boundary cannot be applied. Thus, Bestvina [Bes96] defined $\mathcal{Z}$-boundaries in an attempt to bring the existing theories together and create an axiomatized method of approaching group boundaries for larger classes of groups (see Section 2.2).

$\mathcal{Z}$-boundaries, being closed subsets of a compact metric space, are compact. Thus, if we wish to study their geometry, it does not make sense to approach it from a large-scale point of view. Instead, we must focus more on the small-scale geometry of the boundary. Therefore, there are two approaches in studying the geometry of these metric spaces: large-scale properties of the space or small-scale properties of the boundary attached to the space.

The particular large and small-scale properties that we will discuss here are different notions of dimension. Large-scale dimension theories include macroscopic dimension and asymptotic dimension, while small-scale dimension theories include covering dimension and linearly controlled dimension. These are explained in greater detail in Section 3.2.3. A natural question to ask is if the large-scale dimension of the space affects the small-scale dimension of the boundary, or vice-versa. In particular, in what follows, we aim to answer (or partially answer) the following two questions:

1. Does knowing finiteness of large-scale dimension of a metric space allow us to conclude anything about the small-scale dimension of the boundary?

2. Does knowing finiteness of the small-scale dimension of the boundary allow us to conclude the space has finite large-scale dimension?

Before discussing our contributions to these questions, we first discuss what is known. For a hyperbolic group, G, $\text{asdim} G = \dim \partial G + 1 = \ell\text{-}\dim \partial G + 1 < \infty$ [BS07, BL07]. Within this seemingly simple equation are many deep and important results about hyperbolic groups:

- The covering dimension of hyperbolic group boundaries is finite (first shown by [Gro87, Swe95]).
- The linearly controlled dimension of hyperbolic group boundaries is finite.
- The linearly controlled dimension of the boundary of a hyperbolic group is equal to its covering dimension.
- Hyperbolic groups have finite asymptotic dimension (first shown by [Gro87]).
- There is a clear relationship between the large scale dimension a group and the small scale dimension of its boundary.

With these results as motivation, we outline this paper and detail the results that provide some answers to these two questions for groups that may not be hyperbolic.

In Chapter Two, we discuss $\mathcal{Z}$-boundaries and generalize the definition given by Bestvina. We then use the existence of a single open cover of the space with finite order to prove that generalized $\mathcal{Z}$-boundaries must have finite covering dimension.

Theorem 2.3.7. *If a proper metric ANR (X, d) admits a metric $\mathcal{Z}$-structure $(\hat{X}, Z)$, then Z is finite-dimensional.*

Thus, knowing something finite macroscopic dimension of the space allows us to conclude finite small-scale dimension of the boundary. We then use finite-dimensionality of $\mathcal{Z}$-boundaries to unify the theory of group boundaries.

In Chapter Three, we focus our attention on CAT(0) boundaries so that we may study their linearly controlled dimension. Since linearly controlled dimension is a metric invariant, we first develop two families of metrics on the boundary. One of the families was discussed in [Kap07], where B. Kleiner asked whether the induced action on ∂X of a geometric action on a proper CAT(0) space X is "nice". We give one answer to his question with the following:

Theorem 3.3.6. *Suppose G acts geometrically on a proper CAT(0) space X, $x_0 \in X$ and $A > 0$. Then the induced action of G on $(\partial X, d_{A,x_0})$ is by quasi-symmetries.*

We also use this family to show finiteness of the linearly controlled dimension of the boundary, thereby providing a positive answer to the first question.

Theorem 3.3.7. *If G is a CAT(0) group, then $(\partial G, d_{A,x_0})$ has finite linearly-controlled dimension.*

Using the second family of metrics, we obtain a partial answer to question two:

Theorem 3.4.3. *Suppose X is a geodesically complete CAT(0) space and, when endowed with the $\overline{d}_{x_0}$ metric for $x_0 \in X$, $\ell\text{-}\dim \partial X \leq n$. Then the macroscopic dimension of X is at most $2n + 1$.*

We hope that the framework we have built will lead to a more complete answer to the second question in the future.

Chapter 2

Finite-Dimensionality of Z-boundaries

2.1 Introduction

It is easy to construct a proper CAT(0) space with infinite-dimensional boundary, but a result by Swenson [Swe99] shows that such a space cannot admit a cocompact action by isometries. Thus, every boundary of a CAT(0) group must be finite-dimensional. This observation mirrors an earlier theorem by Gromov [Gro87,Swe95] which asserts that boundaries of hyperbolic groups are finite-dimensional.

The rich study of CAT(0) and hyperbolic group boundaries led Bestvina to formalize the concept of group boundaries for wider classes of groups [Bes96]. Included in his definition is a hypothesis which forces these boundaries, known as $\mathcal{Z}$-boundaries, to be finite-dimensional. Later, when Dranishnikov generalized Bestvina's work to allow for groups with torsion [Dra06], he omitted the requirement in Bestvina's original definition that forced the boundaries to be finite-dimensional. As a result, some of the results in [Dra06] are, a priori, not as strong as their analogs in [Bes96]. In particular, Dranishnikov related the cohomological dimension of a group to the cohomological dimension of its $\mathcal{Z}$-boundary, but not to the Lebesgue covering dimension of that boundary.

In this chapter, we prove a generalization of Swenson's theorem that applies to a more general class of spaces. One of the consequences of this result is a more unified treatment of group boundaries put forth by Bestvina and Dranishnikov. We show that there is no advantage in restricting our attention to finite-dimensional spaces as in [Bes96]. In particular, we may weaken the requirement that the group act on an ER to an analogous action on an AR, without losing any applications. In regards to [Dra06], all conclusions about the cohomological dimension of group boundaries can be extended to results about the Lebesgue covering dimension of these boundaries.

We close with statements of some of our main results that may be found in this chapter.

Theorem 2.3.7. [Mor14] *If a proper metric ANR (X, d) admits a metric $\mathcal{Z}$-structure $(\hat{X}, Z)$, then Z is finite-dimensional.*

Theorem 2.4.1. [Mor14] *If a torsion-free group G admits an AR $\mathcal{Z}$-structure, then G admits a $\mathcal{Z}$-structure, as defined in [Bes96].*

Due to its relevance to our work, at the end of this chapter, we provide the details for an alternative proof of Bestvina's Boundary Swapping Theorem as suggested by Ferry [Fer00].

2.2 Preliminaries

We begin with a few preliminary definitions, leading to Bestvina's original definition of a $\mathcal{Z}$-structure. We then present various generalizations of this definition with explanations, justifications, and consequences of the changes.

We suppose that our spaces are locally compact, separable, and metrizable. We will focus our attention on special types of separable metric spaces: AR, ANR, ER, and ENR's. Recall that a separable metric space X is an **absolute retract** (or AR) if, whenever X is embedded as a closed subset of another separable metric space Y, its image is a retract of Y. X is an **absolute neighborhood retract** (or ANR) if, whenever X is embedded as a closed subset of another separable metric space Y, some neighborhood of X in Y retracts onto X. A **Euclidean retract** (or ER) and a **Euclidean neighborhood retract** (or ENR) are finite-dimensional AR's or ANR's, respectively.

Definition 2.2.1. A closed subset, A, of an ANR, X, is a ***$\mathcal{Z}$-set*** if either of the following equivalent conditions hold:

- There exists a homotopy $H : X \times [0,1] \to X$ such that $H_0 = id_X$ and $H_t(X) \subset X - A$ for every $t > 0$.
- For every open subset U of X, the inclusion $U - A \hookrightarrow U$ is a homotopy equivalence.

The standard example of a $\mathcal{Z}$-set is the boundary of an n-manifold, M^n. We can use a collared neighborhood of the boundary in M^n to define a homotopy, which instantly pushes the boundary off itself.

Definition 2.2.2. A ***$\mathcal{Z}$-compactification*** of a space Y is a compactification $\hat{Y}$ such that $\hat{Y} - Y$ is a $\mathcal{Z}$-set in $\hat{Y}$. We call $\hat{Y} - Y$ a ***$\mathcal{Z}$-boundary*** for Y.

Implicitly in this definition, we are assuming that $\hat{Y}$ is an ANR. Because open subsets of an ANR are ANRs, we require a space Y to be an ANR before beginning to ask whether or not Y is $\mathcal{Z}$-compactifiable. Once we do know $\hat{Y}$ is a $\mathcal{Z}$-compactification of Y, then $\hat{Y}$ and Y will have the same homotopy type. In the realm of compactifications, $\mathcal{Z}$-compactifications are particularly nice as they are sensitive to the overall geometry of the original space. This preservation of geometry explains the choice to use $\mathcal{Z}$-compactifications in the theory of group boundaries as we now see with Bestvina's original definition of a $\mathcal{Z}$-structure on a group G.

Definition 2.2.3. [Bes96] A $\mathcal{Z}$-***structure on a group*** G is a pair of spaces $(\hat{X}, Z)$ satisfying the following four conditions:

1. $\hat{X}$ is a compact ER,
2. $\hat{X}$ is a $\mathcal{Z}$-compactification of $X = \hat{X} - Z$,
3. G acts properly, cocompactly, and freely on X, and

4. $\hat{X}$ satisfies a *nullity condition* with respect to the action of G on X. That is, for every compactum C of X and any open cover $\mathcal{U}$ of $\hat{X}$, all but finitely many G translates of C lie in some element of $\mathcal{U}$.

There are a few things to notice about this definition. First, since an ER is a finite-dimensional AR and an AR is a contractible ANR, $\hat{X}$ is a compact, contractible, finite-dimensional ANR. The requirement that $\hat{X}$ be finite-dimensional is what forces $Z \subseteq \hat{X}$ to be finite-dimensional. In fact, a simple argument using ϵ-maps in the next section will show that $\dim Z \leq \dim X < \infty$.

Secondly, our spaces are metrizable, but we must take care to distinguish between the metric on X and the metric on the compactification $\hat{X}$. These two metrics will look very different and are not necessarily related. In fact, in the next chapter, we will develop a family of metrics, $\overline{d}_{x_0}$ on the compactification $\hat{X}$ in the case that X is a CAT(0) space. Providing the explicit metrics will emphasize the importance of distinguishing these two metrics. Having a metric on $\hat{X}$ does allow us to restate the nullity condition as follows: for every $\epsilon > 0$, all but finitely many G-translates of any compact subset C of X have diameter less than ϵ (in the metric from the compactification). Thus, if we look at the translates using the metric on X, they may stay the same size (in particular if G acts by isometries), but with the metric on $\hat{X}$, the nullity condition forces the translates of every compact set to become arbitrarily small when pushed towards the boundary.

Lastly, we say that Z is a **boundary** (or **$\mathcal{Z}$-boundary**) of G if there is a $\mathcal{Z}$-structure $(\hat{X}, Z)$ on G. This boundary is not unique; there can be multiple $\mathcal{Z}$-structures for a given group G. However, any two boundaries of G will have the same shape [Bes96].

Example 2.2.4.

1. Suppose G acts properly, freely and cocompactly on a finite-dimensional CAT(0) space, X. Then $\hat{X} = X \cup \partial X$ is a $\mathcal{Z}$-structure on G, where ∂X denotes the visual boundary. (More details about CAT(0) spaces and the visual boundary may be found in Section 3.2.1). We take care here to not say that all torsion-free CAT(0) groups admit $\mathcal{Z}$-structures as it is still an open question whether

or not a group acting geometrically on an infinite-dimensional CAT(0) space also acts geometrically on a finite-dimensional CAT(0) space.

2. [BM91] If G is a torsion-free δ-hyperbolic group, G admits a $\mathcal{Z}$-structure. Let $P_\rho(G)$ be an appropriately chosen Rips complex of G and ∂G the Gromov boundary. Then, with an appropriate topology, $\hat{P}_\rho(G) = P_\rho(G) \cup \partial G$ is a $\mathcal{Z}$-structure for G.

3. In [Bes96], Bestvina outlines a method of placing a $\mathcal{Z}$-structure on the Baumslag-Solitar group BS$(1,2)$ by modifying the traditional universal cover of the presentation 2-complex. As BS$(1,2)$ is neither CAT(0) nor δ-hyperbolic, this example illustrates how $\mathcal{Z}$-structures allow us to approach the study of group boundaries for different classes of groups.

4. Osajda and Przytycki in [OP09] prove that all torsion-free systolic groups admit $\mathcal{Z}$-structures.

5. Tirel [Tir11] showed that if two groups G and H admit $\mathcal{Z}$-structures, so do $G \times H$ and $G \star H$.

6. Dahmani [Dah03] showed that if a group G is hyperbolic relative to a collection of subgroups, and each of these subgroups admit a $\mathcal{Z}$-structure, then G admits a $\mathcal{Z}$-structure.

7. Martin [Mar14] provides conditions for building a $\mathcal{Z}$-structure for the fundamental group of a complex of groups over a finite simplicial complex.

As each of the above examples illustrates, groups must be torsion-free if they are to admit a $\mathcal{Z}$-structure. In [Dra06], Dranishnikov generalized Bestvina's definition to allow for groups with torsion. In particular, he omitted the requirement that G act freely on X and replaced it with the requirement that the action of G is geometric (that is, proper, cocompact, and by isometries). He also loosened the restriction that $\hat{X}$ be an ER to being an AR. This change permits $\hat{X}$ to be infinite-dimensional. Using Dranishnikov's definition then, we need not be as restrictive in our above example for CAT(0) spaces. We may now say that CAT(0) groups admit $\mathcal{Z}$-structures (in the sense of Dranishnikov). There is one immediate drawback in permitting infinite-dimensionality of $\hat{X}$: Z could potentially be infinite-dimensional. We will show in the next section that this is not the case, but the proof is not immediate as in the case of Bestvina's original definition. Because it was unknown if $\mathcal{Z}$-boundaries had finite covering dimension, Dranishnikov used cohomological dimension of the boundary to state and prove many of his results, unlike Bestvina's results which used Lebesgue covering dimension. After proving Theorem 2.3.7, one can then easily go back and restate Dranishnikov's results from [Dra06] by replacing the cohomological dimension of the boundary with the Lebesgue covering dimension of the boundary. (We note that this replacement is valid as it is a standard fact that in a space with finite Lebesgue covering dimension, covering dimension and cohomological dimension coincide. See for example [Wal81, Theorem 3.2(b)]. Thus, Theorem 2.3.7 provides a connection between the results on $\mathcal{Z}$-boundaries as presented by Bestvina and Dranishnikov.

Since we will be working with many of deviations from Bestvina's original definition of a $\mathcal{Z}$-structure, we introduce some notation in hopes of highlighting what conditions have been changed or removed. We will always use the notation **$\mathcal{Z}$-structure** to denote Bestvina's original definition. If we remove the requirement that the action be free, we say G admits a **$\mathcal{Z}^{n.f.}$-structure**. If $\hat{X}$ is an AR, rather than an ER, we say G admits a **$\mathcal{Z}_{AR}$-structure**. Thus, a $\mathcal{Z}_{AR}^{n.f}$-structure on a group is a $\mathcal{Z}$-structure in which $\hat{X}$ is an AR and the group need not be torsion-free.

