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ABSTRACT

The reliability of a simple production - storage system in meeting a constant known demand at rate D is analyzed. This system consists of a production facility subject to stochastic failure and repair processes and a storage facility of fixed capacity. When operable, the production facility is assumed to produce at rate $M > D$ until the storage facility is full and then at rate D. Expressions are developed for the fraction of time demand is met and the probability distribution of the contents of the storage facility.

1. Introduction

The output of a production facility may be subject to occasional interruptions, but if the product can be stored, then the consumers of that product can be protected in whole or in part from the effects of the interruptions. Inventories, unreliability in meeting demand, and reliability and capacity of production facilities all have economic consequences. In order to strike an economic balance between these consequences, we need to know how they are interrelated. To that end, we present an analysis of the reliability of a simple production-storage system in meeting a constant known demand at rate D. This system consists of a storage facility (tank) of capacity X (alternatively, X can be interpreted as a self-imposed inventory limit.) and a production facility subject to random failures of random length. The production facility has a maximum production rate of $M > D$ when it is operating, and a zero production rate when it is failed. We assume that the operating policy when the production facility is operable is to produce at rate M until the storage facility is full and then to produce at rate D. We assume that demand during the period of failure is met from inventory until the production facility becomes operable again or the inventory is exhausted. If the storage facility is empty and the production facility is failed, demand is lost rather than backordered. We analyze a model with an exponential distribution of failure duration and with a constant failure duration. This is done both for a Poisson failure process and for constant "up times". This analysis obtains expressions for the fraction of demand met and for the steady state probability distribution of the amount of inventory.

Models of this kind are related to reliability theory, inventory theory, and queueing theory. Yet our work is not closely related to any of the familiar work in these fields. Inventory theory has concentrated on models with supply that is deterministic or in which the only supply uncertainty is uncertain lead times (see for example [1,6,9]). We could think of our models as involving continuous order placing and uncertain "lead times", but this will not be fruitful since the "lead times" for successive orders will overlap and not be independent random variables. Reliability theory has tended to deal with issues such as preventive maintenance, redundancy, and spare parts, rather than product inventories (see for example [3]).

There is an interesting relationship between our models and those of queueing theory. Our storage facility can be viewed as a queue of limited capacity in which products wait to be "serviced" by the market place. In our model the "service time" is constant (1/D). Our arrivals, however, form a compound process. Part of the time arrivals occur with constant interarrival times at rate M and the rest of the time at rate zero. There is a random process that switches the arrival rate from M to 0 and from 0 back to M. Recent work in queueing theory has dealt with queueing processes that change (see for example [2,4,7,8]) but these have been viewed as controlable changes such as adjustments in the service rate or the number of servers rather than random changes. The abstract of one exception [5] in the Russian literature has recently come to our attention; this paper appears to deal with a compound service process and also to be motivated by reliability considerations.

2. Poisson Failures

We consider first models involving a Poisson failure process.

Specifically, the lengths of the "up-times" between successive failures are assumed to have the exponential probability density $\lambda' e^{-\lambda' t}$, so that the probability of an up time of length t^* or less is

$$\int_0^{t^*} \lambda' e^{-\lambda' t} dt = 1 - e^{-\lambda' t^*}.$$

It can be argued that a complex production unit that can fail as a result of any one of numerous factors not primarily of an aging nature would be characterized by such a distribution. With respect to down times resulting from failures, we have considered two basic cases: (A) independent exponentially distributed down-times, and (B) constant down-times. As with up-times, the assumption of the exponential distribution (allowing for a different mean, however) for down-time seems reasonable provided that a variety of different repairs (and possibly the repair of more than one failed item during a given repair period) may be necessary at different times. The constant down-time assumption would be appropriate for production units in which the time required for a restart of an operable unit dominated the time required to remedy most types of failures to which the unit was subject. The consideration of the two types of cases (A) and (B) is also valuable in that the exponential and constant distributions are, in a sense, extreme cases that "bracket" the numerical results that would be obtained by assuming a variety of other probability distributions since the exponential and constant distributions with the same mean $1/\mu'$ represent the forms of the Gamma distribution with mean $1/\mu'$ when the shape parameter is assigned the values 1 and ∞ respectively. This point will be discussed further in the analysis of the numerical results below.

