Removing Singularities of the Generalized Logistic Equation on Time Scales

Amy Wells and Tristan Williams

Faculty Mentor: Chris Ahrendt

University of Wisconsin-Eau Claire

1. Introduction and Background

Goal: To unify and extend the results of ordinary differential equations and difference equations.

A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.
Examples: the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the Cantor set $\mathcal{C}$, and the set $\bigcup_{n\in\mathbb{Z}}[2n, 2n+1]$.
Note: A time scale may be bounded or even finite, but we will hereafter assume that all time scales are unbounded above.

The **forward jump operator** $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$, and we define the **backward jump operator** $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}$.

The **graininess function**, $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$.
Examples: If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ for all points t in the time scale. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ for all points t in the time scale.

Let $t \in \mathbb{T}$.

- t is **right-scattered** if $\sigma(t) > t$,
- t is **right-dense** if $\sigma(t) = t$,
- t is **left-scattered** if $\rho(t) > t$, and
- t is **left-dense** if $\rho(t) = t$.

A point t is **isolated** if it is right-scattered and left-scattered at the same time.

2. Delta-Derivatives and Integrals

Assume $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. Then f is **delta-differentiable** at t provided there exists a number, denoted $f^{\Delta}(t)$, such that

$$f^{\Delta}(t) := \lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

If f is differentiable at $t \in \mathbb{T}$, then

$$f^{\Delta}(t) = \begin{cases} \displaystyle\lim_{s\to t} \frac{f(t) - f(s)}{t - s}, & \mu(t) = 0, \\ \displaystyle\frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) > 0. \end{cases}$$

If $F : \mathbb{T} \to \mathbb{R}$ is such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$, then we say F is a **delta-antiderivative** of the function f.

If a function f has a delta-antiderivative F on $\mathbb{T}$, then for any $a, b \in \mathbb{T}$, we define

$$\int_a^b f(t)\,\Delta t := F(b) - F(a).$$

3. Generalized Exponential and Logistic Equations

The exponential function is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(0) = 1$. On the real numbers $\mathbb{R}$ the unique solution is the function $y(t) = e^{P(t)}$ where $P'(t) = p(t)$. On an arbitrary time scale, the **generalized exponential** is defined as

$$e_p(t, t_0) := \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau\right)$$

where $\xi_{\mu(\tau)}$ is a cylinder transformation of the Hilger complex plane.

Definition. *The first logistic equation is defined by* $y^{\Delta} = [\ominus(p + fy)]y$ *with* $1 + \mu(t)(p(t) + f(t)y(t)) \neq 0$ *for all* $t \in \mathbb{T}$.

Note: The symbol $\ominus$, represents **circle minus subtraction** defined by $(\ominus p)(t) = \frac{-p(t)}{1+p(t)\mu(t)}$.

Make the substitution $p(t) = Nf(t)$ for all t in the time scale and the solution to this equation is

$$y(t) = \frac{1}{e_p(t, t_0)\left[\frac{1}{y_0} + \frac{1}{N}\right] - \frac{1}{N}}.$$

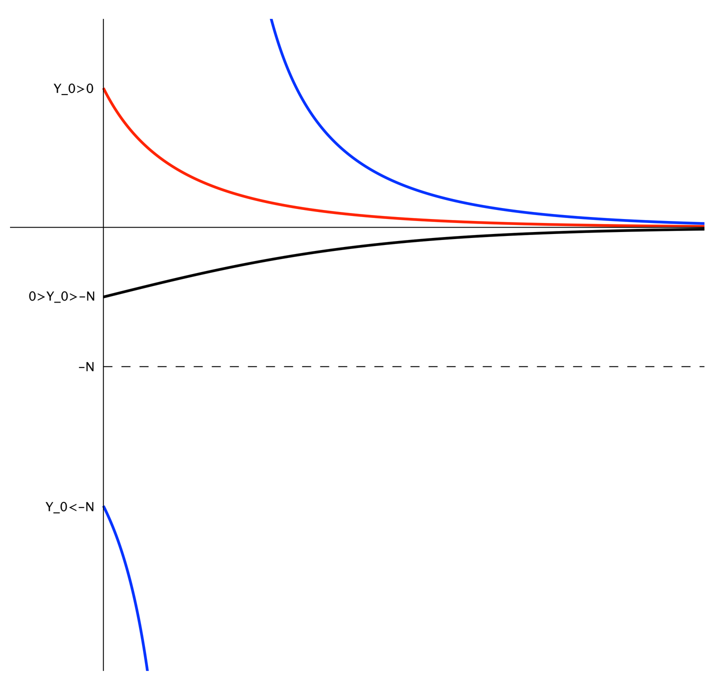
It has two equilibrium solutions: $y(t) \equiv 0$ and $y(t) \equiv -N$.
One of the things we are interested in is the asymptotic properties of this solution $y(t)$. Let $\mathbb{T}$ be a time scale that is unbounded above, and fix $t_0 \in \mathbb{T}$. We are only interested in $[t_0, \infty)_{\mathbb{T}}$ (all points in the time scale greater than t_0). Let $p(t) \equiv P > 0$ and $N > 0$.