As mentioned above, we show in the next section that the dimension of $\mathcal{Z}$-boundaries (in the sense of Dranishnikov) is finite. This fact was already known in the case of CAT(0) and hyperbolic group boundaries (see [Swe99], [Gro87]). Because these two special cases served as models for the definition of $\mathcal{Z}$-boundaries, proving finite-dimensionality of Dranishnikov's $\mathcal{Z}$-boundaries seemed promising. In fact, our main result was motivated by attempting to generalize the following theorem of Swenson on boundaries of CAT(0) spaces:

Theorem 2.2.5. *[Swe99] If X is a CAT(0) space which admits a cocompact action* *by isometries, then ∂X is finite-dimensional.*

In the statement of Swenson's theorem, the action of the group on X is not required to be proper. Thus, to obtain a full generalization of Swenson's theorem, we must also omit the properness condition that is contained in both Bestvina and Dranishnikov's definition of a $\mathcal{Z}$-structure. Simply removing properness from either definition will not suffice because the statement of the nullity condition is dependent on having a proper group action. In particular, it would be possible for infinitely many translates of any compact set C to intersect C. Therefore, we present one final generalized definition of a $\mathcal{Z}$-structure which will be used in Theorem 2.3.7 in the next section.

Definition 2.2.6. Let (X, d) be a metric space. A ***metric $\mathcal{Z}$-structure*** on X, denoted $M\mathcal{Z}$-structure, is a pair of spaces $(\hat{X}, Z)$ satisfying the following conditions:

1. $\hat{X}$ is a compact AR,
2. $\hat{X}$ is a Z-compactification of $X = \hat{X} - Z$,
3. X admits a cocompact action by isometries by some group G, and
4. $\hat{X}$ satisfies a *nullity condition* with respect to the action of G: for every $\epsilon > 0$ and for each bounded subset U of X (bounded in the d metric), there exists a compact subset C of X such that any G-translate of U that does not intersect C has diameter less than ϵ (in the metric on the compactification).

2.3 Finite-Dimensionality Results

Recall that with Bestvina's original definition of a $\mathcal{Z}$-structure, the boundary must be finite-dimensional. In fact, if $(\hat{X}, Z)$ is a $\mathcal{Z}$-structure on a group G, the dimension of $X = \hat{X} - Z$ serves as an upper bound for the dimension of Z. This argument is rather simple and we present it now for completeness and to highlight why such a simple argument cannot be used in the case of $M\mathcal{Z}$-structures. The rest of the section is dedicated to obtaining finite-dimensionality in the more general case.

We first recall the definition of Lebesgue covering dimension.

Definition 2.3.1. The ***Lebesgue covering dimension***, or ***covering dimension***, of a topological space X is the minimal integer n such that for every open cover $\mathcal{U}$ of X, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ where the order of $\mathcal{V}$ is at most $n+1$. In this case, we write $\dim X = n$. If no such integer exists, X is said to be infinite-dimensional.

When we say that the **order** of an open covering $\mathcal{U}$ of a space X is at most n, we mean that each $x \in X$ is in at most $n+1$ elements of $\mathcal{U}$. If (X, d) is a metric space, we define the **mesh** of a cover $\mathcal{U}$ as $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) | U \in \mathcal{U}\}$. One can easily check that an equivalent definition of the covering dimension of a compact metric space can be formulated as follows:

Lemma 2.3.2. *For a compact metric space X, $\dim X \leq n$ if, for every $\epsilon > 0$, there is an open cover $\mathcal{U}$ of X with $\text{mesh}\mathcal{U} < \epsilon$ and $\text{order}\mathcal{U} \leq n+1$.*

There is another method to determine if $\dim X < \infty$ for X a compact metric space using ϵ-maps. This method provides a quick proof that the dimension of $\mathcal{Z}$-boundaries is bounded above by the dimension of the space X.

Definition 2.3.3. Let $\epsilon > 0$, (X, d) a metric space, and Y a topological space. A continuous map $f : X \to Y$ is an ***ϵ-mapping*** if $\text{diam}(f^{-1}(\{y\})) < \epsilon$ for every $y \in Y$.

Theorem 2.3.4. *(See [Eng78, Theorem 1.10.12]) If X is a compact metric space and for every $\epsilon > 0$ there exists an ϵ-mapping $f : X \to Y$ where Y is a compact metric space with $\dim Y \leq n$, then $\dim X \leq n$.*

Proposition 2.3.5. *If a group G admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$, then $\dim Z \leq \dim X$.*

Proof. Let $\epsilon > 0$ and $H : \hat{X} \times [0,1] \to \hat{X}$ be a homotopy associated to the $\mathcal{Z}$-compactification. Since H is uniformly continuous and $H_0 = id_{\hat{X}}$, there is some $t_\epsilon \in (0,1]$ such that $H_{t_\epsilon}|_Z^{H_{t_\epsilon}} : Z \to H_{t_\epsilon}(Z)$ is an ϵ-map. Z is compact, being a closed subset of $\hat{X}$, and thus, by continuity, $H_{t_\epsilon}(Z)$ is also compact. Since $t_\epsilon > 0$,

$H_{t_\epsilon}(Z) \subseteq X$ and $\dim X < \infty$, then $\dim H_{t_\epsilon}(Z) \leq \dim X < \infty$. Applying Theorem 2.3.4, $\dim Z \leq \dim X$.

□

Remark 2.3.6. The above statement is true in the case that $\hat{X}$ is a $\mathcal{Z}$-compactification and Z is a $\mathcal{Z}$-set, as we only made use of the existence of a homotopy. Furthermore, the inequality in Proposition 2.3.5 is strict. Bestvina proves this in [Bes96] using a cohomological approach and Guilbault and Tirel prove this in [GT13] using tools from standard dimension theory.

The goal of the remainder of this section is to prove:

Theorem 2.3.7. *[Mor14] Let (X, d) be a metric space which admits a $M\mathcal{Z}$-structure $(\hat{X}, Z)$. Then* $\dim Z < \infty$.

Notice that we cannot use Theorem 2.3.4 to prove the main result as it requires the range space of the ϵ-mapping to be finite-dimensional. Our proof relies on the existence of a particular uniformly bounded open cover, $\mathcal{U}$, with finite order. Once such a cover exists, because of the nullity condition, elements of the cover near infinity become arbitrarily small. Thus, we can think of small neighborhoods of infinity of the boundary as being like finite-dimensional sets. Using these covers of neighborhoods of infinity, we define covers of the boundary with arbitrarily small mesh and order bounded above by the order of $\mathcal{U}$.

With the sketch of the proof in mind, we are now ready to provide the details, beginning with the existence of such a cover. We will use the notation that $B(x, r)$ is the open ball of radius r centered at x and $\overline{B(x, r)}$ is the closed ball of radius r centered at x.

Lemma 2.3.8. *[Mor14] Suppose G acts cocompactly by isometries on a proper metric space X. Then there exists a uniformly bounded open cover $\mathcal{U}$ of X with finite order.*

Proof. As the action of G on X is cocompact, there exists a compact subset C of X such that $GC = X$. Choose $r > 0$ large enough so that $C \subset B(x_0, r)$ for some $x_0 \in X$. Let $G_x = \{gx_0 | g \in G\}$ be the orbit of x_0 and let $A \subset G_x$ be a maximal r-separated subset of G_x. That is for all $x, y \in A$ with $x \neq y$, $d(x, y) \geq r$ and A is maximal with respect to this property. Let $\mathcal{U} = \{B(x, 2r) | x \in A\}$. Clearly, $\mathcal{U}$ consists of uniformly bounded open sets, which are just translates of $B(x_0, 2r)$. To show that $\mathcal{U}$ is a cover, let $y \in X$. There is some isometry $g \in G$ so that $gy \in C$. As $C \subseteq B(x_0, r)$, then $d(gy, x_0) < r$. Since g is an isometry, $d(y, g^{-1}x_0) < r$. The element $g^{-1}x_0 \in G_x$, so by maximality of A, there is some $x \in A$ such that $d(x, g^{-1}x_0) < r$. Applying the triangle inequality: $d(x, y) \leq d(x, g^{-1}x_0) + d(g^{-1}x_0, y) < r + r = 2r$. Hence, $y \in B(x, 2r)$ for some $x \in A$. To see that $\mathcal{U}$ has finite order, first observe that the number of r-separated points in $\overline{B(x_0, 4r)}$ must be finite (by compactness). If we let n be this maximal number of r-separated points, then order$\mathcal{U} \leq n$. Otherwise, there are points $x_1, x_2, ...x_{n+1} \in A$ with $\cap_{i=1}^{n+1} B(x_i, 2r) \neq \emptyset$. Thus, $r \leq d(x_i, x_j) < 4r$ for $i \neq j$ and $i, j \in \{1, 2, ..., n+1\}$. Choosing an isometry $g \in G$ with $gx_1 = x_0$, the points $gx_1, gx_2, ..., gx_{n+1}$ are r-separated and contained in $B(x_0, 4r)$, a contradiction. Hence, order$\mathcal{U} \leq n$. □

We will call a cover as described in the proof of Lemma 2.3.8 an **r-separated covering of order n**.

Proof of Theorem 2.3.7. Let $H : \hat{X} \times [0, 1] \to \hat{X}$ be a Z-set homotopy with $H_0 = id_{\hat{X}}$ and $H_t(\hat{X}) \cap Z = \emptyset$ for every $t > 0$. Let $\epsilon > 0$ and fix a metric $\hat{d}$ on $\hat{X}$.

Using Lemma 2.3.8, choose an r-separated covering $\mathcal{U}$. Let $k < \infty$ be the order of $\mathcal{U}$ and choose $U \in \mathcal{U}$. Recall that all remaining elements in the cover are certain G-translates of U. Thus, by the nullity condition, we may choose a compact set $K \subseteq X$ such that $\text{diam}_{\hat{d}} V < \epsilon/2$ for every $V \in \mathcal{U}$ with $V \cap K = \emptyset$.

Choose $\delta_1 \in (0, 1]$ small enough such that $H_\delta(Z)$ is covered by open sets $V \in \mathcal{U}$ with $\text{diam}_{\hat{d}} V < \epsilon/2$ for all $\delta \leq \delta_1$. This may be accomplished because of the nullity condition. As $\mathcal{U}$ is a bounded open cover of X, there are open sets in $\mathcal{U}$ that do not intersect K and cover a neighborhood of infinity $X - K'$ where K' is a compact set containing K. Thus, we can choose δ_1 so that for every $\delta \leq \delta_1$, $H_\delta(Z) \subseteq X - K'$.

Moreover, $H : \hat{X} \times [0,1] \to \hat{X}$ is uniformly continuous, so we may choose a $\delta_2 \in (0,1]$ so that for every $\delta \leq \delta_2$ and for each $z \in Z$, $\hat{d}(z, H_\delta(z)) < \epsilon/4$.

Take t_ϵ =min$\{\delta_1, \delta_2\}$. Note then that:

1. $H_{t_\epsilon}(Z)$ is covered by open sets $V \in \mathcal{U}$ having diameters less than $\epsilon/2$ because $t_\epsilon \leq \delta_1$ and

2. $\hat{d}(z, H_{t_\epsilon}(z)) < \epsilon/4$ for every $z \in Z$, since $t_\epsilon \leq \delta_2$.

Consider $\mathcal{V}_\epsilon = \{V \in \mathcal{U} | V \cap H_{t_\epsilon}(Z) \neq \emptyset \text{ and } V \cap K = \emptyset\}$. That is, $\mathcal{V}_\epsilon$ is an open cover of $H_{t_\epsilon}(Z)$ with mesh bounded by $\epsilon/2$ and order bounded by k.
Define $\mathcal{W}_\epsilon = \{H_{t_\epsilon}|_Z^{-1}(V) | V \in \mathcal{V}_\epsilon\}$.

Clearly, $\mathcal{W}_\epsilon$ forms a cover of Z since $\mathcal{V}_\epsilon$ forms a cover of $H_{t_\epsilon}(Z)$. Each $W \in \mathcal{W}_\epsilon$ is also open as it is the pre-image of an open set under a continuous map.

We now show that diam$W < \epsilon$ for every $W \in \mathcal{W}_\epsilon$. Let $z_1, z_2 \in W$. Then

$$\hat{d}(z_1, z_2) \leq \hat{d}(z_1, H_{t_\epsilon}(z_1)) + \hat{d}(H_{t_\epsilon}(z_1), H_{t_\epsilon}(z_2)) + \hat{d}(H_{t_\epsilon}(z_2), z_2) < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon$$

The last inequality is due to (1) and (2) from above. Every pair of points in V_g is within a distance of ϵ from one another, so diam$W < \epsilon$ for all $W \in \mathcal{W}_\epsilon$.

Lastly, the order of the cover $\mathcal{V}_\epsilon$ of $H_{t_\epsilon}(Z)$ is at most k. Since $\mathcal{W}_\epsilon$ is the set of pre-images of $\mathcal{V}_\epsilon$ under the continuous map $H_{t_\epsilon}|_Z$, then $\mathcal{W}_\epsilon$ also has order at most k.

□

Remark 2.3.9. Theorem 2.2.5 now follows directly from Theorem 2.3.7.

Remark 2.3.10. If desired, we could give an upper bound on the dimension of Z by letting m be the minimum over all orders of r-separated coverings of X. Then dim$Z \leq m - 1$.

In practice, we do not need the complete generality of Theorem 2.3.7. In particular, we usually require our group actions to be proper, cocompact, and by isometries (i.e. geometric). Adding properness eliminates the need to use Lemma 2.3.8 to obtain a single uniformly bounded cover with finite order. We conclude this section with a few words about this particular case and what it means for the dimension of $\mathcal{Z}$-boundaries in Dranishnikov's definition of a $\mathcal{Z}$-structure.

Suppose G admits a $\mathcal{Z}^{n.f.}_{AR}$-structure. Since G acts cocompactly on X, there is an open subset U of X such that $\overline{U}$ is compact and the set of all G-translates of U cover X, that is $\cup_{g\in G} gU = X$. Furthermore, since the action of G on X is proper, the set $\{g \in G | gU \cap U \neq \emptyset\}$ is finite. Let $k \in \mathbb{Z}^+$ be this finite number of G-translates of U that intersect U. Since the cover of X was formed in a "nice" way by the group action, the cover looks the same everywhere. That is, any translate of U can also intersect only k other translates. Thus, the order of a cover constructed in this way is at most k. We will call this type of cover a **covering by translations of order at most** k.

Theorem 2.3.11. *Let G be a group which admits a $\mathcal{Z}^{n.f.}_{AR}$-structure $(\hat{X}, Z)$. Then the dimension of Z is bounded above by $k-1$, where k is the minimum over orders of covering by translations of X.*

Proof. Repeat the proof of Theorem 2.3.7 choosing the cover $\mathcal{U}$ to be a covering by translations of order at most k. □

Remark 2.3.12. We could in fact lower the bound in Theorem 2.3.11 to $k-2$ by combining the strategy used here with a technique found in [GT13].

Corollary 2.3.13. *If G admits a $\mathcal{Z}$-structure $(\hat{X}, Z)$ in the sense of [Dra06], then* $\dim Z < \infty$.