In the derivation below, we will obtain steady-state probabilities for various states of the system, including empty and full storage tank states. It should be kept in mind that the steady-state probabilities characterize the long-term average behavior of the system, as opposed to transient behavior given an initial state for the system. No attempt has been made here to study the transient behavior of the production storage system, but such an investigation would be an interesting extension of the work reported here.

A. Exponentially Distributed Repair Times

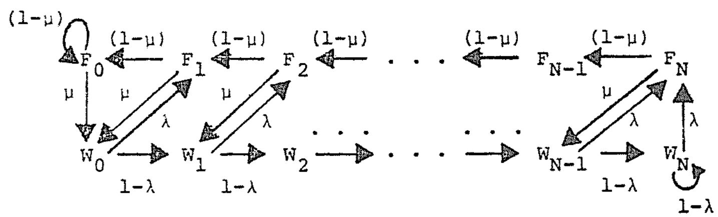
The exponential repair time case is easier to deal with analytically than the constant repair time case and yields more compact expressions, hence we will consider it first. Let the probability density function for repair time be $\mu' e^{-\mu' t}$, and approximate the system at steady-state by defining discrete time increments Δ and Δ' for the operating and failed states of the system respectively. These time increments are determined by the relations

$$\Delta = X/(M-D)N \qquad \Delta' = X/DN$$

where N is a large integer (below we will let $N \to \infty$ to obtain expressions for the continuous case). These relations indicate that it requires N of the time units Δ to fill an empty tank when the production unit is operating, and N of the time units Δ' to empty a full storage tank when the production unit is down. Defining

$$\omega \equiv \frac{\Delta}{\Delta'} = \frac{D}{M-D},$$

we have $\omega\Delta' = \Delta$, so that there are ω of the time units Δ' in a time unit Δ (This relation is independent of the values of X or N.) Let W_i denote the state of the system in which the production unit is working (up) and the amount of product in the tank is $i(M-D)\Delta$, and F_i denote the state in which the production unit is down (failed) and the amount of product in the tank is $i \cdot D \cdot \Delta'$. The diagram in Figure 1 indicates the transition probabilities between these states:



$\mu = \mu'\Delta' \quad \lambda = \lambda'\Delta$

FIGURE 1

The diagram assumes that a transition can take place only at the end of one of the discrete time intervals Δ' or Δ, and the transition probabilities at each state are approximated only to first order. (For example, the probability of a failure occurring during any interval Δ is given by

$$\int_t^{t+\Delta} \lambda' e^{-\lambda'\tau} d\tau \Big/ \int_t^{\infty} \lambda' e^{-\lambda'\tau} d\tau = (e^{-\lambda' t} - e^{-\lambda'(t+\Delta)})/e^{-\lambda t}$$

$$= 1 - e^{-\lambda'\Delta} = \lambda'\Delta - (\lambda'\Delta)^2/2! + (\lambda'\Delta)^3/3! \ \ldots$$

When the system has achieved steady-state, for each state the probability of a new arrival in any given unit of time is equal to the probability of a departure in the same unit of time. Letting $P(x)$ be the probability of being in any state x, the probability of an arrival at F_0 in time Δ is ω times the probability of a new arrival in time Δ' or $\omega(1-\mu)P(F_1)$ and similarly the probability of a departure is $\omega\mu P(F_0)$, so that

$$\omega(1-\mu)P(F_1) = \omega\mu P(F_0) \tag{A.1}$$

or

$$P(F_0) = \frac{(1-\mu)}{\mu} P(F_1) \ . \tag{A.2}$$

The analogous equation for state W_0 is

$$\omega\mu P(F_0) + \omega\mu P(F_1) = P(W_0)$$

or, substituting for $\omega\mu P(F_0)$ from A.1 ,

$$\omega P(F_1) = P(W_0). \tag{A.3}$$

For state F_1 we have

$$\omega P(F_1) = \omega(1-\mu)P(F_2) + \lambda P(W_0)$$

or, substituting for $P(W_0)$ from A.3 and dividing by $\omega(1-\mu)$,

$$P(F_2) = \frac{(1-\lambda)}{(1-\mu)} P(F_1). \tag{A.4}$$

For state W_1 the equation is

$$P(W_1) = \omega\mu P(F_2) + (1-\lambda)P(W_0)$$

which becomes, upon substitution of A.3 and A.4,

$$P(W_1) = \omega P(F_2). \tag{A.5}$$

By considering equations for $F_2, W_2, F_3, W_3, \ldots, F_{N-1}, W_{N-1}$, we can proceed as above to show

