Arbitrarily Large Regions of Invisible Integer Lattice Points

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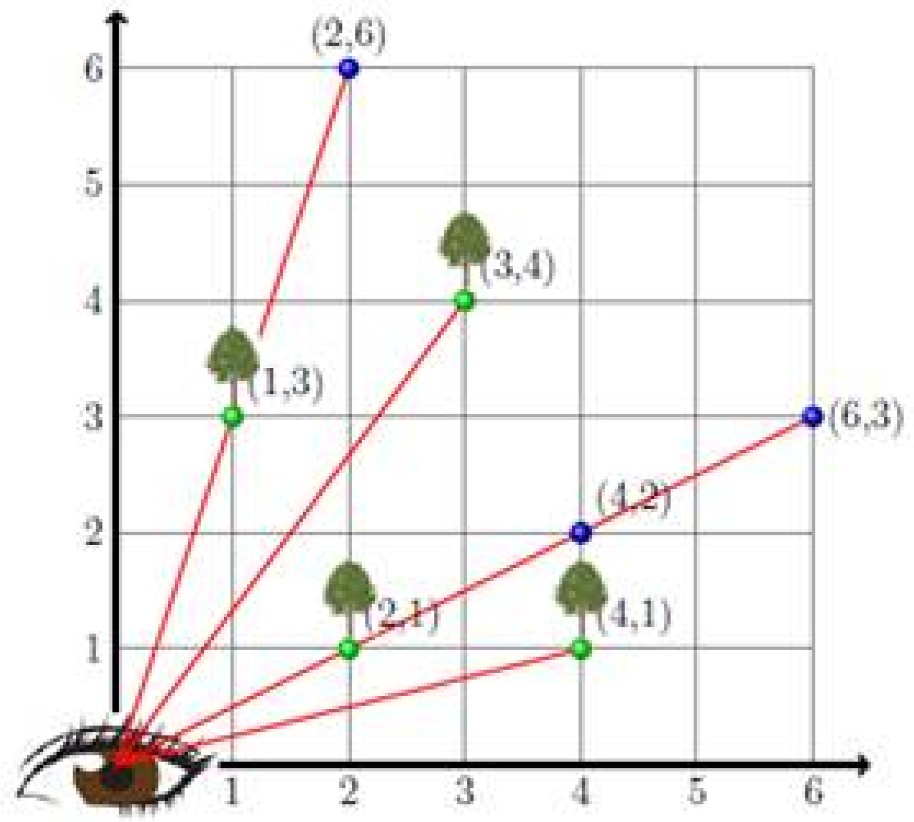
Introduction

Consider the integer lattice in the plane (i.e., all the points (x, y) such that x and y are integer values). Imagine for a moment that each integer lattice point is an infinitely thin tree. Which trees can you see from the origin? Surely on the x-axis, one can only see the points $(1, 0)$ to the right and $(-1, 0)$ to the left. But all other "trees" behind these two points are obscured from view. For example, $(2, 0)$ cannot be seen because $(1, 0)$ is blocking it. See the diagram to the right, where the blue vertices are examples of three points which are not visible since they are obscured from view by a visible point (denoted by a tree). A natural question to ask is, what fraction of integer lattice points are visible from the origin? It turns out that this is a well-known question with an answer involving a value that is ubiquitous in mathematics, namely, the Riemann-zeta function $\zeta(s)$ evaluated at $s = 2$. Phrased in equivalent terms, the probability that a randomly selected lattice point is visible from the origin is ; that is, approximately 60%. This classic result was proven in 1883 by Cesáro [1]. So what can be said about the 40% of invisible lattice points? Are there arbitrarily large patches of invisible lattice points? Yes.



Background

Definition: A point (x, y) in the integer lattice $\mathbb{Z}^2$ is invisible from the origin if $\gcd(x, y) > 1$.

Theorem: *The fraction of pairs (x, y) in the integer lattice $\mathbb{Z}^2$ such that $\gcd(x, y) = 1$ is $\zeta(2)^{-1}$ which equals $6/\pi^2 \approx .607927$. Hence approximately 40% of $\mathbb{Z}^2$ is invisible.*

$\zeta(s) = \sum_{n=1}^{\infty} 1/(n^s)$
$\zeta(2) = \sum_{n=1}^{\infty} 1/(n^2) = \pi^2/6 \approx 1.644934$
$\zeta(3) = \sum_{n=1}^{\infty} 1/(n^3) \approx 1.202057$

Definition: Let $n \in \mathbb{N}$. Then $n = p_1^{k_1} p_2^{k_2} \ldots p_{\alpha(n)}^{k_{\alpha(n)}}$ by the unique factorization theorem, where $\alpha(n)$ is the number of distinct prime factors dividing n. Define the *prime set* of n to be $\Pr(n) = \{p_i\}_{i=1}^{\alpha(n)}$.