2.4 Consequences of Finite-Dimensionality of $\mathcal{Z}$-Boundaries

The goal of this section is to demonstrate how knowing the covering dimension of the various formulations of $\mathcal{Z}$-boundaries can serve to unify the theories of group boundaries presented by Bestvina and Dranishnikov. First, any result about the cohomological dimension of the boundary may now be replaced with a statement concerning the Lebesgue covering dimension (or vice-versa). This fact allows us to see the results from [Bes96] and [Dra06] within the context of a consistent dimension theory. Secondly, we show that there is no advantage in restricting ourselves to working with an ER rather than an AR by proving the following:

Theorem 2.4.1. *[Mor14] Suppose a group G admits a $\mathcal{Z}_{AR}$-structure. Then G admits a $\mathcal{Z}$-structure.*

The proof of Theorem 2.4.1 relies on a more general version of Bestvina's boundary swapping theorem. Given that G admits a $\mathcal{Z}$-structure, the original version of boundary swapping [Bes96, Lemma 1.4] provides a method to take the $\mathcal{Z}$-boundary from the $\mathcal{Z}$-structure and place it on another finite-dimensional space admitting an action by G to obtain a new $\mathcal{Z}$-structure on G. Since Bestvina only worked in the ER setting, if we are to prove Theorem 2.4.1, we need a more general version of boundary swapping that allows one of the spaces to be infinite-dimensional. We can easily obtain this more general version by knowing finite-dimensionality of $\mathcal{Z}_{AR}$-boundaries. We should note here that in the next section, we present an alternative approach to boundary swapping suggested by Ferry [Fer00] that does not rely on finite-dimensionality of the boundary. This second version could also be used to prove Theorem 2.4.1.

Theorem 2.4.2 (Boundary Swapping: Version 1)**.** *Let G be a group acting properly, cocompactly, and freely on an ER X_1 and an AR X_2. Assume that X_1 and X_2 are G-homotopy equivalent and $\hat{X}_2 = X_2 \cup Z$ is a $\mathcal{Z}_{AR}$-structure on G. Then $(\hat{X}_1, Z)$ is a $\mathcal{Z}$-structure on G.*

Proof. Since G admits a $\mathcal{Z}_{AR}$-structure, by Theorem 2.3.11, $\dim Z < \infty$. A key component in Lemma 1.4 of [Bes96] is proving that $\hat{X}_1$ is an ANR. This is accomplished by showing that $\hat{X}_1$ is locally-contractible. The equivalence of local-contractibility and being an ANR requires finite-dimensionality (see [Hu65, Page 168, Theorem 7.1]). Since we know Z is finite-dimensional, even if $(\hat{X}_2, Z)$ in an infinite-dimensional $\mathcal{Z}_{AR}$-structure, $\hat{X}_1$ is finite-dimensional, so Bestvina's original proof is still valid in this more general setting. □

The last ingredient in the proof of Theorem 2.4.1 is the following deep result by West from ANR theory which will provide the ER onto which we may add the desired boundary:

Theorem 2.4.3. *[Wes77] Every compact ANR is homotopy equivalent to a finite complex.*

Proof of Theorem 2.4.1. Let $(\hat{X}, Z)$ be a $\mathcal{Z}_{AR}$-structure for G. The map $p : X \to X/G$ is a covering projection, so X/G is a compact ANR. Theorem 2.4.3 says that X/G is homotopy equivalent to a finite complex Y. Lifting the homotopies to the universal cover $\tilde{Y}$, an ER, we obtain a G-equivariant homotopy equivalence between X and $\tilde{Y}$. Applying Theorem 2.4.2, $\tilde{Y} \cup Z$ is a $\mathcal{Z}$-structure for G. □

The proof of Theorem 2.4.1 provides a different upper bound for the dimension of the boundary. Recall that a group G has **finite geometric dimension** if there exists a finite-dimensional $K(G, 1)$ space. In that case,
$\text{gd}G = \min\{\dim K | K \text{ is a } K(G, 1)\}$. Thus, we obtain the following:

Corollary 2.4.4. *Suppose a group G admits a $\mathcal{Z}_{AR}$-structure $(\hat{X}, Z)$, then*
$\dim Z \leq \text{gd}G$

Proof. Suppose that $(\hat{X}, Z)$ is a $\mathcal{Z}_{AR}$-structure on G. The finite complex Y in the proof of Theorem 2.4.1 is a finite $K(G, 1)$ space for G. Since $\tilde{Y} \cup Z$ is a $\mathcal{Z}$-structure for G, Proposition 2.3.5 ensures $\dim Z \leq \dim Y$, and therefore the dimension of Z is bounded above by the geometric dimension of G. □

Remark 2.4.5. Again, we can get the inequality in Corollary 2.4.4 to be strict using the result that $\dim Z < \dim Y$ [Bes96, GT13].

While Theorem 2.4.1 shows that there is no reason to limit our attention to ER's, and is therefore one step closer towards bridging Bestvina and Dranishnikov's definitions, it does not completely bridge the gap. One would hope to generalize

Theorem 2.4.1 as follows: If a group G, not necessarily torsion-free, admits a $\mathcal{Z}$-structure in the sense of [Dra06], then G admits a $\mathcal{Z}$-structure in the sense of [Bes96] (modulo the freeness requirement found therein.)

However, notice that the proof of Theorem 2.4.1 relies heavily on the use of covering space theory. In particular, once we permit groups with torsion, we cannot obtain the required equivariant homotopies using lifting theorems. One idea to fix this complication is to use the theory of $\underline{E}G$ complexes. Here, a group may have torsion, but stabilizers of all finite subgroups must be contractible subcomplexes. When $\underline{E}G$ complexes exist, they are well-defined up to G-equivariant homotopy equivalence. Thus, we leave the reader with an important open question that, if answered in the affirmative, would further serve to unify the theory of $\mathcal{Z}$-boundaries found in [Bes96] and [Dra06].

Question 2.4.6. If G admits a $\mathcal{Z}_{AR}^{n.f}$-structure $(\hat{X}, Z)$, does there exists a cocompact $\underline{E}G$ complex? Furthermore, must X be G-equivariantly homotopic to that complex?

2.5 An Alternative Approach to Boundary Swapping

In [Fer00, Remark 1.7(ii)], Ferry suggests an alternative approach to boundary swapping. In that remark, he restricts attention to $\mathcal{Z}$-compactifications of universal covers of finite $K(G, 1)$ complexes. However, we will show that the suggested proof applies more generally. Furthermore, since Hanner's Criterion [Han51], rather than local-contractibility, is used to detect the ANR property, the hypothesis that the boundary be finite-dimensionality is not required. As a corollary, one may obtain an alternative proof of Theorem 2.4.1 and also a new approach to proving finite-dimensionality of $\mathcal{Z}_{AR}$-boundaries. Because of its relevance to this paper, we dedicate this section to filling in the details of Ferry's approach to boundary swapping and discuss its connections to finite-dimensionality of boundaries.

We begin with the statement of the second version of boundary swapping and then introduce a few results that are needed for the proof.

Theorem 2.5.1 (Boundary Swapping: Version 2)**.** *Let G be a group acting properly and cocompactly on ARs X and Y. Suppose that $f : X \to Y$ is a G-equivariant homotopy equivalence. If $\hat{Y} = Y \cup Z$ is a $\mathcal{Z}_{AR}$-structure on G, then we may topologize $\hat{X} = X \cup Z$ so that $(\hat{X}, Z)$ is also a $\mathcal{Z}_{AR}$-structure on G.*

We first describe the topology on $\hat{X} = X \cup Z$:

Definition 2.5.2. Let $f : X \to Y$ be a proper map between ANRs. If $\hat{Y} = Y \cup Z$ is a $\mathcal{Z}$-compactification of Y, define $\overline{f} : \hat{X} = X \cup Z \to \hat{Y}$ to be the identity on Z and f on X. The topology on $\hat{X}$ is generated by the open subsets of X and sets of the form $\overline{f}^{-1}(U)$ where $U \subset \hat{Y}$ is open.

The foundation of the proof of the second version of boundary swapping is the following theorem from Ferry which describes when we know a closed subset of a space is a $\mathcal{Z}$-set. We point out here that Ferry's definition of a $\mathcal{Z}$-set is not restricted to ANR's, but allows for any metric space. We use the same terminology for both cases, but will take care to distinguish between the two in the proof of Theorem 2.5.1 as we ultimately need to work in the ANR setting.

Theorem 2.5.3. *[Fer00] Let $(\hat{X}, Z)$ and $(\hat{Y}, Z)$ be compact metric spaces that are homotopy equivalent $rel Z$ by maps and homotopies which are the identity on Z and which take the complement of Z to the complement of Z. Then Z is a $\mathcal{Z}$-set in $\hat{X}$ if and only if Z is a $\mathcal{Z}$-set in $\hat{Y}$.*

Now we are ready to give the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. By assumption we have G-equivariant maps and homotopies

$$f : X \to Y$$

$$h : Y \to X$$

$$H : Y \times [0, 1] \to Y\,,\ H_0 = f \circ h\,,\ H_1 = id_Y$$

$$F : X \times [0, 1] \to X\,,\ F_0 = h \circ f\,,\ F_1 = id_X$$

We claim that $\overline{f} = f \cup id_Z$ is a homotopy equivalence rel$Z$.

Define maps:

$$\overline{h} : \hat{Y} \to \hat{X} \text{ , by } \overline{h} = h \cup id_Z$$

$$\overline{H} : \hat{Y} \times [0,1] \to \hat{Y} \text{ , by } \overline{H}(y,t) = H(y,t) \text{ for } y \notin Z \text{ and identity else.}$$

$$\overline{F} : \hat{X} \times [0,1] \to \hat{X} \text{ , by } \overline{F}(x,t) = F(x,t) \text{ for } x \notin Z \text{ and identity else.}$$

Since the maps are the identity on Z, send complements of Z to complements of Z, and $\overline{H}_0 = \overline{f} \circ \overline{h}$, $\overline{H}_1 = id_{\hat{Y}}$, $\overline{F}_0 = \overline{H} \circ \overline{f}$, and $\overline{F}_1 = id_{\hat{X}}$, all we must check is the continuity of the maps at points of Z. Then, applying Theorem 2.5.3, we will have a homotopy $H : \hat{X} \times [0,1] \to \hat{X}$ such that $H_0 = id_{\hat{X}}$ and $H_t(\hat{X}) \cap Z = \emptyset$ for all $t > 0$. As mentioned above, Theorem 2.5.3 only requires $\hat{X}$ to be a metric space, but the existence of such a homotopy together with an application of Hannner's Criterion for ANRs [Han51] proves that $\hat{X}$ is indeed an ANR and thus $\hat{X}$ is a $\mathcal{Z}$-compactification of X.

First, note that any open set containing $z \in Z \cap \hat{X}$ must be of the form $\overline{f}^{-1}(U)$ where U is an open set in $\hat{Y}$ containing z. So, rather than picking arbitrary open sets in $\hat{X}$ for continuity arguments on Z, we can pick arbitrary open sets in $\hat{Y}$.

Let $z \in Z$ and U any open neighborhood of z in $\hat{Y}$. Choose $\epsilon > 0$ such that $\overline{B(z,\epsilon)} \subseteq U$. By cocompactness, choose compact sets $K_Y \subset Y$ and $K_X \subset X$ with $GK_Y = Y$ and $GK_X = X$. Let $L = \overline{f} \circ \overline{F}(K_X \times [0,1]) \cup \overline{H}(K_Y \times [0,1])$, a compact subset of Y. Since $\mathcal{U} = \{U, \hat{Y} - \overline{B(z,\epsilon)}\}$ forms an open cover of $\hat{Y}$, by the nullity condition, there exists a finite subset $\Gamma \subset G$ such that $\forall g \in G - \Gamma$, $gL \subset U$ or $gL \cap \overline{B(z,\epsilon)} = \emptyset$. Set

$$V = B(z,\epsilon) - \left(\bigcup_{g \in \Gamma} gL \right)$$

<u>Claim 1:</u> $\overline{h}(V) \subseteq \overline{f}^{-1}(U)$, so $\overline{h}$ is continuous at $z \in Z$.

If $y \in V \cap Y$, choose $g \in G$ such that $y \in gK_Y$. Then

$$\overline{f} \circ \overline{h}(y) \in \overline{f} \circ \overline{h}(gK_Y)$$

$$= g\overline{f} \circ \overline{h}(K_Y)$$

$$\subseteq gL$$

$$\subseteq U$$

Claim 2: $\overline{H}(V \times [0,1]) \subseteq U$, so $\overline{H}$ is continuous at $z \times [0,1]$ for all $z \in Z$.
If $y \in V \cap Y$, choose $g \in G$ such that $y \in gK_Y$. Then

$$\overline{H}(y \times [0,1]) \subseteq \overline{h}(gK_Y \times [0,1])$$

$$= g\overline{H}(K_Y \times [0,1])$$

$$\subseteq gL$$

$$\subseteq U$$

Claim 3: $\overline{F}(\overline{f}^{-1}(V) \times [0,1]) \subseteq \overline{f}^{-1}(U)$, so $\overline{F}$ is continuous at $z \times [0,1]$ for $z \in Z$.
If $x \in \overline{f}^{-1}(V) \cap X$, choose $g \in G$ such that $x \in gK_X$. Then

$$\overline{f} \circ \overline{F}(x \times [0,1]) \subseteq \overline{f} \circ \overline{F}(gK_X \times [0,1])$$

$$= g\overline{f} \circ \overline{F}(K_X \times [0,1])$$

$$\subseteq gL$$

$$\subseteq U$$

All that remains to be shown is the nullity condition. Let K be a compact subset of X and $\mathcal{U}$ any open cover of $\hat{X}$. Let

$$\mathcal{V} = \{U \in \mathcal{U} | U = \overline{f}^{-1}(W)\,,\, W \text{ open in } \hat{Y}\,,\, U \cap Z \neq \emptyset\}$$

$$\mathcal{W} = \{W | \overline{f}^{-1}(W) \in \mathcal{V}\}$$

$\mathcal{V}$ and $\mathcal{W}$ are open covers of Z in $\hat{X}$ and $\hat{Y}$, respectively.

$$A = \hat{X} - \bigcup_{V \in \mathcal{V}} V$$

is a compact subset of X, so by properness of the action, there exists a finite subset $\Gamma_1 \subseteq G$ such that $gK \cap A = \emptyset$ for all $g \in G - \Gamma_1$.
Now we must fill out the open cover in $\hat{Y}$. Let

$$B = \hat{Y} - \bigcup_{W \in \mathcal{W}} W$$

$$\mathcal{B} = \{B(y, r_y) | y \in B\,,\; r_y = \frac{1}{2} d(y, Z)\}$$

Set $\mathcal{W}' = \mathcal{W} \cup \mathcal{B}$, an open cover of $\hat{Y}$. Since B is compact and $\mathcal{B}$ is an open cover of B, there exists a finite subcover $\mathcal{B}' \subseteq \mathcal{B}$. Set

$$C = \bigcup_{\mathcal{B}'} \overline{B(y, r_y)}$$

By properness, there exists a finite subset $\Gamma_2 \subseteq G$ such that for all $g \in G - \Gamma_2$, $g\overline{f}(K) \cap C = \emptyset$. Furthermore, by the nullity condition, there exists a finite subset $\Gamma_3 \subseteq G$ such that for all $g \in G - \Gamma_3$, $g\overline{f}(K) \subseteq W$ for $W \in \mathcal{W}'$.
Thus, let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, a finite subset of G. Then $\forall g \in G - \Gamma$, $g\overline{f}(K) \subseteq W$ for $W \in \mathcal{W}$. By equivariance of the action, $gK \subseteq \overline{f}^{-1}(W) \in \mathcal{V} \subseteq \mathcal{U}$.