$$P(F_k) = \left(\frac{1-\lambda}{1-\mu}\right)^{k-1} P(F_1) \tag{A.6}$$

and

$$P(W_{k-1}) = \omega P(F_k) \ . \tag{A.7}$$

Finally, the equation for state F_N is

$$\lambda P(W_{N-1}) + \lambda P(W_N) = \omega P(F_N)$$

so that $$P(W_N) = \omega\left(\frac{1-\lambda}{\lambda}\right) P(F_N). \tag{A.8}$$

Equations A.2, A.6, A.7, A.8 can be used to express all $P(W_i)$ and $P(F_i)$ in terms of $P(F_1)$. Since the sum of the probabilities $P(W_i)$ and $P(F_1)$ must equal 1, we have (letting $\rho = (\frac{1-\lambda}{1-\mu})$):

$$1 = \sum_{i=0}^{N} (P(W_i) + P(F_i)) = (\frac{1-\mu}{\mu})P(F_1) + \sum_{i=0}^{N-1} (\omega+1)\rho^i P(F_1) + \frac{\omega(1-\lambda)}{\lambda}P(F_N)$$

$$= \frac{(1-\mu)}{\mu}P(F_1) + \frac{(\omega+1)(1-\rho^N)}{(1-\rho)}P(F_1) + \frac{\omega(1-\lambda)}{\lambda}\rho^{N-1}P(F_1)$$

$$= [\frac{(1-\mu)}{\mu} + \frac{(\omega+1)(1-\rho^N)(1-\mu)}{\lambda-\mu} + \frac{\omega(1-\mu)\rho^N}{\lambda}]P(F_1)$$

$$= \frac{(1-\mu)(\lambda+\omega\mu)(\lambda-\mu\rho^N)}{\mu(\lambda-\mu)\lambda}P(F_1). \qquad \text{A.9}$$

Substituting from A.9 in A.2 we derive

$$P(F_0) = \frac{\lambda}{\lambda+\omega\mu} \cdot \frac{\lambda - \mu}{\lambda-\mu\rho^N} .$$

To obtain an expression for $P(F_0)$ in terms of λ' and μ', we use the equations $\lambda = \lambda'\Delta$ and $\mu = \mu'\Delta'$:

$$P(F_0) = \frac{\lambda'\Delta}{\lambda'\Delta + \omega\mu'\Delta'} \cdot \frac{\lambda'\Delta - \mu'\Delta'}{\lambda'\Delta - \mu'\Delta'\rho^N}$$

or, since $\Delta = \Delta'\omega$,

$$P(F_0) = \frac{\lambda'}{\lambda' + \mu'} \cdot \frac{\lambda'\omega - \mu'}{\lambda'\omega - \mu'\rho^N} . \qquad \text{A.10}$$

As $\Delta, \Delta' \to 0$ (maintaining $\Delta = \Delta'\omega$), the only term in A.10 that is affected is ρ^N, so we must compute $\lim_{N\to\infty} \rho^N$.

Since

$$\rho^N = \frac{(1-\lambda'\Delta)^N}{(1-\mu'\Delta')^N} = \frac{(1-\lambda'\omega X/DN)^N}{(1-\mu'X/DN)^N} ,$$

$$\lim_{N\to\infty} \rho^N = \frac{\exp\ (-\lambda'\omega X/D)}{\exp\ (-\mu'X/D)} , \qquad \text{A.11}$$

and in the limit

$$P(F_0) = \frac{\lambda'}{\lambda' + \mu'} \cdot \frac{\lambda'\omega - \mu'}{\lambda'\omega - \mu'\ \exp\ (-(\lambda'\omega - \mu')X/D)} . \qquad \text{A.12}$$

