The Moduli Space of 5-dimensional Complex Associative Algebras

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The Power of AND

What is a Complex Associative Algebra?

A complex associative algebra is a vector space over the complex numbers equipped with a multiplication operation which satisfies the following:

- $a * (b + c) = a * b + a * c$, (Left Distributive Law)
- $(a + b) * c = a * c + b * c$, (Right Distributive Law)
- $r(a * b) = (ra) * b = a * (rb)$ for any r in $\mathbb{C}$ (Homogeneity)
- $a * (b * c) = (a * b) * c$. (Associativity)

It is often useful to encode the multiplication structure of an algebra into a ***codifferential***. An algebra is a map $V \otimes V \to V$, and this gives rise to a coderivation of the tensor coalgebra of the parity reversion $W = \Pi(V)$. Here is a quick example of an Algebra. This is the multiplication structure for the 36th 5-dimensional algebra:

$$v_4 \cdot v_4 = v_4, \qquad v_5 \cdot v_5 = v_5, \qquad v_3 \cdot v_3 = v_3,$$
$$v_3 \cdot v_1 = v_1$$

We represent this algebra structure as the codifferential d_{36}.

$$d_{36} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1}$$

The terms of the algebra correspond to the terms in the codifferential.

Constructing the Algebras

The Fundamental Theorem of Finite Dimensional Associative Algebras says that if an associative algebra structure on a finite dimensional vector space V is non nilpotent, then there is an exact sequence of algebras

$$0 \to M \to V \to W \to 0$$

where M is the *maximal nilpotent ideal* in V, and W is a *semisimple* algebra. Over the complex numbers, we have a stronger result, namely that $V = M \rtimes W$, that is, V is a semidirect product of the maximal nilpotent ideal M and a semisimple subalgebra W.

Using this theorem, we build an algebra structure on $V = M \oplus W$ from an algebra structure μ on M and an algebra structure δ on W. The extended algebra structure d is given by

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda : M \otimes W \oplus W \otimes M \to M$ is the ***module structure*** on M
$\psi : W \otimes W \to M$ is the ***cocycle***

The associativity condition $[d, d] = 0$ is equivalent to the conditions below.

$[\mu, \lambda] = 0$, compatibility condition
$[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0$, Maurer-Cartan condition
$[\delta + \lambda, \psi] = 0$, cocycle condition
We use these conditions to put constraints on the extended algebra structure d, and eventually obtain our algebras.

Cohomology and Families

Let $C^n = \hom(T^n(V), V)$ be the space of n-cochains of V. If d is an algebra, then the associativity condition $[d, d] = 0$ means that the map $D : C^n \to C^{n+1}$ given by

$$D(\varphi) = [d, \varphi]$$

satisfies $D^2 = 0$, so that we can define the *cohomology* of d by

$$H^n(d) = \ker(D : C^n \to C^{n+1}) / \mathrm{Im}(D : C^{n-1} \to C^n).$$

The cohomology in degree 2, $H^2(d)$, plays an important role in determining the deformations of d.

Some algebras are actually families of distinct algebras determined by parameters. In the moduli space of 5-dimensional algebras, each such family is naturally parametrized either by $\mathbb{P}$ or the *projective orbifold* $\mathbb{P}/S_2$, where S_2 acts by permuting the projective coordinates.

A family parametrized by $\mathbb{P}/S_2$ is

$$d_{218}(p : q) = \psi_5^{55} + p\psi_3^{21} + q\psi_3^{12} + \psi_1^{44}.$$

The isomorphism between $d_{218}(p : q)$ and $d_{218}(q : p)$ is not immediately obvious.

Deforming the Algebras

Suppose that d is an algebra, and

$$d_t = d + \psi_1 t + \psi_2 t^2 + \mathrm{ho}$$

is a 1-parameter deformation of d, where ho stands for higher order terms in the variable t. This means that d_t is a (possibly formal) power series in t, and the associativity condition is that

$$[d_t, d_t] = 0,$$

where [,] is the *Gerstenhaber* bracket.

We say that d_t is a jump deformation of d if there is some algebra given by d' such that $d_t \sim d'$ for all $t \neq 0$ in some neighborhood of $t = 0$.

We say that d_t is a smooth deformation if $d_t \nsim d_{t'}$ for $t \neq t'$ in some neighborhood of $t = 0$.

If $\langle \delta^i \rangle$ is a basis for H^2, then there is a deformation with multiple parameters t_i, called a *versal* deformation, which encodes all of the deformations of d. The computability of a versal deformation makes it possible to determine all jumps and smooth deformations of an algebra at once. The versal deformation d^∞ is of the form

$$d^\infty = d + \delta^i t_i + \mathrm{ho}$$

where we use the Einstein summation convention for repeated indices.

Example of a Versal Deformation

The algebra d_{36} is given by the formula $d_{36} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1}$. Its versal deformation is given by:

$$d^\infty = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1} + t_1\psi_1^{1,2} + t_2\psi_2^{2,2}$$

We find there is a relation on the parameters t_1 and t_2 that must be satisfied in order for the expression above to give an associative algebra:

$$t_1^2 - t_1 t_2 = 0$$

- One solution occurs when $t_1 = 0$. In this case, as long as $t_2 \neq 0$, we find the deformation is isomorphic to

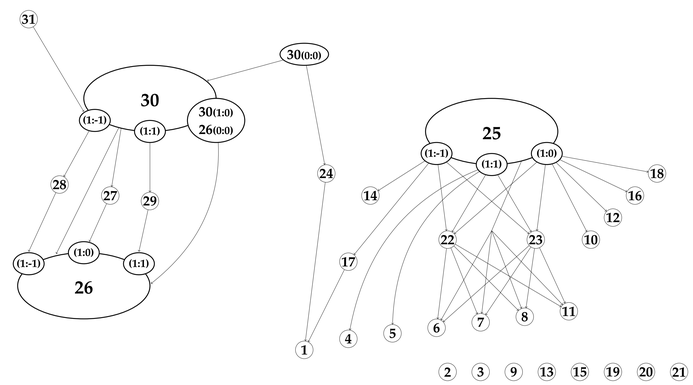
$$d_4 = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_2^{2,2} + \psi_1^{2,1}$$

- Another solution occurs when $t_1 = t_2$. In this case, again as long as $t_2 \neq 0$, we find the deformation is isomorphic to

$$d_6 = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_2^{2,2} + \psi_1^{2,1} + \psi_1^{1,3}$$

Deformations of 1|3-Dimensional Algebras

This is an example of the end goal of our project. The figure below illustrates how the moduli space is glued together by deformations for 1|3-Dimensional Algebras.



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