□

Using the second version of Boundary Swapping, we recover finite-dimensionality of the boundary, but in a much less generalized form than is found in the statement of Theorem 2.3.7. In particular, the second version of Boundary Swapping proves that a $\mathcal{Z}_{AR}$-boundary must be finite-dimensional. The proof is essentially the same as the proof of Corollary 2.4.4. Suppose that $(\hat{X}, Z)$ is a $\mathcal{Z}_{AR}$-structure on G. Then X/G is homotopy equivalent to a finite $K(G, 1)$ space for G. Lifting the homotopies gives a proper G-equivariant homotopy equivalence between X and $\tilde{K}$, the universal cover of K. By Theorem 2.5.1, $(\tilde{K} \cup Z, Z)$ is a $\mathcal{Z}$-structure on G. Proposition 2.3.5 ensures $\dim Z \leq \dim K < \infty$.

Chapter 3

Metrics on Visual Boundaries of CAT(0) Spaces

3.1 Introduction

In the previous chapter, we showed that coarse (large-scale) dimension properties of a space X can impose restrictions on the classical (small-scale) dimension of boundaries attached to X. A natural question to ask is if the converse is true. For example, one might hope to use the finite-dimensionality of ∂G, proved first in [Swe99] and following as a corollary of Theorem 2.3.7, to attack the following well-known open question:

Question 3.1.1. Does every CAT(0) group have finite asymptotic dimension?

This question provides motivation for much of the work in this chapter. Although we do not answer Question 3.1.1, a framework is developed that we hope will lead to future progress. Along the way, we prove some results that we hope are of independent interest; one such result is a weak solution to Question 3.1.1 that captures the spirit of our approach.

As is often the case with questions about CAT(0) groups, Question 3.1.1 is rooted in known facts about hyperbolic groups. Gromov observed that all hyperbolic groups have finite asymptotic dimension. A more precise bound on the asymptotic dimension, which helps to establish our point of view, is the following:

Theorem 3.1.2. *[BS07, BL07] For a hyperbolic group,* $\text{asdim}G \leq \dim\partial G + 1 = \ell\text{-}\dim\partial G + 1 < \infty$.

In this theorem 'asdim' denotes *asymptotic dimension*, 'dim' denotes *covering dimension*, and 'ℓ-dim' denotes *linearly controlled dimension*. All of these terms will be explained in Section 3.2.3. For now, we note that linearly controlled dimension is similar to, but stronger than, covering dimension; both are small-scale invariants defined using fine open covers. The difference is that ℓ-dim is a metric invariant, requiring a linear relationship between the mesh and the Lebesgue numbers of the covers used.

Implicit in the statement of Theorem 3.1.2 is that ∂G be endowed with a *visual metric*. There is a family of naturally occurring visual metrics on ∂G, but all are *quasi-symmetric* to one-another. That is enough to make $\ell\text{-}\dim\,\partial G$ well-defined. This also will be explained shortly.

We can now summarize the content of this chapter. We begin by reviewing a number of key definitions and properties from CAT(0) geometry. Next, we recall definitions of quasi-isometry and quasi-symmetry, and then we discuss variations, both small- and large-scale, on the notion of dimension. To bring the utility of linearly controlled dimension to CAT(0) spaces, it is necessary to have specific metrics on their visual boundaries. Although CAT(0) boundaries are important, well-understood, and metrizable, specific metrics have seldom been used in a significant way. In Sections 3 and 4, we develop two natural families of metrics for CAT(0) boundaries and verify a number of their basic properties. One of these families $\{d_{A,x_0}\}_{x_0\in X}^{A>0}$ was discussed in [Kap07], where B. Kleiner asked whether the induced action on ∂X of a geometric action on a proper CAT(0) space X is "nice" in some sense. After first showing that all metrics in the family $\{d_{A,x_0}\}_{x_0\in X}^{A>0}$ are quasi-symmetric in Section 3.3.1, we provide an affirmative answer to Kleiner's question with the following:

Theorem 3.3.6. *Suppose G acts geometrically on a proper CAT(0) space X, $x_0 \in X$ and $A > 0$. Then the induced action of G on $(\partial X, d_{A,x_0})$ is by quasi-symmetries.*

In Section 3.3.2, we look to prove analogs of Theorem 3.1.2 for CAT(0) spaces. The question of whether ℓ-dimension of a CAT(0) group boundary agrees with its covering dimension (under either of our metrics) is still open, but we can prove.

Theorem 3.3.7. *If G is a CAT(0) group, then $(\partial G, d_{A,x_0})$ has finite ℓ-dimension.*

As for the inequality in Theorem 3.1.2, we are thus far unable to use the ℓ-dimension of $(\partial X, d_{A,x_0})$ to make conclusions about the asymptotic dimension of X. Instead we turn to our other family of metrics $\{\overline{d}_{x_0}\}$. In some sense, these boundary metrics retain more information about the interior space X. That additional information allows us to prove the following theorem, which we view as a weak solution to Question 3.1.1. It is our primary application of the $\overline{d}_{x_0}$ metrics.

Theorem 3.4.3. *Suppose X is a geodesically complete CAT(0) space and, when endowed with the $\overline{d}_{x_0}$ metric for $x_0 \in X$, $\ell\text{-}\dim \partial X \leq n$. Then the macroscopic dimension of X is at most $2n+1$.*

In Section 5, we compare the d_{A,x_0} and $\overline{d}_{x_0}$ metrics to each other by applying them to some simple examples. We also compare them to the established visual metrics when we have a space that is both CAT(0) and hyperbolic.

Much work remain in this area. We conclude the chapter with a list of open questions.

3.2 Preliminaries

Before discussing the possible metrics and their properties, we first review CAT(0) spaces and the visual boundary, quasi-symmetries, and the various dimension theories that will be discussed. The study of metrics on the boundary begins in Section 3.3.

3.2.1 CAT(0) Spaces and their Geometry

In this section, we review the definition of CAT(0) spaces, some basic properties of these spaces, and the visual boundary. For a more thorough treatment of CAT(0) spaces, see [BH99].

Definition 3.2.1. A geodesic metric space (X, d) is a ***CAT(0) space*** if all of its geodesic triangles are no fatter than their corresponding Euclidean comparison triangles. That is, if $\Delta(p, q, r)$ is any geodesic triangle in X and $\overline{\Delta}(\overline{p}, \overline{q}, \overline{r})$ is its comparison triangle in $\mathbb{E}^2$, then for any $x, y \in \Delta$ and the comparison points $\overline{x}, \overline{y}$, then $d(x, y) \leq d_{\mathbb{E}}(\overline{x}, \overline{y})$.

A few important properties worth mentioning are that proper CAT(0) spaces are contractible, uniquely geodesic, balls in the space are convex, and the distance function is convex. Furthermore, we now record a very simple geometric property that will be used repeatedly throughout the rest of the paper.

Lemma 3.2.2. *Let (X, d) be a proper CAT(0) space and suppose $\alpha, \beta : [0, \infty) \to X$ are two geodesic rays based at the same point $x_0 \in X$. Then for $0 < s \leq t < \infty$, $d(\alpha(s), \beta(s)) \leq \frac{s}{t} d(\alpha(s), \beta(t))$.*

Proof. Let $p = \alpha(t)$, $q = \beta(t)$, $x = \alpha(s)$, and $y = \beta(s)$. Consider the geodesic triangle $\Delta(x_0, p, q)$ in X and its comparison triangle $\overline{\Delta}(\overline{x_0}, \overline{p}, \overline{q})$ in $\mathbb{E}^2$. Let $\overline{x}, \overline{y}$ be the corresponding points to x, y on $\overline{\Delta}$. (See picture below.)

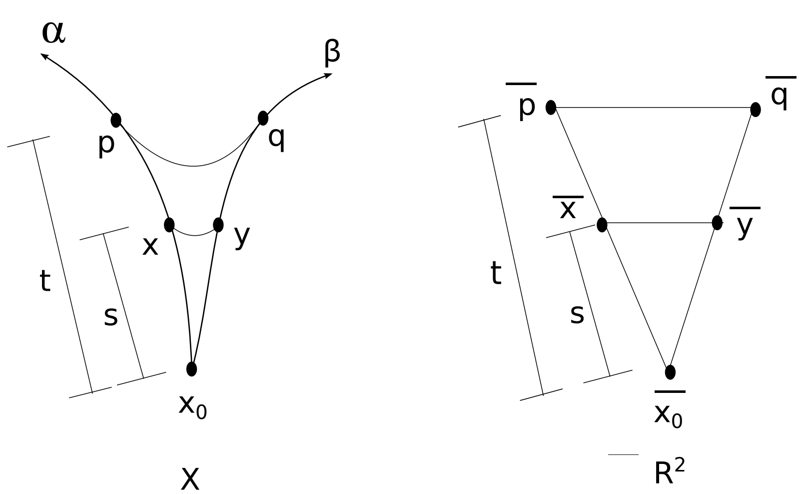

Figure 3.1: Comparison Triangle

By similar triangles in $\mathbb{E}^2$,

$$\frac{d_{\mathbb{E}}(\overline{p}, \overline{q})}{d_{\mathbb{E}}(\overline{x}, \overline{y})} = \frac{d_{\mathbb{E}}(\overline{x_0}, \overline{p})}{d_{\mathbb{E}}(\overline{x_0}, \overline{x})} = \frac{t}{s}$$

Thus, $d_{\mathbb{E}}(\overline{x}, \overline{y}) = \frac{s}{t} d_{\mathbb{E}}(\overline{p}, \overline{q}) = \frac{s}{t} d(p, q)$

Applying the CAT(0)-inequality, we obtain the desired inequality:

$$d(x, y) \leq \left(\frac{s}{t}\right) d(p, q)$$

□

We now review the definition of the boundary of CAT(0) spaces:

Definition 3.2.3. The ***boundary*** of a proper CAT(0) space X, denoted ∂X, is the set of equivalence classes of rays, where two rays are equivalent if and only if they are asymptotic. We say that two geodesic rays $\alpha, \alpha' : [0, \infty) \to X$ are ***asymptotic*** if there is some constant k such that $d(\alpha(t), \alpha'(t)) \leq k$ for every $t \geq 0$.

Once a base point is fixed, there is a a unique representative geodesic ray from each equivalence class by the following:

Proposition 3.2.4 (See [BH99] Proposition 8.2)**.** *If X is a complete CAT(0) space and $\gamma : [0, \infty) \to X$ is a geodesic ray with $\gamma(0) = x$, then for every $x' \in X$, there is a unique geodesic ray $\gamma' : [0, \infty) \to X$ asymptotic to γ and with $\gamma'(0) = x'$.*

Remark 3.2.5. In the construction of the asymptotic ray for Proposition 3.2.4, it is easy to verify that $d(\gamma(t), \gamma'(t)) \leq d(x, x')$ for all $t \geq 0$.

We may endow $\overline{X} = X \cup \partial X$, with the **cone topology**, described below, which makes ∂X a closed subspace of $\overline{X}$ and $\overline{X}$ compact (as long as X is proper). With the topology on ∂X induced by the cone topology on $\overline{X}$, the boundary is often called the **visual boundary**. In what follows, the term 'boundary' will always mean 'visual boundary'. Furthermore, we will slightly abuse terminology and call the cone topology restricted to ∂X simply the cone topology if it is clear that we are only interested in the topology on ∂X.

One way in which to describe the cone topology on $\overline{X}$, denoted $\mathcal{T}(x_0)$ for $x_0 \in X$, is by giving a basis. A basic neighborhood of a point at infinity has the following form: given a geodesic ray c and positive numbers $r > 0$, $\epsilon > 0$, let

$$U(c, r, \epsilon) = \{x \in X | d(x, c(0)) > r, d(p_r(x), c(r)) < \epsilon\}$$

where p_r is the natural projection of $\overline{X}$ onto $\overline{B}(c(0), r)$. Then a basis for the topology, $\mathcal{T}(x_0)$, on $\overline{X}$ consists of the set of all open balls $B(x, r) \subset X$, together with the collection of all sets of the form $U(c, r, \epsilon)$, where c is a geodesic ray with $c(0) = x_0$.

Remark 3.2.6. For all $x_0, x_0' \in X$, $\mathcal{T}(x_0)$ and $\mathcal{T}(x_0')$ are equivalent [BH99, Proposition 8.8].

3.2.2 Quasi-Symmetries

As we are interested in both large-scale and small-scale properties of metric spaces, we briefly discuss two different types of maps that may be used to capture the particular scale we care about. The first type of map is a quasi-isometry.

Definition 3.2.7. A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is a ***quasi-isometric embedding*** if there exists constants $A, B > 0$ such that for every $x, y \in X$, $\frac{1}{A} d_X(x, y) - B \leq d_Y(f(x), f(y)) \leq A d_X(x, y) + B$. Moreover, if there exists a $C > 0$ such that for every $z \in Y$, there is some $x \in X$ such that $d_Y(f(x), z) \leq C$, then we call f a ***quasi-isometry***.

Quasi-isometries capture the large-scale geometry of a metric space, but ignore the small scale-behavior. Thus, they are ideal when studying large scale notions of dimension, which we will discuss briefly in the next section. Since small-scale behavior is ignored, all compact metric spaces turn out to be quasi-isometric because they are all quasi-isometric to a point. Thus, quasi-isometries are not particularly useful when studying compact metric spaces. When interested in compact metric spaces and small-scale behavior, we can turn to a second type of map: quasi-symmetry.

Quasi-symmetric maps were defined to extend the notion of quasi-conformality. Since these maps care about local behavior, they are ideal when studying small scale notions of dimension, in particular, linearly controlled dimension. Quasi-symmetric maps have also played a large role in the the study of hyperbolic group boundaries. For example, it has been shown that all visual metrics on the boundary are quasi-symmetric.

We review the definition and properties that will be needed in later sections. For more information, see [TV80] or [Hei01].

Definition 3.2.8. A map $f : X \to Y$ between metric spaces is said to be ***quasi-symmetric*** if it is not constant and there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for any three points $x, y, z \in X$ satisfying $d(x, z) \leq td(y, z)$, it follows that $d(f(x), f(z)) \leq \eta(t)d(f(y), f(z))$ for all $t \geq 0$. The function η is often called a ***control function*** of f. A ***quasi-symmetry*** is a quasi-symmetric homeomorphism.