Letting $1-S'$ = stream factor = $S = \frac{\mu'}{\mu' + \lambda'}$, A.12 may be written as

$$P(F_0) = S'\left[\frac{\omega S' - S}{\omega S' - S\ \exp[(1 - \frac{\omega S'}{S})\frac{\mu' X}{D}]}\right] . \qquad \text{A.13}$$

$P(F_0)$ may be thought of as the fraction of the time that the demand is <u>not</u> met because no product is available. In equation A.13 it is expressed as the down time factor for the manufacturing facility (S') multiplied by the fraction of down time during which the storage tank is empty. The quantity $\frac{\mu' X}{D}$ that occurs in the expression for this fraction is the ratio of the storage capacity to the average demand during a "failed" period. Note also that $P(F_0)$ is a function of only three composite variables: S, ω, and $\mu'X/D$.

When $\omega S' = S$, the expression in A.13 is indeterminate. The value of $P(F_0)$ in this case is given by

$$P(F_0) = \frac{S'}{1 + \mu' X/D} . \qquad \text{A.14}$$

The quantity $S' - P(F_0)$ is the additional fraction of time demand is met due to the presence of the storage tank. An expression for $P(W_N)$, the probability that the tank is full and the production facility is operating at rate D, may be derived in a similar fashion:

$$P(W_N) = S \cdot \frac{\omega S' - S}{\omega S' \exp[(\frac{\omega S'}{S} - 1)\frac{\mu' X}{D}] - S} . \qquad \text{A.15}$$

In order to have a more direct measure of the value of storage in terms of satisfying demand when the production unit is down, we denote by I the "improvement" or fraction of time in <u>addition</u> to the stream factor during which the demand is met:

$$I = S' - P(F_0) = \begin{cases} \frac{S}{\omega} \cdot \frac{S'(1-\exp[(1-\frac{\omega S'}{S})\frac{\mu' X}{D}])}{S' - \frac{S}{\omega}\exp[(1-\frac{\omega S'}{S})\frac{\mu' X}{D}]} & \text{if } \omega S' \neq S \\ \frac{S}{\omega} \cdot \frac{\mu' X/D}{1 + \mu' X/D} & \text{if } \omega S' = S \ . \end{cases} \qquad \text{A.16}$$

By setting k equal to αN in (A.6), taking the limit as $N \to \infty$ and using (A.7), (A.9) and (A.11), we can show that for $0 < \alpha < 1$ the steady state probability density of the storage tank having contents $x = \alpha X$ while the plant is up (and $x \neq X$) is given by

$$P'_W(x) = \frac{\mu'(\lambda'\omega-\mu')\lambda'\exp[(\mu'-\lambda'\omega)x/D]}{(\lambda'+\mu')(\lambda'w-\mu'\exp[(\mu'-\lambda'w)X/D])(M-D)} \ . \qquad \text{A.17}$$

By using (A.7), it is easily seen that the steady state probability density for the tank having contents x while the plant is down (and $x \neq 0$) is

$$P'_F(x) = P'_W(x)/\omega \ . \qquad \text{A.18}$$

B. Constant Repair Times

As before, we assume a maximum production rate of M, a demand rate of D, a storage capacity X, and an exponentially distributed "up-time" with mean $1/\lambda'$. The time required for the repair of a breakdown is denoted by T'. The system is discretized by considering a small time interval Δ. We assume that

$$\frac{X}{(M-D)\Delta} = N, \text{ an integer} \qquad \text{B.1}$$

and that

$$K \equiv \min\left\{\frac{T'D}{(M-D)\Delta}, N\right\}, \text{ an integer} \ . \qquad \text{B.2}$$

(Thus N of the time intervals Δ are required to fill an empty tank and K time intervals are required to replace the amount lost during a failure.)

Figure 2 denotes the transition probabilities between the states of the discretized system, where

$$\lambda = \lambda'\Delta \qquad \text{B.3}$$

and W_i indicates that the production unit is up and the storage tank contains $i(M-D)\Delta$ units of product. (In figure 2 it is the case that $N > K > N/2$, but the analysis is carried out below for the cases $K = N$ and $K \leq N/2$ also.)