Definition: An $n \times n$ forest with bottom-left corner (x, y) in the quadrant $\mathbb{Z}^+ \times \mathbb{Z}^+$ is denoted $F^n_{(x,y)}$. If $F^n_{(x,y)}$ is not visible from the origin, then we call $F^n_{(x,y)}$ a *hidden forest* and denote it by $H^n_{(x,y)}$.

Theorem: *For every $n \in \mathbb{N}$, there exists disjoint sets A_1 and A_2 each containing n consecutive natural numbers such that $\gcd(a_1, a_2) > 1$ whenever $a_i \in A_i$. Hence an $n \times n$ hidden forest exists for each $n \in \mathbb{N}$.*

Definition: An $n \times n \times n$ forest with the bottom-left corner (x, y, z) in the quadrant $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ is denoted as $F^n_{(x,y,z)}$ and if hidden then denoted $H^n_{(x,y,z)}$.

Results

2×2 : For $H^2_{(x,y)}$ consider the four points (x, y), $(x, y + 1)$, $(x + 1, y)$, and $(x + 1, y + 1)$. Then $\alpha(x + i) \geq 2$ and $\alpha(y + j) \geq 2$ for $i, j \in \{0, 1\}$. In particular, we need at least 4 distinct prime factors $p_1, \ldots, p_4$ such that:

$$p_1, p_2 \in \Pr(x) \qquad p_1, p_3 \in \Pr(y)$$
$$p_3, p_4 \in \Pr(x+1) \qquad p_2, p_4 \in \Pr(y+1)$$

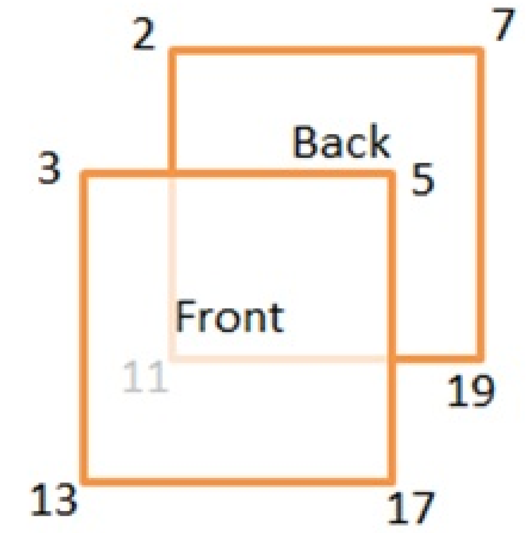
3×3 : For $H^3_{(x,y)}$ there are three distinct cases. Optimal Case: $x \in 2\mathbb{Z}$, $y \in 2\mathbb{Z}$. Let $p_1 = 2$. Then

$$p_1, p_2 \in \Pr(x) \qquad p_1, p_3 \in \Pr(y)$$
$$p_3, p_4, p_5 \in \Pr(x+1) \qquad p_2, p_4, p_6 \in \Pr(y+1)$$
$$p_1, p_6 \in \Pr(x+2) \qquad p_1, p_5 \in \Pr(y+2)$$

4×4 : For $H^4_{(x,y)}$ there are multiple distinct cases. Optimal Case: $3|x$, $3|y$, $x \in 2\mathbb{Z}$, $y \in 2\mathbb{Z}$. Let $p_1 = 2$, and $p_2 = 3$. Then

$$p_1, p_2, p_3 \in \Pr(x) \qquad p_1, p_2, p_4 \in \Pr(y)$$
$$p_4, p_5, p_6, p_7 \in \Pr(x+1) \qquad p_3, p_5, p_8, p_{10} \in \Pr(y+1)$$
$$p_1, p_8, p_9 \in \Pr(x+2) \qquad p_1, p_6, p_{11} \in \Pr(y+2)$$
$$p_2, p_{10}, p_{11} \in \Pr(x+3) \qquad p_2, p_7, p_9 \in \Pr(y+3)$$

$2 \times 2 \times 2$: We applied the Chinese Remainder Theorem to locate $H^2_{(x,y,z)}$ with $x = x_0 + 1$, $y = y_0 + 1$, and $z = z_0 + 1$. The eight corner points have coordinates $(x_0 + i, y_0 + j, z_0 + k)$ where $1 \leq i, j, k \leq 2$. In the diagram below, the prime numbers correspond to each $\gcd(x_0 + i, y_0 + j, z_0 + k)$.