Theorem 3.2.9. *[Hei01, Proposition 10.6] If $f : X \to Y$ is η-quasi-symmetric, then $f^{-1} : f(X) \to X$ is η'-quasi-symmetric where $\eta'(t) = 1/\eta^{-1}(t^{-1})$ for $t > 0$. Moreover, if $f : X \to Y$ and $g : Y \to Z$ are η_f and η_g quasi-symmetric, respectively, then $g \circ f : X \to Z$ is $\eta_g \circ \eta_f$ quasi-symmetric.*

Theorem 3.2.10. *[Hei01, Theorem 11.3]A quasi-symmetric embedding f of a uniformly perfect space X is η-quasi-symmetric with η of the form $\eta(t) = c{*}\max\{t^{\delta}, t^{1/\delta}\}$ where $c \geq 1$ and $\delta \in (0, 1]$ depends only on f and X.*

We say that a metric space X is **uniformly perfect** if there exists a $c > 1$ such that for all $x \in X$ and for all $r > 0$, the set $B(x, r) - B(x, \frac{r}{c}) \neq \emptyset$ whenever $X - B(x, r) \neq \emptyset$. Some examples of uniformly perfect spaces include connected spaces and the Cantor ternary set. Being uniformly perfect is a quasi-symmetry invariant [Hei01].

3.2.3 A Review of Various Dimension Theories

Recall that the **covering dimension** of a space X is at most n, denoted $\dim X \leq n$, if every open cover of X has an open refinement of order at most $n+1$. The covering dimension can be studied for any topological space, in particular, spaces need not be metrizable. However, if X is a compact metric space, we may use the Lemma 2.3.2 which says that $\dim X \leq n$ if, for every $\epsilon > 0$, there is an open cover of X with mesh smaller than ϵ and order at most $n + 1$.

We now review some terminology associated to covers of a metric space. Given a cover $\mathcal{U}$ of a metric space X, we define $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) | U \in \mathcal{U}\}$. We say that the cover $\mathcal{U}$ is **uniformly bounded** if there exists some $D > 0$ such that $\text{mesh}(\mathcal{U}) \leq D$. The **order** of $\mathcal{U}$ is the smallest integer n for which each element $x \in X$ is contained in at most n elements of $\mathcal{U}$. The **Lebesgue number** of $\mathcal{U}$, denoted $\mathcal{L}(\mathcal{U})$, is defined as $\mathcal{L}(\mathcal{U}) = \inf_{x\in X}\mathcal{L}(\mathcal{U}, x)$, where $\mathcal{L}(\mathcal{U}, x) = \sup\{d(x, X-U)|U \in \mathcal{U}\}$ for each $x \in X$.

One reason for pointing out the alternate characterization of covering dimension for compact metric spaces is that the other dimension theories that we discuss here are restricted to metric spaces. These restrictions are due to the need for control of Lebesgue numbers as well as the mesh of covers. In particular, we record two properties for covers that will be used to characterize the different notions of dimension.

Let $\mathcal{U}$ be a uniformly bounded open cover of a metric space X. We say that $\mathcal{U}$ has

- Property $\mathcal{P}^n_\lambda$ if $\mathcal{L}(\mathcal{U}) \geq \lambda$ and $\text{order}(\mathcal{U}) \leq n + 1$.
- Property $\mathcal{P}^n_{\lambda,c}$ if $\mathcal{L}(\mathcal{U}) \geq \lambda$, $\text{mesh}(\mathcal{U}) \leq c\lambda$, and $\text{order}(\mathcal{U}) \leq n + 1$

This second property requires not only a given Lebesgue number, but also a linear relationship between the mesh of the cover and the Lebesgue number. We now provide definitions for the remaining dimension theories, which are grouped in terms of large-scale and small-scale properties.

Definition 3.2.11. Let X be a metric space.

1. The ***macroscopic dimension*** of X is at most n, denoted $\text{mdim}X \leq n$, if there exists a single uniformly bounded open cover of X with order $n + 1$.

2. The ***asymptotic dimension*** of X is at most n, denoted $\text{asdim}X \leq n$, if for every $\lambda > 0$, there exists a cover $\mathcal{U}$ with Property ${\mathcal{P}_\lambda}^n$.

3. The ***linearly-controlled asymptotic dimension*** of X is at most n, denoted $\ell\text{-asdim}X \leq n$, if there exists $c \geq 1$ and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, there is a cover $\mathcal{U}$ with Property $\mathcal{P}^n_{\lambda,c}$.

4. The ***Assouad-Nagata dimension*** of X is at most n, denoted $\text{ANdim}X \leq n$, if there exists $c \geq 1$, such that for all $\lambda > 0$, there is a cover $\mathcal{U}$ with Property $\mathcal{P}^n_{\lambda,c}$.

5. The ***linearly-controlled dimension*** of X is at most n, denoted $\ell\text{-dim}X \leq n$, if there exists $c \geq 1$ and $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, there is a cover $\mathcal{U}$ with Property $\mathcal{P}^n_{\lambda,c}$.

We wish to record a few facts about the various dimension theories, as well as some relationships between them:

1. Asymptotic dimension and linearly-controlled asymptotic dimension are quasi-isometry invariants of a metric space. For a nice survey of asymptotic dimension, see [BD11]. It has become widely studied due in part to its relationship to the Novikov Conjecture.

2. Assouad-Nagata dimension is a quasi-symmetry invariant [LS05]. Linearly-controlled metric dimension is a quasi-symmetry invariant for bounded metric spaces since, in this case, $\ell\text{-dim}X = \text{ANdim}X$ (see proof below).

3. In fact, linearly-controlled metric dimension is a quasi-symmetry invariant of a larger class of metric spaces: uniformly perfect metric spaces [BS07].

4. For a metric space X, we have the following comparisons:

$$\text{mdim}X \leq \text{dim}X \leq \ell\text{-dim}X \leq \text{ANdim}X$$

$$\text{mdim}X \leq \text{asdim}X \leq \ell\text{-asdim}X \leq \text{ANdim}X$$

Lemma 3.2.12. *Let X be a bounded metric space with $\ell\text{-dim}X = n$. Then $\text{ANdim}X = n$.*

Proof. We need only prove that $\text{ANdim}X \leq n$. Let $\lambda > 0$ and set $D = \text{diam}X$. Since $\ell\text{-dim}x \leq n$, there exists constants $b \geq 1$ and $\lambda_0 > 0$ such that for all $\tau \leq \lambda_0$, there is an open cover $\mathcal{U}$ of X with $\text{order}\mathcal{U} \leq n+1$, $\mathcal{L}(\mathcal{U}) \geq \tau$ and $\text{mesh}(\mathcal{U}) \leq b\tau$. Set $c = \max\{b, \frac{D}{\lambda_0}\}$.

By above, if $\lambda \leq \lambda_0$, there is an open cover $\mathcal{U}$ of X with order$\mathcal{U} \leq n+1$, $\mathcal{L}(\mathcal{U}) \geq \lambda$ and mesh$(\mathcal{U}) \leq c\lambda$. Thus, we need only verify this condition for $\lambda > \lambda_0$. To do so, let $\mathcal{U} = X$. Since the cover consists of the entire space only, the order is 1, which is clearly less than $n+1$. Trivially, $\mathcal{L}(\mathcal{U}) = \infty > \lambda$. Lastly, mesh$(\mathcal{U})$ =diam$X = D = \frac{D}{\lambda_0}\lambda_0 \leq c\lambda_0 \leq c\lambda$.

Thus, there is a $c \geq 1$ such that for every $\lambda > 0$, there is an open cover $\mathcal{U}$ of X with order at most $n+1$, mesh bounded above by $c\lambda$ and Lebesgue number at least λ, proving ANdim$X \leq n$. □

For more on the above dimension theories, see [BS07]

3.3 The d_{A,x_0} metrics

We are now ready to define the first family of metrics on the visual boundary of a CAT(0) space: the d_{A,x_0} metrics.

Fix a base point $x_0 \in X$ and choose $A > 0$. For $[\alpha], [\beta] \in \partial X$, let $\alpha : [0, \infty) \to X$ and $\beta : [0, \infty) \to X$ be the geodesic rays based at x_0 and asymptotic to $[\alpha]$ and $[\beta]$, respectively. Let $a \in (0, \infty)$ be such that $d(\alpha(a), \beta(a)) = A$. If such an a does not exist, set $a = \infty$. Then, define $d_{A,x_0} : \partial X \times \partial X \to \mathbb{R}$ by

$$d_{A,x_0}([\alpha], [\beta]) = \frac{1}{a}$$

3.3.1 Basic Properties of the d_{A,x_0} metrics

Before discussing any properties of the d_{A,x_0} metrics, we must first show that each member of the family is indeed a metric and induces the cone topology on ∂X.

Lemma 3.3.1. *If (X, d) is a CAT(0) space and $x_0 \in X$, then d_{A,x_0} for any $A > 0$ is a metric on ∂X.*

Proof. Fix a base point $x_0 \in X$ and choose $A > 0$. Let $[\alpha], [\beta], [\gamma] \in \partial X$ and $\alpha, \beta, \gamma : [0, \infty) \to X$ be the geodesic rays based at x_0 and asymptotic to $[\alpha], [\beta], [\gamma]$, respectively.

Clearly, $d_{A,x_0}([\alpha],[\alpha]) = 0$ since $d(\alpha(t),\alpha(t)) = 0$ for every $t \geq 0$ and hence $a = \infty$. If $d_{A,x_0}([\alpha],[\beta]) = 0$, then there is no $a \in (0,\infty)$ such that $d(\alpha(a),\beta(a)) = A$. By convexity of CAT(0) metric, this means $d(\alpha(t),\beta(t)) = 0$ for every $t \geq 0$. Hence, $\alpha = \beta$, which means $[\alpha] = [\beta]$. Also, $d_A([\alpha],[\beta]) = d_A([\beta],[\alpha])$ since $d(\alpha(t),\beta(t)) = d(\beta(t),\alpha(t))$. Finally, to verify the triangle inequality, suppose $a, b, c \in (0,\infty]$ satisfy

$$d_{A,x_0}([\alpha],[\beta]) = \frac{1}{a}\,,\; d_{A,x_0}([\beta],[\gamma]) = \frac{1}{b}\,,\; d_{A,x_0}([\alpha],[\gamma]) = \frac{1}{c}$$

If $c \geq a$ or $c \geq b$, then

$$d_{A,x_0}([\alpha],[\gamma]) = \frac{1}{c} \leq \frac{1}{a} \leq \frac{1}{a} + \frac{1}{b} = d_{A,x_0}([\alpha],[\beta]) + d_{A,x_0}([\beta],[\gamma])$$

or

$$d_{A,x_0}([\alpha],[\gamma]) = \frac{1}{c} \leq \frac{1}{b} \leq \frac{1}{a} + \frac{1}{b} = d_{A,x_0}([\alpha],[\beta]) + d_{A,x_0}([\beta],[\gamma])$$

Thus, the only interesting case is if $c < a$ and $c < b$. By Lemma 3.2.2

$$d(\alpha(c),\beta(c)) \leq \frac{c}{a}A$$

and

$$d(\beta(c),\gamma(c)) \leq \frac{c}{b}A$$

Then,

$$A = d(\alpha(c),\gamma(c)) \leq d(\alpha(c),\beta(c)) + d(\beta(c),\gamma(c)) \leq \frac{c}{a}A + \frac{c}{b}A = Ac\left(\frac{a+b}{ab}\right)$$

Thus,

$$c \geq \frac{ab}{a+b}$$

which proves:

$$d_{A,x_0}([\alpha],[\gamma]) = \frac{1}{c} \leq \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b} = d_{A,x_0}([\alpha],[\beta]) + d_{A,x_0}([\beta],[\gamma])$$

□

Lemma 3.3.2. *The topology induced by the d_{A,x_0} metric on ∂X is equivalent to the cone topology on ∂X.*

Proof. Fix $A > 0$ and $x_0 \in X$. Since the base point is fixed, we will simplify d_{A,x_0} to d_A. Consider the basic open set $B_{d_A}([\alpha], \epsilon)$ for $[\alpha] \in \partial X$ and $\epsilon > 0$ and let $[\beta] \in B_{d_A}([\alpha], \epsilon)$. Let $\alpha, \beta : [0, \infty) \to X$ be the unique geodesic rays based at x_0 corresponding to $[\alpha]$ and $[\beta]$, respectively. Choose $\delta > 0$ such that $B_{d_A}([\beta], \delta) \subset B_{d_A}([\alpha], \epsilon)$ and consider the basic open set in the cone topology $U(\beta, \frac{1}{\delta}, A) \cap \partial X$. Let $[\gamma] \in U(\beta, \frac{1}{\delta}, A) \cap \partial X$. Then $d(\beta(\frac{1}{\delta}), \gamma(\frac{1}{\delta})) < A$. If $a > 0$ is the point such that $d(\beta(a), \gamma(a)) = A$, then $a > \frac{1}{\delta}$. Thus, $d_A([\beta], [\gamma]) = \frac{1}{a} < \delta$. Thus, $[\gamma] \in B_{d_A}([\beta], \delta) \subset B_{d_A}([\alpha], \epsilon)$, proving $[\beta] \in U(\beta, r, A) \cap \partial X \subset B_{d_A}([\alpha], \epsilon)$.

Now consider a basic open set $U(\alpha, r, \epsilon) \cap \partial X$ in the cone topology where $r > 0$, $A > \epsilon > 0$ and $\alpha : [0, \infty) \to X$ is a geodesic ray based at x_0 . Let $[\beta] \in U(\alpha, r, \epsilon) \cap \partial X$. Choose $\delta > 0$ such that $B_d(\beta(r), \delta) \cap S(x_0, r) \subset B_d(\alpha(r), \epsilon) \cap S(x_0, r)$ and consider the basic open set in the metric topology $B_{d_A}([\beta], \frac{\delta}{Ar})$. Let $[\gamma] \in B_{d_A}([\beta], \frac{\delta}{Ar})$. Then $d_A([\beta], [\gamma]) = \frac{1}{a} < \frac{\delta}{Ar}$ where $a > 0$ is such that $d(\beta(a), \gamma(a)) = A$, which means $a > r$ since $A > \epsilon \geq \delta$. By Lemma 3.2.2, $d(\gamma(r), \beta(r)) \leq \frac{r}{a}A < r\frac{\delta}{Ar}A = \delta$. Thus, $\gamma(r) \in B_d(\beta(r), \delta) \cap S(x_0, r) \subset B_d(\alpha(r), \epsilon) \cap S(x_0, r)$, proving $[\gamma] \in U(\alpha, r, \epsilon)$. Thus $[\beta] \in B_{d_A}([\beta], \frac{\delta}{Ar}) \subset U(\alpha, r, \epsilon)$. □

Remark 3.3.3. Recall that the cone topology is defined on $\overline{X} = X \cup \partial X$. However, the preceding lemma restricts the cone topology to the boundary since there is not a natural extension of d_{A,x_0} to $\overline{X}$.

We now answer two important questions: what happens if we change A and what happens if we move the base point? It turns out that in both cases, the metrics are quasi-symmetric. Thus, by transitivity, all members of the d_{A,x_0} family are quasi-symmetric.