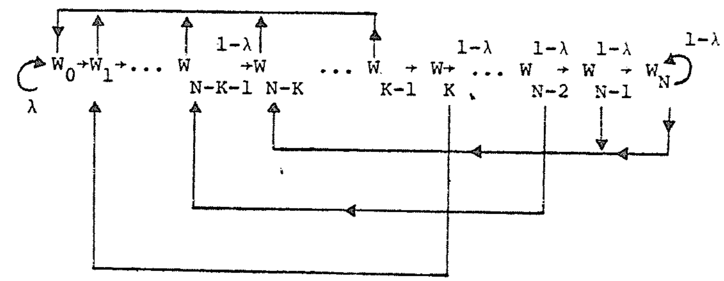


FIGURE 2

The procedure for deriving expressions for the discrete and continuous state probabilities is basically similar to case (A), and details may be found in the Appendix.

If $T'D \geq X$, so that the storage tank runs dry whenever there is a failure, the probability of a full tank is

$$P(X) = Se^{-\hat{\lambda}X} \qquad \text{B.4}$$

where the stream factor $S = (1+\lambda'T')^{-1}$ and $\hat{\lambda} \equiv \lambda/(M-D)$; and the improvement is given by

$$I = S(1-e^{-\hat{\lambda}X})/\omega. \qquad \text{B.5}$$

If $X/(m+1) \le X' \le X/m$, where $X' = T'D$ and m is a positive integer, we denote the probability of a full tank and the improvement by $P_m(X)$ and I_m respectively, and can represent these characteristics by the formulae:

$$P_m(X) = e^{-\hat{\lambda}X}/N_m$$

$$I_m = \frac{SN_m - e^{-\hat{\lambda}X}}{\omega N_m} \qquad \text{B.6}$$

$$N_m = (1+\lambda'T') \sum_{i=0}^{m} [-\hat{\lambda}(X-iX')e^{-\hat{\lambda}X'}]^i/i!\ .$$

Note that to apply these formulae, which represent an infinite number of subcases ($m = 1,2,3,\ldots$), it is necessary to determine the integer m such that $X/(m+1) \le X' \le X/m$. When X' is quite small compared to X, m will be quite large and N_m will contain many terms, and care must be taken to preserve numerical accuracy. Probability density expressions may be found in the Appendix.

Extensive tables for the fraction of time the tank is not empty (and hence product is available) and the fraction of time the tank is full (and hence production is limited) are contained in the Appendix. Results for both case A and case B are tabulated.

C. Sample Computation

Consider the following example:

(a) average length of up-time = 12 months = $1/\lambda'$

(b) average duration of failure = three months (= $1/\mu'$ in case A, = T' in case B)

(c) storage capacity = 2 x demand rate x average duration of failure ($X = 2D/\mu' = 2T'D$)

(d) demand rate = 2 x fill rate ($D = 2(M-D)$)

From this information, we can compute the fraction of the time the tank is full and the fraction of the time the demand is satisfied as follows:

Stream Factor = $S = [1/\lambda'/(1/\lambda'+1/\mu')] = 0.80$

Demand-Fill Ratio = $D/(M-D) \equiv \omega = 2$

Capacity-Loss Ratio = $\mu'X/D = X/T'D = X/X' = 2$ ($m = 1$) .

The values in Tables 1 and 2 in the Appendix yield:

$$\text{Fraction of time tank full} = \begin{cases} 0.422 & \text{case (B)} \\ 0.490 & \text{case (A)} \end{cases}$$

$$\begin{array}{l}\text{Fraction of time demand met} \\ = \text{Stream Factor plus Improvement}\end{array} = \begin{cases} 0.989 & \text{case (B)} \\ 0.955 & \text{case (A)} \end{cases}$$

Figure 3 illustrates the rather small difference between cases A and B for a broad range of values of the parameters.

3. Constant Up-Times

Reliability expressions are easily obtained in two additional cases: (1) a constant "up-time" $\bar{T}$ with exponential "down-time" and (2) constant "up-time" and constant "down-time". In the former case, essentially the same derivation as in section B may be used, since the only changes required in the state transition diagram are to substitute D_{N-r} for $\dot{W}_r$ ($r=0,\ldots,N$) and to replace λ by $\mu \equiv \mu'\Delta'$. (For notational convenience in the formulae we define $\hat{\mu} \equiv \mu'/D$ and $\bar{X} = \bar{T}(M-D)$.) The revised diagram is shown as Figure 4.