4. Behavior of the Logistic Solution

The behavior of the solution changes depending on what y_0 is chosen to be. There are three different cases resulting in three different behaviors.

1. Let $\mathbb{T}$ be *any* time scale. If $y_0 > 0$, then $y(t) > 0$ and $y^{\Delta}(t) < 0$ for all $t \in \mathbb{T}$, $t > y_0$. Thus $y(t)$ stays in the positive realm and tends downward to 0.
2. If $0 < y_0 < -N$, then $0 < y(t) < -N$ and $y^{\Delta}(t) > 0$ for all $t \in \mathbb{T}$, $t > y_0$. Thus $y(t)$ stays in the region between $-N$ and 0 and tends upward to 0.
3. If $\mathbb{T} = \mathbb{R}$ and $y_0 < -N$, then $y(t)$ has a singularity at $k = \frac{\ln\left(\frac{y_0}{y_0+N}\right)}{P}$.

In this case the solution is only defined on the interval $[t_0, k)$. We are MOST interested in how this initial condition affects the behavior of the solution on other time scales.



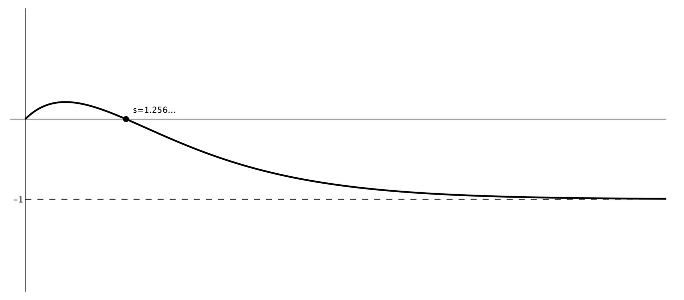
5. Our Question and Method

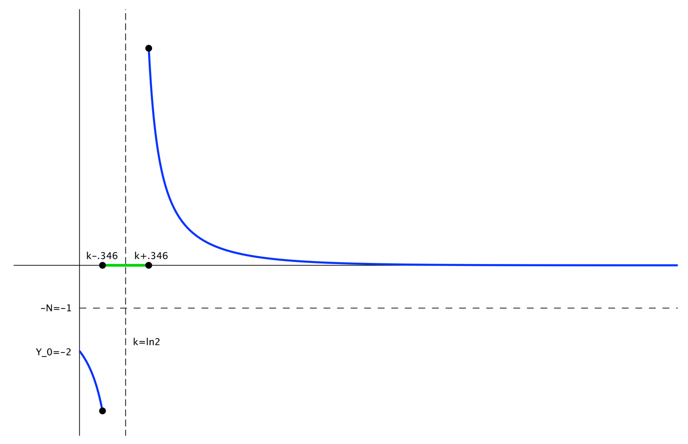
The Question: How can a time scale be constructed so that it is unbounded above and $y(t)$ is defined on $[t_0, \infty)_{\mathbb{T}}$, and further, what is the behavior of the solution on such a time scale?

The Method: At each singularity, remove from the time scale a ball of radius ϵ.
Let $\mathbb{T} = [t_0, k - \epsilon] \cup [k + \epsilon, \infty)$. Then

$$y(k + \epsilon) = \frac{N}{\frac{1+2P\epsilon}{e^{P\epsilon}} - 1}.$$

To get a better handle on this equation, we will substitute $P\epsilon = x$ and set it equal to 0. **The root of this equation shall be called** s.





6. Behavior of the Solution

Depending on the ϵ ball chosen, the equation yields three different scenarios.

1. If $0 < \epsilon < \frac{s}{P}$, then $y(k_0 + \epsilon) > 0$.
2. If $\epsilon > \frac{s}{P}$, then $y(k_0 + \epsilon) < -N$.
3. If $\epsilon = \frac{s}{P}$, then $y(k_0 + \epsilon)$ is undefined.

If $\frac{1}{P}\ln\frac{y_0}{y_0 + N} < \frac{s}{P}$, then $y(k + \epsilon) > 0$ after the first jump.

If $y(k+\epsilon) < -N$, then there is another singularity, k_1, and potentially more singularities thereafter.

Theorem.

$$k_n = \frac{1}{P}\ln\left(\prod_{i=0}^{n-1}\left(\frac{e^{2P\epsilon_i}}{1 + 2P\epsilon_i}\right)\left(\frac{y_0}{y_0 + N}\right)\right) \qquad y(k_n + \epsilon_n) = \frac{N}{\left(\frac{1+2P\epsilon_n}{e^{P\epsilon_n}}\right) - 1}.$$

Theorem. *Let* k_n *be given and let* ϵ_n *be chosen so that* k_{n+1} *is determined. Then*

$$\epsilon_n < k_{n+1} - k_n < 2\epsilon_n.$$

This result allows us to formulate the following conjecture:

Conjecture. *There exists* N *such that* $k_{N+1} - k_N < \frac{s}{P}$, *and therefore* $y(k_{N+1} + \epsilon_{N+1}) > 0$.

References

[1] M. Bohner and A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, Boston (2001).

[2] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston (2003).

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