Lemma 3.3.4. *Let X be a proper CAT(0)-space. For all $A, A' > 0$,* $id_{\partial X} : (\partial X, d_{A,x_0}) \to (\partial X, d_{A',x_0})$ *is a quasi-symmetry.*

Proof. Fix a base point $x_0 \in X$ and suppose, without loss of generality, that $A < A'$. Clearly the identity map is a homeomorphism, so we need only verify that $id_{\partial X}$ is a quasi-symmetric map. Let $\eta(t) = \frac{A'}{A}t$; we will show this a control function for $id_{\partial X}$. Suppose that $[\alpha], [\beta], [\gamma] \in \partial X$ with $d_{A,x_0}([\alpha], [\gamma]) \leq d_{A,x_0}([\beta], [\gamma])$ for $t > 0$. Let

$\alpha, \beta, \gamma : [0, \infty) \to X$ be geodesic rays based at x_0 that are asymptotic to $[\alpha], [\beta], [\gamma]$, respectively. Let $a, b, a', b' > 0$ be such that

$$d_{A,x_0}([\alpha],[\gamma]) = \frac{1}{a}\ ,\ d_{A,x_0}([\beta],[\gamma]) = \frac{1}{b}$$

$$d_{A',x_0}([\alpha],[\gamma]) = \frac{1}{a'}\ ,\ d_{A',x_0}([\beta],[\gamma]) = \frac{1}{b'}$$

By convexity of CAT(0) metric and since $A' > A$, then $a \le a'$ and $b \le b'$. Furthermore, applying Lemma 3.2.2,

$$A = d(\beta(b), \gamma(b)) \le d_{\mathbb{E}}(\overline{\beta(b)}, \overline{\gamma(b)}) = \frac{A'b}{b'}$$

Thus, $\frac{Ab'}{A'} \le b$. Applying the above, we obtain the following inequalities:

$$d_{A',x_0}([\alpha],[\gamma]) = \frac{1}{a'} \le \frac{1}{a} = d_{A,x_0}([\alpha],[\gamma])$$

$$\le t d_{A,x_0}([\beta],[\gamma]) = t\frac{1}{b} \le t\frac{A'}{A}\frac{1}{b'} = \eta(t) d_{A',x_0}([\beta],[\gamma])$$

□

Lemma 3.3.5. *Suppose X is a complete CAT(0) space. For all $x_0, x_0' \in X$, $id_{\partial X} : (\partial X, d_{A,x_0}) \to (\partial X, d_{A,x_0'})$ is a quasi-symmetry.*

Proof. Let $x_0, x_0' \in X$ with $x_0 \ne x_0'$. We begin by assuming $A > 2d(x_0, x_0')$. We show that $\eta(t) = \left(\frac{A}{A - 2d(x_0, x_0')}\right)^2 t$ is a control function for $id_{\partial X}$. Suppose that $[\alpha], [\beta], [\gamma] \in \partial X$ and satisfy the inequality $d_{A,x_0}([\alpha],[\gamma]) \le t d_{A,x_0}([\beta],[\gamma])$ for $t > 0$. Let $\alpha, \beta, \gamma : [0, \infty) \to X$ be geodesic rays based at x_0 and asymptotic to the corresponding points in ∂X. Let $a, b \in (0, \infty)$ be such that $d_{A,x_0}(\alpha(a), \gamma(a)) = A$ and $d_{A,x_0}(\beta(b), \gamma(b)) = A$.

Since X is a complete CAT(0) space, there exists unique geodesic rays α', β', γ' in X based at x_0' and asymptotic to α, β, γ, respectively. Let $a', b' \in (0, \infty)$ be such that $d_{A,x_0'}(\alpha'(a'), \gamma'(a')) = A$ and $d_{A,x_0'}(\beta'(b'), \gamma'(b')) = A$. There are four cases to consider:

Case 1: $a' \ge a$ and $b \ge b'$. Then

$$d_{A,x_0'}([\alpha],[\gamma]) = \frac{1}{a'} \le \frac{1}{a} = d_{A,x_0}([\alpha],[\gamma]) \le t d_{A,x_0}([\beta],[\gamma]) = t\frac{1}{b}$$

$$\leq t\frac{1}{b'} = td_{A,x_0'}([\beta],[\gamma]) \leq \eta(t)d_{A,x_0'}([\beta],[\gamma])$$

<u>Case 2</u>: $a' \geq a$ and $b < b'$. Applying Lemma 3.2.2, $d(\beta'(b),\gamma'(b)) \leq \frac{Ab}{b'}$. Thus, $\frac{b'}{A}d(\beta'(b),\gamma'(b)) \leq b$. Furthermore, by Remark 3.2.5,

$$A = d(\beta(b),\gamma(b)) \leq d(\beta(b),\beta'(b)) + d(\beta'(b),\gamma'(b)) + d(\gamma'(b),\gamma(b))$$

$$\leq 2d(x_0,x_0') + d(\beta'(b),\gamma'(b))$$

Thus, $A - 2d(x_0,x_0') \leq d(\beta'(b),\gamma'(b))$

Applying all of the above,

$$d_{A,x_0'}([\alpha],[\gamma]) = \frac{1}{a'} \leq \frac{1}{a} = d_{A,x_0}([\alpha],[\gamma]) \leq td_{A,x_0}([\beta],[\gamma]) = t\frac{1}{b}$$

$$\leq t\frac{A}{d(\beta'(b),\gamma'(b))}\frac{1}{b'} \leq t\frac{A}{A-2d(x_0,x_0')}d_{A,x_0'}([\beta],[\gamma]) \leq \eta(t)d_{A,x_0'}([\beta],[\gamma])$$

<u>Case 3</u>: $a' < a$ and $b \geq b'$ Using Lemma 3.2.2, $d(\alpha(a'),\gamma(a')) \leq \frac{Aa'}{a}$. Furthermore, by Remark 3.2.5,

$$A = d(\alpha'(a'),\gamma'(a')) \leq d(\alpha'(a'),\alpha(a')) + d(\alpha(a'),\gamma(a')) + d(\gamma(a'),\gamma'(a'))$$

$$\leq 2d(x_0,x_0') + d(\alpha(a'),\gamma(a'))$$

Applying the above,

$$d_{A,x_0'}([\alpha],[\gamma]) = \frac{1}{a'} \leq \frac{A}{d(\alpha(a'),\gamma(a'))}\frac{1}{a} \leq \frac{A}{A-2d(x_0,x_0')}\frac{1}{a} = \frac{A}{A-2d(x_0,x_0')}d_{A,x_0}([\alpha],[\gamma])$$

$$\leq \frac{A}{A-2d(x_0,x_0')}td_{A,x_0}([\beta],[\gamma]) = \frac{A}{A-2d(x_0,x_0')}t\frac{1}{b} \leq \frac{A}{A-2d(x_0,x_0')}t\frac{1}{b'}$$

$$= \frac{A}{A-2d(x_0,x_0')}td_{A,x_0'}([\beta],[\gamma]) \leq \eta(t)d_{A,x_0'}([\beta],[\gamma])$$

<u>Case 4</u>: $a' < a$ and $b < b'$. Using the computations in Cases 2 and 3:

$$d_{A,x_0'}([\alpha],[\gamma]) = \frac{1}{a'} \leq \frac{A}{A-2d(x_0,x_0')}d_{A,x_0}([\alpha],[\gamma])$$

$$\leq \frac{A}{A-2d(x_0,x_0')} t d_{A,x_0}([\beta],[\gamma]) = \frac{A}{A-2d(x_0,x_0')} t \frac{1}{b} \leq t \left(\frac{A}{A-2d(x_0,x_0')}\right)^2 \frac{1}{b'}$$

$$= t \left(\frac{A}{A-2d(x_0,x_0')}\right)^2 d_{A,x_0'}([\beta],[\gamma]) = \eta(t) d_{A,x_0'}([\beta],[\gamma])$$

Thus, $\eta(t) = \left(\frac{A}{A-2d(x_0,x_0')}\right)^2 t$ is a control function for $id_{\partial X}$ for $A > 2d(x_0, x_0')$.

Now, suppose we are given any $A > 0$. Since X is a CAT(0) space, it is path connected. Let $\gamma : [0, d(x_0, x_0')] \to X$ be a geodesic segment connecting x_0 to x_0'. Let $\{y_0, y_1, ..., y_{n-1}, y_n\}$ be a partition of $[0, d(x_0, x_0')]$ where $|x_k - x_{k-1}| < \frac{A}{2}$ for $k = 1, 2, ...n$ and set $x_k = \gamma(y_k)$ for $k = 0, 1, ..., n-1$ and $x_0' = \gamma(y_n)$. From above, we know $id^k_{\partial X} : (\partial X, d_{A,x_k}) \to (\partial X, d_{A,x_{k-1}})$ is a quasi-symmetry for each k. Theorem 3.2.9 guarantees that $id_{\partial X} = id^n_{\partial X} \circ ... \circ id^1_{\partial X} : (\partial X, d_{A,x_0}) \to (\partial X, d_{A,x_0'})$ is a quasi-symmetry. □

In the future, we will use d_A to denote an arbitrary representative of the family of metrics $\{d_{A,x_0}\}$. When specific calculations are to be done, $A > 0$ should be fixed and a base point x_0 should be chosen.

In problem 46 of [Kap07], B. Kleiner asked whether the group of isometries of a CAT(0) space acts in a "nice" way on the boundary. The following theorem provides one answer.

Theorem 3.3.6. *Suppose G is a finitely generated group that acts by isometries on a complete CAT(0) space X. Then the induced action of G on $(\partial X, d_{A,x_0})$ is a quasi-symmetry. In other words, G acts by quasi-symmetries on ∂X.*

Proof. Fix a base point $x_0 \in X$ and $A > 0$. Notice that proving this theorem relies on knowing that changing base point is a quasi-symmetry, since if $\alpha, \beta, \gamma : [0, \infty) \to X$ are geodesic rays based at x_0, then

$$d_{A,x_0}([\alpha],[\gamma]) = d_{A,gx_0}([g\alpha],[g\gamma])$$

$$d_{A,x_0}([\beta],[\gamma]) = d_{A,gx_0}([g\beta],[g\gamma]).$$

This is a simple consequence of the action being by isometries. Hence, to obtain the desired inequality for a quasi-symmetric map, all we need to do is find the distances of the translated rays with respect to the base point x_0 rather than gx_0. A simple application of Theorem 3.3.5 proves g is a quasi-symmetry.

□

3.3.2 Dimension Results Using the d_A metrics

In [BL07], it is shown that the linearly controlled dimension of every compact locally self-similar metric space X is finite and ℓ-dim$X$ =dim$X$. Since hyperbolic group boundaries are compact and locally self-similar, we obtain the equality of linearly controlled dimension and covering dimension of hyperbolic group boundaries in Theorem 3.1.2. Swenson shows in [Swe99] that the boundary of a proper CAT(0) space admitting a cocompact action by isometries has finite topological dimension. Since topological dimension can be defined for arbitrary topological spaces, there was no need for a metric on the boundary to prove this fact. Now that we have the d_A family of metrics on the boundary, we can examine the linearly controlled metric dimension. We have been unable to show equality of the two dimensions, but we do show that linearly controlled dimension of a CAT(0) group boundary must be finite. This proof was motivated by the proof of Theorem 2.3.7.

Theorem 3.3.7. *Suppose G acts geometrically on a proper CAT(0)-space X. Then* $\ell\text{-dim}(\partial X, d_A) < \infty$.

This proof relies on the existence of a single cover with Property $\mathcal{P}^n_{R,4R}$ for some $R, n > 0$.

Lemma 3.3.8. *Suppose a group G acts geometrically on a proper CAT(0) space (X, d). Then for all sufficiently large R, there exists a finite order open cover $\mathcal{V}$ of X with* $\text{mesh}(\mathcal{V}) \leq 4R$ *and* $\mathcal{L}(\mathcal{V}) \geq R$.

Proof. Let $C \subseteq X$ be a compact set with $GC = X$ and choose R large enough so that $C \subseteq B(x_0, R)$ for some $x_0 \in X$. Then $\mathcal{V} = \cup_{g\in G} B(gx_0, 2R)$ is a finite order open cover of X with mesh bounded above by $4R$. Notice that the order of $\mathcal{V}$ is finite since the action of G is proper, that is only finitely many G-translates of any compact set C can intersect C. Since the cover is obtained by this nice geometric action, it must look the same everywhere. Thus, the order of $\mathcal{V}$ is bounded above by the finite number of translates of $gB(x_0, 2R)$ intersecting $B(x_0, 2R)$. Furthermore, the Lebesgue number of $\mathcal{V}$ is at least R. For if we take $x \in X$ and let $g \in G$ such that $gx \in C \subseteq B(x_0, R)$. Then $d(gx, X - B(x_0, 2R)) \geq R$. As the action is by isometries:

$$R \leq d(gx, X - B(x_0, 2r)) = d(x, g^{-1}(X - B(x_0, 2R))) = d(x, X - g^{-1}(B(x_0, 2R)))$$

$$= d(x, X - B(g^{-1}x_0, 2R))$$

Since $B(g^{-1}x_0, 2R) \in \mathcal{V}$, and $d(x, B(g^{-1}x_0, 2R)) \geq R$, then $\mathcal{L}(\mathcal{V}) \geq R$. □

Remark 3.3.9. Lemma 3.3.8 proves that $\text{mdim} X < \infty$ for a CAT(0) space admitting a geometric action.

Proof of Theorem 3.3.7. Fix $A > 0$. By Lemma 3.3.8, we may choose a sufficiently large $R > A$ so that there is a finite order open cover $\mathcal{V}$ of X with $\text{mesh}(\mathcal{V}) \leq 4R$ and $\mathcal{L}(\mathcal{V}) \geq R$. Set $n = \text{order}(\mathcal{V})$.
Set $t_\lambda = \frac{1}{\lambda}$ for each $\lambda \in (0, \infty)$, and for each $V \in \mathcal{V}$, define

$$U_V = \{[\gamma] | \gamma \text{ is a geodesic ray based at } x_0 \text{ with } \gamma(t_\lambda) \in V\}$$

We will show that $\mathcal{U} = \cup_{V \in \mathcal{V}} U_V$ is an open cover of ∂X with order bounded above by n, Lebesgue number at least λ and mesh at most $\frac{4R}{A}\lambda$.

Clearly $\mathcal{U}$ is an open cover since $\mathcal{V}$ is an open cover of X. Furthermore, since $\gamma(t_\lambda)$ can be in at most n-elements of $\mathcal{V}$, then $[\gamma]$ can be in at most n elements of $\mathcal{U}$.

We now show the Lebesgue number must be at least λ. Let $[\gamma] \in \partial X$ and γ a geodesic ray in X based at x_0 and asymptotic to $[\gamma]$. Since $\mathcal{L}(\mathcal{V}) \geq R$, there is some $V \in \mathcal{V}$ such that $d(\gamma(t_\lambda), X - V) \geq R$. Consider then $d_A([\gamma], \partial X - U_V)$. If $[\beta] \in \partial X - U_V$, then $\beta(t_\lambda) \notin V$ and hence $d(\gamma(t_\lambda), \beta(t_\lambda)) \geq R$. Letting $a \in (0, \infty)$ be such that $d(\gamma(a), \beta(a)) = A$, then $a \leq t_\lambda$ since $R \geq A$. Hence,

$$d_A([\gamma], [\beta]) = \frac{1}{a} \geq \frac{1}{t_\lambda} = \lambda$$

Hence, $d_A([\gamma], \partial X - U_V) \geq \lambda$, so $\mathcal{L}(\mathcal{V}) \geq \lambda$.