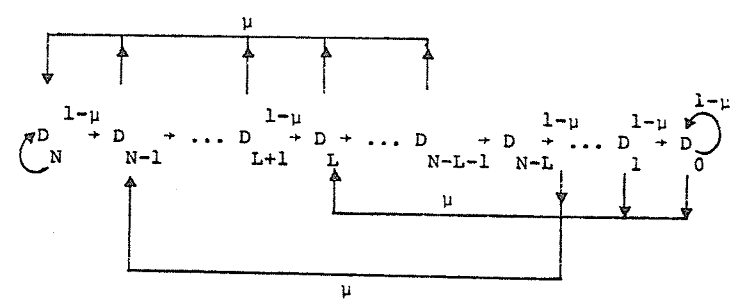
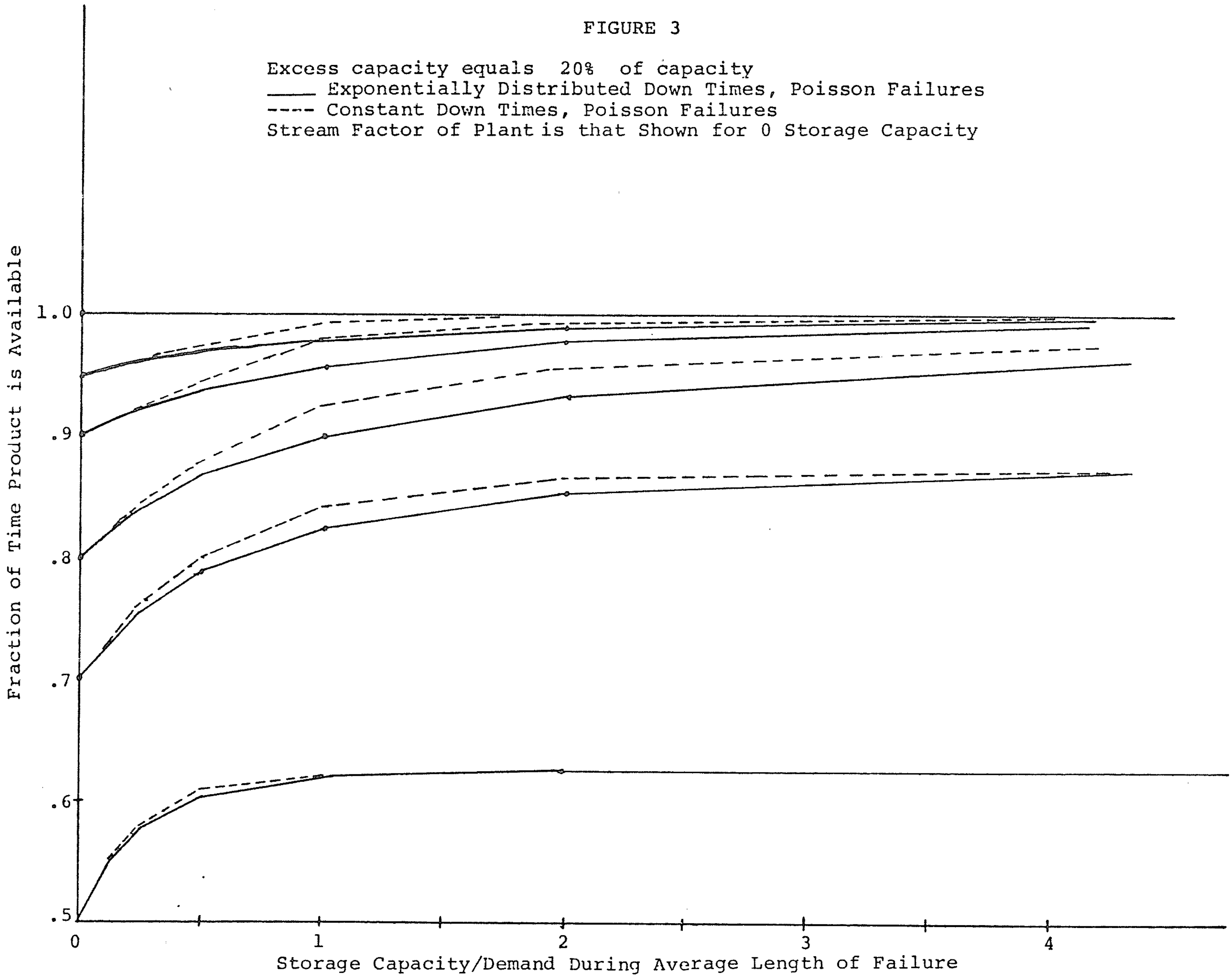