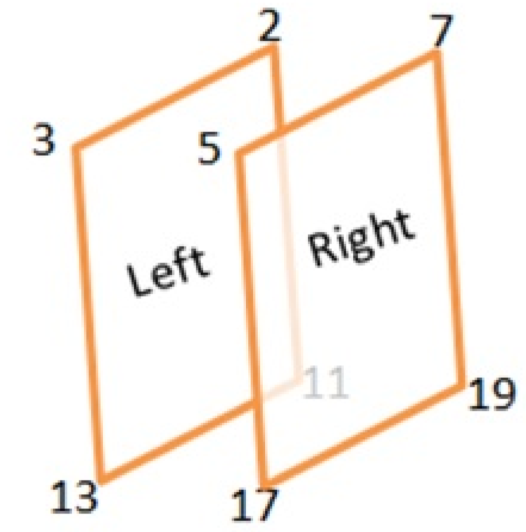


$\begin{cases} x+1 \equiv 0 \pmod{\text{Front}} \\ x+2 \equiv 0 \pmod{\text{Back}} \end{cases}$

$\begin{cases} x \equiv -1 \pmod{3315} \\ x \equiv -2 \pmod{2926} \end{cases}$

$x_0 = 573{,}494$

$x_0 + 1 = 3 \cdot 5 \cdot 13 \cdot 17 \cdot 173$
$x_0 + 2 = 2^3 \cdot 7^3 \cdot 11 \cdot 19$

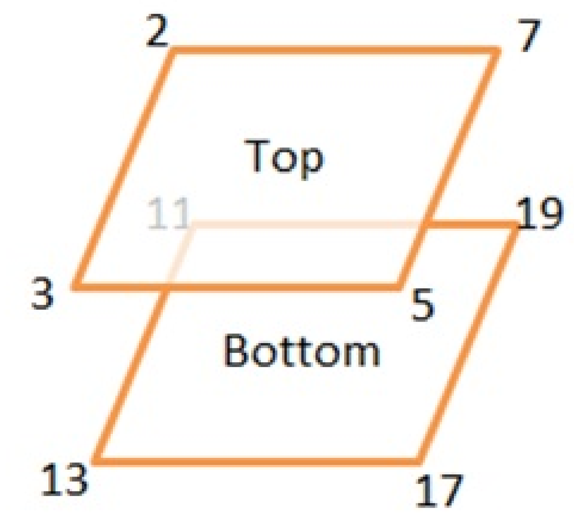


$\begin{cases} y+1 \equiv 0 \pmod{\text{Left}} \\ y+2 \equiv 0 \pmod{\text{Right}} \end{cases}$

$\begin{cases} y \equiv -1 \pmod{858} \\ y \equiv -2 \pmod{11305} \end{cases}$

$y_0 = 1{,}413{,}124$

$y_0 + 1 = 5^4 \cdot 7 \cdot 17 \cdot 19$
$y_0 + 2 = 2 \cdot 3^4 \cdot 11 \cdot 13 \cdot 61$



$\begin{cases} z+1 \equiv 0 \pmod{\text{Top}} \\ z+2 \equiv 0 \pmod{\text{Bottom}} \end{cases}$

$\begin{cases} z \equiv -1 \pmod{210} \\ z \equiv -2 \pmod{46189} \end{cases}$

$z_0 = 877{,}589$

$z_0 + 1 = 2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 199$
$z_0 + 2 = 11 \cdot 13 \cdot 17 \cdot 19^2$

References

[1] Cesáro, E. *Question 75 (Solution)*. Mathesis 3 (1883), 224-225.

[2] Mbirika, Aba. *Hidden trees in the forest: On lattice points and prime labeling of graphs*. (unpublished), 2012.

[3] *Problem Number 47*. Project Euler. An online resource for a series of challenging mathematical problems.

Future Research

More investigation will be done to find the closest $H^4_{(x,y)}$ and $H^2_{(x,y,z)}$. Also, determining a theorem for the minimal required prime factors of an arbitrary $H^n_{(x,y)}$ appears to be fairly reasonable. When using the Chinese Remainder Theorem to determine various $H^n_{(x,y)}$ different permutations of the same $n \times n$ matrix yield different results. We conjecture that the closest hidden forests can be obtained in this way.

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