Lastly, we show mesh$(\mathcal{U}) \leq \frac{4R}{A}\lambda$. Let $[\alpha], [\beta] \in U_V$ for some $U_V \in \mathcal{U}$. Let α, β be geodesic rays in X based at x_0 and asymptotic to $[\alpha]$ and $[\beta]$, respectively. Let $a \in (0, \infty)$ be such that $d(\alpha(a), \beta(a)) = A$. Since $\alpha(t_\lambda), \beta(t_\lambda) \in V$, then $d(\alpha(t_\lambda), \beta(t_\lambda)) \leq 4R$. There are then two cases to consider:
Case 1: $d(\alpha(t_\lambda), \beta(t_\lambda)) \leq A$. Then $a \geq t_\lambda$, so

$$d_A([\alpha], [\beta]) = \frac{1}{a} \leq \frac{1}{t_\lambda} = \lambda \leq \frac{4R}{A}\lambda$$

Case 2: $A \leq d(\alpha(t_\lambda), \beta(t_\lambda)) \leq 4R$. Then $a \leq t_\lambda$, and by Lemma 3.2.2, $d(\alpha(a), \beta(a)) \leq \frac{a}{t_\lambda} d(\alpha(t_\lambda), \beta(t_\lambda))$. Thus,

$$A = d(\alpha(a), \beta(a)) \leq \frac{a}{t_\lambda} d(\alpha(t_\lambda), \beta(t_\lambda)) \leq \frac{a}{t_\lambda}(4R)$$

Rearranging, we obtain that $a \geq \frac{At_\lambda}{4R}$, and thus:

$$d_A([\alpha], [\beta]) = \frac{1}{a} \leq \frac{4R}{At_\lambda} = \frac{4R}{A}\lambda$$

Thus, there exists a $c \geq 1$ such that for every $\lambda > 0$, there is an open cover $\mathcal{U}$ of ∂X with order$(\mathcal{U}) \leq n$, $\mathcal{L}(\mathcal{U}) \geq \lambda$ and mesh$(\mathcal{U}) \leq c\lambda$, proving ℓ-dim$(\partial X, d_A) < \infty$.

□

The above proof really only required the existence of a single finite order uniformly bounded open cover with large Lebesgue number. Thus, if we know a proper CAT(0) space has finite asymptotic dimension, we do not need a group action to provide such a cover. We point out that there are some CAT(0) spaces that are known to have finite asymptotic dimension: $\mathbb{R}^n$ for all $n \geq 0$, Gromov hyperbolic CAT(0) spaces, and CAT(0) cube complexes [Wri12]. Thus, there are spaces for which the following proposition will apply.

Proposition 3.3.10. *Suppose (X, d) is a proper CAT(0) space with finite asymptotic dimension. Then ℓ-dim$(\partial X, d_A) \leq$ asdim$X$.*

Proof. Fix $A > 0$. Since asdim$X \leq n$ for some $n > 0$, there exists a uniformly bounded cover $\mathcal{V}$ with order $\mathcal{V} \leq n+1$ and $\mathcal{L}(\mathcal{V}) \geq R$ for some $R \geq A$. We may assume that this cover is also open, because if it is not, we can simply choose a larger R, "push in" the cover $\mathcal{V}$ using the Lebesgue number, and obtain a smaller open cover with the desired properties. Repeat the same argument as in the proof of Theorem 3.3.7 to obtain an open cover $\mathcal{U}$ of ∂X with order at most $n+1$, $\mathcal{L}(\mathcal{U}) \geq \lambda$ and mesh$\mathcal{U} \leq \frac{\text{mesh}\mathcal{V}}{A}\lambda$. □

3.4 The $\overline{d}_{x_0}$-metrics

To define the second family of metrics on ∂X, fix a base point $x_0 \in X$. For $[\alpha], [\beta] \in \partial X$, let $\alpha : [0, \infty) \to X$ and $\beta : [0, \infty) \to X$ be the unique representatives of $[\alpha]$ and $[\beta]$ based at x_0. Define $\overline{d}_{x_0} : \partial X \times \partial X \to \mathbb{R}$ by

$$\overline{d}_{x_0}([\alpha], [\beta]) = \int_0^\infty \frac{d(\alpha(r), \beta(r))}{e^r}\, dr$$

This family of metrics, unlike the d_A metrics, takes into account the entire timespan of the geodesic rays. Due to this fact, it can naturally be extended to $\overline{X} = X \cup \partial X$. To do so, consider $x, y \in X$. Let $c_x : [0, d(x_0, x)] \to X$ be the geodesic from x_0 to x and $c_y : [0, d(x_0, y)] \to X$ the geodesic segment from x_0 to y. Extend c_x to $c'_x : [0, \infty) \to X$ by letting $c'_x(r) = x$ for all $r > d(x_0, x)$ and $c'(r) = c(r)$ otherwise. Extend c_y to $c'_y : [0, \infty) \to X$ in a similar fashion. Then

$$\overline{d}_{x_0}(x, y) = \int_0^\infty \frac{d(c'_x(r), c'_y(r))}{e^r}\, dr$$

3.4.1 Basic Properties of the $\overline{d}_{x_0}$ metrics

The following lemma that $\overline{d}_{x_0}$ is a metric is trivial.

Lemma 3.4.1. *If (X, d) is a proper CAT(0) space and $x_0 \in X$, then $\overline{d}_{x_0}$ is a metric on ∂X.*

Lemma 3.4.2. *The topology induced on $\overline{X} = X \cup \partial X$ by the $\overline{d}_{x_0}$ metric is equivalent to the cone topology on $\overline{X}$.*

Proof. Fix $x_0 \in X$. We will denote $\overline{d}_{x_0}$ by $\overline{d}$. We first show the cone topology is finer than the metric topology by considering points in X and ∂X, respectively.

Let $y \in X$ and $B_{\overline{d}}(x, \epsilon)$ be a basic open set in $\overline{X}$ containing y for some $\epsilon > 0$ and $x \in \overline{X}$. Choose $\delta > 0$ such that $B_{\overline{d}}(y, \delta) \subset B_{\overline{d}}(x, \epsilon)$ and $B_{\overline{d}}(y, \delta) \cap \partial X = \emptyset$. Consider the basic open set $B_d(y, \delta)$ in the cone topology. Clearly, $y \in B_d(y, \delta)$ and if $z \in B_d(y, \delta)$, then $z \in B_{\overline{d}}(y, \delta)$ since $\overline{d}(y, z) < d(y, z)$. Thus,

$$y \in B_d(y, \delta) \subset B_{\overline{d}}(y, \delta) \subset B_{\overline{d}}(x, \epsilon)$$

Now, let $[\beta] \in \partial X$, and consider the basic open set $B_{\overline{d}}(x, \epsilon)$ for $\epsilon > 0$ and $x \in \overline{X}$. Choose $\delta > 0$ such that $B_{\overline{d}}([\beta], \delta) \subset B_{\overline{d}}(x, \epsilon)$. Let $t > 0$ be such that $e^{-t} < \delta/4$ and consider the basic open set $U(\beta, t, \frac{\delta}{2})$ in the cone topology. Clearly $[\beta] \in U(\beta, t, \frac{\delta}{2})$, so if $[\gamma] \in U(\beta, t, \frac{\delta}{2}) \cap \partial X$, then

$$\begin{aligned}
\overline{d}([\beta], [\gamma]) &= \int_0^t \frac{d(\beta(r), \gamma(r))}{e^r}\, dr + \int_t^\infty \frac{d(\beta(r), \gamma(r))}{e^r}\, dr \\
&\leq \int_0^t \frac{d(\gamma(t), \beta(t))}{e^r}\, dr + \int_t^\infty \frac{2(r-t) + d(\gamma(t), \beta(t))}{e^r}\, dr \\
&= d(\gamma(t), \beta(t)) + \frac{2}{e^t} \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned}$$

Moreover, if $y \in U(\beta, t, \frac{\delta}{2}) \cap X$ and $c_y : [0, d(x_0, y)] \to X$ is the geodesic from x_0 to y, then

$$\begin{aligned}
\overline{d}([\beta], x) &= \int_0^t \frac{d(c_y(r), \beta(r))}{e^r}\, dr + \int_t^\infty \frac{d(c_y(r), \beta(r))}{e^r}\, dr \\
&< \int_0^t \frac{d(c_y(t), \beta(t))}{e^r}\, dr + \int_t^\infty \frac{(r-t) + d(c_y(t), \beta(t))}{e^r}\, dr \\
&< \frac{3\delta}{4} < \delta
\end{aligned}$$

These two calculations show $U(\beta, t, \frac{\delta}{2}) \subset B_{\overline{d}}([\beta, \delta])$ and thus,

$$[\beta] \in U(\beta, t, \frac{\delta}{2}) \subset B_{\overline{d}}([\beta, \delta]) \subset B_{\overline{d}}(x, \epsilon)$$

Now, we show the metric topology is finer than the cone topology, again by considering points in X and ∂X.

Let $y \in X$ and B a basic open set in the cone topology. Choose $\delta > 0$ such that $B_d(y, \delta) \subset B \cap X$. Consider the basic open set $B_{\overline{d}}(y, R)$ where $R = \frac{\delta}{e^{d(x_0,y)}}$ (if necessary, choose R smaller so that $B_{\overline{d}}(y, R) \subset X$). Let $z \in B_{\overline{d}}(y, R)$ and c_y and c_z the geodesics connecting x_0 to y and z, respectively. Set $t = \max\{d(x_0, y), d(x_0, z)\}$. Then

$$\begin{aligned}\overline{d}(y, z) &> \int_t^\infty \frac{d(c_z(r), c_y(r))}{e^r}\, dr \\ &= \int_t^\infty \frac{d(y, z)}{e^r}\, dr \\ &= \frac{d(y, z)}{e^t} \\ &\geq \frac{d(y, z)}{e^{d(x_0,y)}}\end{aligned}$$

Since $\overline{d}(y, z) < \frac{\delta}{e^{d(x_0,y)}}$, by the above calculation, $z \in B_d(y, \delta)$ proving

$$y \in B_{\overline{d}}(y, R) \subset B_d(y, \delta) \subset B$$

For a boundary point $[\beta] \in \partial X$, let $U(\alpha, t, \epsilon)$ be a basic open set containing $[\beta]$ for $t, \epsilon > 0$ and α a geodesic ray based at x_0. Choose $1 > \delta > 0$ so that $B_d(\beta(t), \delta) \cap S(x_0, t) \subset B_d(\alpha(t), \epsilon) \cap S(x_0, t)$. Consider the basic open set $B_{\overline{d}}([\beta], \frac{\delta}{e^t})$. If $[\gamma] \in B_{\overline{d}}([\beta], \frac{\delta}{e^t}) \cap \partial X$, then $d(\beta(t), \gamma(t)) < \delta$. Otherwise,

$$\overline{d}([\gamma], [\beta]) \geq \int_t^\infty \frac{\delta}{e^r}\, dr = \frac{\delta}{e^t}$$

Thus, $d(\gamma(t), \beta(t)) < \delta < \epsilon$, so $[\gamma] \in U([\alpha], t, \epsilon)$. If $x \in B_{\overline{d}}([\beta], \frac{\delta}{e^t}) \cap X$, we first notice that $\overline{d}(x, [\beta]) \geq \overline{d}([\beta], \beta(d(x_0, x))) = e^{-d(x_0,x)}$. Thus, $d(x_0, x) \geq t$, otherwise $x \notin B_{\overline{d}}([\beta], \frac{\delta}{e^t})$. By the same argument just given for a boundary point, we see that $d(c_x(t), \beta(t)) < \delta$ proving $x \in U([\alpha], t, \epsilon)$. Thus,

$$[\beta] \in B_{\overline{d}}\left([\beta], \frac{\delta}{e^t}\right) \subset U([\alpha], t, \epsilon)$$

□

Thus far, we have been unable to prove analogs of Lemma 3.3.5 and Theorem 3.3.6 for this family of metrics. However, we will see that there are some significant advantages in using $\overline{d}_{x_0}$ for comparing dimension properties of ∂X and X. In particular, we use the $\overline{d}_{x_0}$ metric to obtain a weak solution to Question 3.1.1 (which we have been unable to accomplish using the d_A metrics).

3.4.2 Dimension Results Using the $\overline{d}_{x_0}$ Metrics

Theorem 3.4.3. *Suppose X is a geodesically complete CAT(0) space and ℓ-dim$\partial X \leq n$, where ∂X is endowed with the $\overline{d}_{x_0}$ metric. Then the macroscopic dimension of X is bounded above by $2n+1$.*

The proof "pushes in" covers of the boundary obtained by knowing finite linearly controlled metric dimension of the boundary to create covers of the entire space.

Proof of Theorem 3.4.3. We will show that there exists a uniformly bounded cover $\mathcal{V}$ of X with order$\mathcal{V} \leq 2n+1$. Fix a base point $x_0 \in X$. Since ℓ-dim$\partial X \leq n$, there exists constants $\lambda_0 \in (0,1)$ and $c \geq 1$ and $n+1$-colored coverings (by a single coloring set A) $\mathcal{U}_k$ of ∂X with

- mesh$\mathcal{U}_k \leq c\lambda_k$
- $\mathcal{L}(\mathcal{U}_k) \geq \lambda_k/2$
- $\mathcal{U}_k^a$ is $\lambda_k/2$-disjoint for each $a \in A$.

where $\lambda_k \leq \lambda_0$. Such a cover is guaranteed by [BS07, Lemma 11.1.3].

Choose $R > 0$ so that $\frac{4}{e^R} < \lambda_0$ and set $\lambda_k = \frac{4}{e^{kR}}$.

Let $B_k = \{x \in X | (k+\frac{1}{2})R \leq d(x,x_0) \leq (k+\frac{3}{2})R$ be an the annulus centered at x_0 for each $k = 1,2,3,....$ We will cover each of these B_k by "pushing in" the cover $\mathcal{U}_k$ of the boundary. To do so, let

$$V_{U_k} = \{\gamma(kR,(k+2)R) | \gamma \text{ is a geodesic ray with } [\gamma] \in U_k\}$$

and $\mathcal{V} = \cup_{U_k \in \mathcal{U}_k} V_{U_k}$. Clearly $\mathcal{V}_k$ is a cover of B_k.

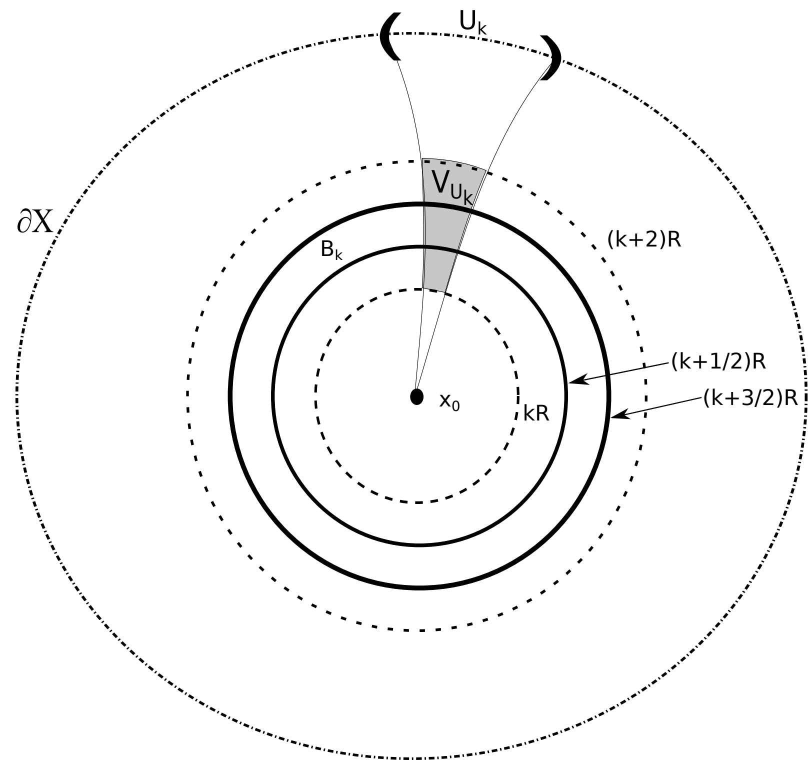

Figure 3.2: Creating Covers

Claim 1: $\mathcal{V}_k$ is $(n+1)$-colored by the same set A. That is, $\mathcal{V}_k^a$ is a disjoint collection of sets for each $a \in A$.