FIGURE 4

FIGURE 3


Excess capacity equals 20% of capacity
____ Exponentially Distributed Down Times, Poisson Failures
---- Constant Down Times, Poisson Failures
Stream Factor of Plant is that Shown for 0 Storage Capacity
Fraction of Time Product is Available
1.0
.9
.8
.7
.6
.5
0
1
2
3
4
Storage Capacity/Demand During Average Length of Failure

Thus the expression developed for $P(X)$ in case B may be easily modified to yield the probability of an empty tank in this case. In particular, if $\bar{X} \geq X$, the fraction of time S^* that demand is met is given by

$$S^* = 1 - e^{-\hat{\mu}X}/(1+\mu'\bar{T});$$

and, if $X/(m+1) \leq \bar{X} < X/m$, where m is a positive integer, the fraction of time that demand is met is given by

$$S^* = 1 - e^{-\hat{\mu}X}/\bar{N}_m ,$$

where

$$\bar{N}_m = (1+\mu'\bar{T})\left(\sum_{i=0}^{m} (-1)^i \frac{(X-i\bar{X})^i}{i!} e^{-\hat{\mu}i\bar{X}}\right).$$

When <u>both</u> up-time and down-time are <u>constant</u> (T and T' respectively) there are three cases to consider:

(1) If $T'D \leq X$ and $T(M-D) \geq T'D$, (this includes cases in which $T(M-D) \geq X$ and $T(M-D) < X$) then, after the first production cycle, the tank will never be empty, so $S^* = 1$.

(2) If $T'D > X$ and $T(M-D) \geq X$, then during each down-time cycle following the first production cycle the tank will contain product for a period of length X/D, and

$$S^* = (T+X/D)/(T+T') = S + (1-S)(X/X') .$$

(3) If $T'D > T(M-D)$ and $T(M-D) < X$, the tank will eventually be empty and never be full afterwards, and

$$S^* = \frac{T + T(M-D)/D}{T + T'} = S + S/\omega = SM/D.$$

In all three cases the dynamic inventory will form a simple sawtooth pattern with flat parts where the storage tank is empty or full. Hence $P'(x)$, the steady state probability of the tank having contents x, will have the same value for all feasible $x \in (0,X)$. In case 1,

$$P'(x) = \frac{(1+\omega)}{(T+T')D}, \text{for} \quad x \in [X-T'D, X),$$

and the probability of a full tank is $(T-\omega T')/(T+T')$. In case 2, the probability of a full tank is $(T-\omega x/D)/(T+T')$, and

$$P'(x) = \frac{(1+\omega)}{(T+T')D}, \text{for} \quad x \in (0, X).$$

In case 3,

$$P'(x) = \frac{(1+\omega)\omega}{(T+T')D}, \text{for} \quad x \in (0, T(M-D)).$$

4. Possible Extensions

Several different kinds of extensions of this work would be interesting. We have already mentioned the possibility of studying the dynamics of the models. Other possibilities include:

1. Extension of the models to random and/or time varying demand patterns,
2. Extension of the models to handle partial failure (e.g., parallel productive facilities) as well as total failure.
3. Extension to tandem versions of the models.
4. Extension to a production facility with both regular preventive maintenance and unscheduled failures,
5. Extension to production facilities with repair time distributions other than constant or exponential,
6. Superimposition of cost structures on the model to find optimal values for M, X, λ' and μ', and
7. Determination of the range of cost structures for which the assumed operating policies are optimal.