Suppose otherwise. That is, that there exists $V_U, V_{U'} \in \mathcal{V}_k^a$ with $V_U \cap V_{U'} \neq \emptyset$. If $x \in V_U \cap V_{U'}$ then there exists geodesic rays α and β passing through x with $[\alpha] \in U$ and $[\beta] \in U'$. Since $U, U' \in \mathcal{U}_k^a$, then $\overline{d}([\alpha],[\beta]) \geq \lambda_k/2$. Thus,

$$\frac{\lambda_k}{2} \leq \overline{d}([\alpha],[\beta]) = \int_0^\infty \frac{d(\alpha(r),\beta(r))}{e^r}\,dr$$

$$= \int_{d(x,x_0}^\infty \frac{d(\alpha(r),\beta(r))}{e^r}\,dr$$

$$\leq \int_{d(x,x_0}^\infty \frac{2(r-d(x,x_0)}{e^r}\,dr$$

$$= \frac{2}{e^{d(x,x_0)}}$$

$$< \frac{2}{e^{kR}} = \frac{\lambda_k}{2}$$

The last line provides the required contradiction. Thus, order$(\mathcal{V}_k) \leq n$ for each k.

Claim 2: For every $x, y \in V_{U_k} \in \mathcal{V}_k$ with $d(x_0, x) = (k+2)R = d(x_0, y)$, then $d(x, y) \leq 4ce^{2R}$. To show this, suppose otherwise. Choose $x, y \in \mathcal{V}_k$ with $d(x_0, x) = (k+2)R = d(x_0, y)$ and $d(x, y) > 4ce^{2R}$. Let γ_x and γ_y be geodesic rays based at x_0 with $[\gamma_x], [\gamma_y] \in U_k$ and such that $\gamma_x((k+2)R) = x$ and $\gamma_y((k+2)R) = y$. Thus,

$$\overline{d}([\gamma_x], [\gamma_y]) \geq \int_{(k+2)R}^{\infty} \frac{d(\gamma_x(r), \gamma_y(r))}{e^r}\, dr$$

$$> \int_{(k+2)R}^{\infty} \frac{4ce^{2R}}{e^r}\, dr$$

$$= \frac{4c}{e^{kR}} = c\lambda_k$$

Since $[\gamma_x], [\gamma_y] \in U_k$ and mesh$\mathcal{U}_k \leq c\lambda_k$, we obtain the desired contradiction.

Claim 3: mesh$\mathcal{V}_k \leq 4ce^{2R} + 2R$. Let $x, y \in V_{U_k} \in \mathcal{V}_k$. Let γ_x and γ_y be geodesic rays based at x_0 passing through x and y, respectively. Suppose $\gamma_x(t) = x$ and $\gamma_y(s) = y$ for $t, s \in (kR, (k+2)R)$. Without loss of generality, suppose $s \leq t$. Then

$$d(x, y) \leq d(x, \gamma_x(s)) + d(\gamma_x(s), \gamma_y(s))$$

$$= (t - s) + d(\gamma_x(s), \gamma_y(s))$$

$$\leq 2R + d(\gamma_x((k+2)R), \gamma_y((k+2)R))$$

$$\leq 2R + 4ce^{2R}$$

Thus, we have shown that mesh$\mathcal{V}_k \leq 4ce^{2R} + 2R$ and order$\mathcal{V}_k \leq n$ for every k. Since $\mathcal{V}_k \cap \mathcal{V}_{k-1} = \emptyset$, then $\cup \mathcal{V}_k$ is a uniformly bounded cover of $X - B(x_0, \frac{3}{2}R)$ with order bounded above by $2n$. Letting $\mathcal{V} = \cup \mathcal{V}_k \cup B(x_0, 2R)$ we obtain our desired cover.

□

The missing piece in the above argument that would prove finite asymptotic dimension is having arbitrarily large Lebesgue numbers for the cover. Thus, this argument is a potential step in finally answering the open asymptotic dimension question.

3.5 Examples

The previous sections highlight important properties and results that can be obtained using the d_A and $\overline{d}$ metrics. Many of the results we obtained with the given techniques worked for one metric, but not the other. That is of course not to say that the same results cannot be obtained using different methods with the other metric. However, the different results do provide interesting comparisons between the two metrics and some insight into each ones strengths or weaknesses. In this section, we highlight some other differences by showing calculations done on T_4, the four valent tree.

Example 3.5.1. *In this example, we show that $\overline{d}_{x_0}$ is a visual metric on ∂T_4, but d_A is not a visual metric on T_4.*

Recall that a metric d on the boundary of a hyperbolic space is called a **visual metric** with parameter $a > 1$ if there exists constants $k_1, k_2 > 0$ such that

$$k_1 a^{-(\zeta,\zeta')_p} \leq d(\zeta,\zeta') \leq k_2 a^{-(\zeta,\zeta')_p}$$

for all $\zeta, \zeta' \in \partial X$. [Here $(\zeta,\zeta')_p$ is the extended Gromov product based at $p \in X$. See [BH99] for more information on visual metrics.]

Fix a base point $x_0 \in X$ and $A > 0$. Let $[\alpha], [\beta] \in \partial T_4$ and let $\alpha, \beta : [0,\infty) \to T_4$ be the corresponding geodesic rays based at x_0. Set $t = \max\{r | d(\alpha(r), \beta(r)) = 0\}$. Then $d(\alpha(r), \beta(r)) = 2(r - t)$ for all $r \geq t$. A simple computation shows:

$$\overline{d}_{x_0}([\alpha],[\beta]) = \int_t^\infty \frac{2(r-t)}{e^r}\, dr = \frac{2}{e^t}$$

Furthermore, since $([\alpha],[\beta])_{x_0} = t$, we see that $\overline{d}_{x_0}$ is a visual metric on T_4 with parameter e.

Now, suppose, by way of contradiction, that d_A is visual with parameter $a > 1$. Then there exists $k_1, k_2 > 0$ such that $k_1 a^{-(\zeta,\zeta')_{x_0}} \leq d_A(\zeta,\zeta') \leq k_2 a^{-(\zeta,\zeta')_{x_0}}$ for all $\zeta, \zeta' \in \partial X$.

Choose $n \in \mathbb{Z}^+$ large enough such that $\frac{a^n}{n+1} > k_2 a^{A/2}$, which is possible since $\lim_{n\to\infty} \frac{a^n}{n+1} = \infty$. Let $\alpha, \beta : [0,\infty) \to X$ be any two proper geodesic rays based at

x_0 with the property that $\alpha(t) = \beta(t)$ for all $t \le \lceil n - \frac{A}{2} \rceil$ and $\alpha(t) \neq \beta(t)$ for all $t > \lceil n - \frac{A}{2} \rceil$ (that is, α and β are two rays that branch at time $t = \lceil n - \frac{A}{2} \rceil$. Notice then that

$$d_A([\alpha], [\beta]) = \frac{1}{\lceil n - \frac{A}{2} \rceil + \frac{A}{2}} \text{ and } ([\alpha], [\beta])_{x_0} = \left\lceil n - \frac{A}{2} \right\rceil$$

By the visibility assumption,

$$d_A([\alpha], [\beta]) \le k_2 a^{-([\alpha],[\beta])_{x_0}}$$

and thus,

$$\frac{1}{\lceil n - \frac{A}{2} \rceil + \frac{A}{2}} \le k_2 a^{-([\alpha],[\beta])_{x_0}}$$

Since $\lceil n - \frac{A}{2} \rceil \ge n - \frac{A}{2}$ and $\lceil n - \frac{A}{2} \rceil \le n - \frac{A}{2} + 1$, we obtain the following inequality:

$$\frac{1}{n+1} \le \frac{1}{\lceil n - \frac{A}{2} \rceil + \frac{A}{2}} \le k_2 a^{-([\alpha],[\beta])_{x_0}} = k_2 a^{-\lceil n - \frac{A}{2} \rceil} \le k_2 a^{-(n - \frac{A}{2})}$$

Rearranging, we see that

$$\frac{a^n}{n+1} \le k_2 a^{A/2},$$

a contradiction to the choice of n.

Proposition 3.5.2. *$id_{\partial X} : (\partial X, d_A) \to (\partial X, \overline{d})$ is not a quasi-symmetry.*

We prove this proposition by showing it in the case that $X = T_4$. For this, we need the following lemma.

Lemma 3.5.3. *$(\partial T_4, d_A)$ is uniformly perfect.*

Proof. Fix a base point $x_0 \in T_4$. It suffices to show $(\partial T_4, d_1)$ is uniformly perfect since $(\partial T_4, d_A)$ is quasi-symmetric to $(\partial T_4, d_1)$ for every $A > 0$ by Lemma 3.3.4. Let $[\alpha] \in \partial T_4$ and $\alpha : [0, \infty) \to T_4$ the ray asymptotic to $[\alpha]$ based at x_0. Since $\operatorname{diam}(T_4, d_1) = 2$, we show that $B([\alpha], r) - B([\alpha], \frac{r}{4}) \neq \emptyset$ for all $0 < r < 2$. Consider the geodesic ray $\beta : [0, \infty) \to T_4$ based at x_0 with $\alpha(t) = \beta(t)$ for all $t \le \lceil \frac{1}{r} \rceil$ and $\alpha(t) \neq \beta(t)$ for all $t > \lceil \frac{1}{r} \rceil$. Then, $d_1([\alpha], [\beta]) = \frac{1}{\lceil 1/r \rceil + 1/2}$ and thus, $d_1([\alpha], [\beta]) < r$. Moreover, $\lceil \frac{1}{r} \rceil + \frac{1}{2} \le \frac{1}{r} + 1 + \frac{1}{2} < \frac{1}{r} + \frac{3}{r}$, so $d_1([\alpha], [\beta]) > \frac{r}{4}$, proving $[\beta] \in B([\alpha], r) - B([\alpha], \frac{r}{4})$. □

Proof of Proposition 3.5.2. Let $X = T_4$. We will show that $id : (\partial T_4, d_A) \to (\partial T_4, \overline{d})$ is not a quasi-symmetry for $A = 1$ and then refer to Lemma 3.3.4 for the full claim. Fix a base point $x_0 \in T_4$ and suppose, by way of contradiction, that $id : (\partial T_4, d_1) \to (\partial T_4, \overline{d})$ is a quasi-symmetry. By Theorem 3.2.10 and Lemma 3.5.3, η must be of the form $\eta(t) = c \max\{t^\delta, t^{1/\delta}\}$ where $c \geq 1$ and $\delta \in (0, 1]$ depends only on f and X. Let $\alpha, \gamma : [0, \infty) \to T_4$ be two proper geodesic rays such that $\alpha(t) \neq \gamma(t)$ for all $t > 0$. Then

$$d_1([\alpha], [\gamma]) = \frac{1}{1/2} = 2$$

$$\overline{d}([\alpha], [\gamma]) = \int_0^\infty \frac{2r}{e^r}\, dr = 2$$

Choose $n \in \mathbb{Z}^+$ large enough such that $n - \frac{1}{\delta} \ln(2n+1) > \ln(c)$, which is possible since $\lim_{n\to\infty} n - \frac{1}{\delta} \ln(2n + 1) = \infty$.

Let $\beta : [0, \infty) \to T_4$ be a proper geodesic ray with the property that $\beta(t) = \gamma(t)$ for all $t \leq n$ and $\beta(t) \neq \gamma(t)$ for all $t > n$. Then

$$d_1([\beta], [\gamma]) = \frac{1}{n + 1/2} = \frac{2}{2n + 1}$$

$$\overline{d}([\beta], [\gamma]) = \int_n^\infty \frac{2(r - n)}{e^r}\, dr = \frac{2}{e^n}$$

Set $t = \frac{d_1([\alpha],[\gamma])}{d_1([\beta],[\gamma])} = 2n + 1$.

By the quasi-symmetry assumption,

$$\overline{d}([\alpha], [\gamma]) \leq \eta(t)\overline{d}([\beta], [\gamma])$$

and thus,

$$2 \leq \eta(2n + 1)\frac{2}{e^n}$$

$$\Rightarrow e^n \leq \eta(2n + 1) = c \max\{(2n + 1)^\delta, (2n + 1)^{1/\delta}\}$$

$$\Rightarrow e^n \leq c(2n + 1)^{1/\delta}$$

$$\Rightarrow n \leq \ln(c) + \frac{1}{\delta} \ln(2n + 1)$$

This last inequality contradicts the choice of n, proving our claim.

□

3.6 Open Questions

Since metrics on visual boundaries of CAT(0) spaces have not been widely studied, there is still much work to be done in this area. We hope that the results here show the development of these metrics is worthwhile and provides the opportunity to study CAT(0) boundaries from a different point of view, which may of course lead to answering interesting unanswered questions about these boundaries. We end with a list of open questions.

Question 3.6.1. Is there an extension of d_A to $\overline{X}$ that is equivalent to the cone topology on $\overline{X}$?

Question 3.6.2. In the proof of Theorem 3.3.6, a different control function is used for each $g \in G$. Is there a single control function for the entire group?

Question 3.6.3. Are all of the members of the $\overline{d}_{x_0}$ family of metrics quasi-symmetric?

The answer to this question is yes in the extreme cases that X is $\mathbb{R}^2$ or the four-valent tree by simple calculations. If it can be shown that the answer is yes for any CAT(0) space X, then we could easily show that the group of isometries of a CAT(0) space acts by quasi-symmetries on the boundary as in Theorem 3.3.6.

Question 3.6.4. Is the linearly controlled dimension of CAT(0) group boundaries finite when the boundary is endowed with the $\overline{d}_{x_0}$ metric? Furthermore, if the answer to Question 3.6.3 is no, can a CAT(0) boundary with two different metrics from the same family $\{\overline{d}_{x_0}\}$ have different linearly controlled dimension?

Question 3.6.5. For a hyperbolic group G, ℓ-dim$\partial X = $ dim∂X. Can the same be said for CAT(0) group boundaries? In particular, can it be shown for a CAT(0) group G, ℓ-dim$\partial X \leq$ dim∂X with respect to either the d_A metric or $\overline{d}$ metric?

Question 3.6.6. In Example 3.5.1, we showed that $\overline{d}_{x_0}$ is a visual metric on ∂T_4. Is $\overline{d}_{x_0}$ a visual metric on the boundary of any δ-hyperbolic space?

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