APPENDIX

Derivation of Expressions in Case B

The steady-state equations for the system in Figure 2 are easily written down, letting $P(x)$ = probability of being in state x:

$$\begin{aligned}
(1-\lambda)P(W_{N-1}) &= \lambda P(W_N) \\
(1-\lambda)P(W_{N-2}) &= P(W_{N-1}) \\
&\vdots \\
(1-\lambda)P(W_{N-K}) &= P(W_{N-K+1}) \\
(1-\lambda)P(W_{N-K-1}) + \lambda(P(W_N) + P(W_{N-1})) &= P(W_{N-K}) \\
(1-\lambda)P(W_{N-K-2}) + \lambda P(W_{N-2}) &= P(W_{N-K-1}) \\
&\vdots \\
(1-\lambda)P(W_0) + \lambda P(W_T) &= P(W_1) \\
(1-\lambda)P(W_0) &= \sum_{i=1}^{T-1} \lambda P(W_i) .
\end{aligned} \tag{1}$$

In order to obtain the continuous analog of these equations we define the probability distribution function

$$P'_W(x) = \lim_{\substack{\Delta\to 0 \\ i(M-D)\Delta \to x}} P(W_i)/(M-D)\Delta \qquad (0 \le x < X) \tag{2}$$

and denote the probability of a full tank by $P(X)$. For notational convenience we let

$$\begin{aligned}
\hat{\lambda} &\equiv \lambda'/(M-D) \\
X' &\equiv \min\{T'D, X\} \\
X^* &\equiv X-X' \\
P'_W(X^*_+) &\equiv \lim_{\substack{x\to X^* \\ x>X^*}} P'_W(x) \\
P'_W(X^*_-) &\equiv \lim_{\substack{x\to X^* \\ x<X^*}} P'_W(x) .
\end{aligned} \tag{3}$$

The system of equations (1) can then be shown to reduce (as $\Delta \to 0$) to:

$$P'_W(X) = \hat{\lambda} P(X) \tag{4}$$
$$P''_W(x) = -\hat{\lambda} P'_W(x) \qquad X^* < x \le X \tag{5}$$
$$\hat{\lambda} P(X) = P'_W(X^*) \tag{6}$$
$$P''_W(x) = -\hat{\lambda} P'_W(x) + \hat{\lambda} P'_W(x+X') \qquad 0 \le x < X^* \tag{7}$$

$$P'_W(0) = \int_0^{X'} \hat{\lambda} P'_W(x)\,dx \ . \tag{8}$$

(The integral in (8) is to be interpreted as a Stieltjes integral if $X' = X$.)

<u>First Case:</u> $(X' = X)$

If $X' = X$, equations (6) and (7) do not apply, and the solution of the differential equations is given by:

$$P(X) = e^{-\hat{\lambda} X}/(1+\lambda' T') \tag{9}$$
$$P'_W(x) = \lambda e^{-\hat{\lambda} x}/(1+\lambda' T') \tag{10}$$

The "improvement" or fraction of the time during which the production unit is down, yet the demand is being supplied from the storage tank in this case, is thus:

$$I \equiv \lambda' X P(X)/D + \int_0^X \frac{\lambda' x P'_W(x)}{D}\,dx = \tag{11}$$
$$\frac{S}{\omega}(\hat{\lambda} X e^{-\hat{\lambda} X} - \hat{\lambda} X e^{-\hat{\lambda} X} + 1 - e^{-\hat{\lambda} X})$$
$$= S \cdot \frac{1-e^{-\hat{\lambda} X}}{\omega}$$

where $\omega = D/(M-D)$ and $S = (1+\lambda' T')^{-1}$.

<u>Second Case:</u> $(X' < X)$

Consider first the subcase $X/2 \le X' \le X$. Under this assumption, the probability density function satisfying (4) - (8) is:

$$P(X) = e^{-\hat{\lambda} X}/N_1 \tag{12}$$

$$P'_W(x) = \begin{cases} f_1(x)/N_1 = [\hat{\lambda} e^{-\hat{\lambda} x}]/N_1 \,, & X^* < x < X \quad (13) \\ f_2(x)/N_1 = [\hat{\lambda} e^{-\hat{\lambda} x}(1-e^{-\hat{\lambda} X'}) + & \\ \quad + \hat{\lambda}(x-X^*) f_1(x+X')]/N_1 \,, & 0 \le x < X^* \quad (14) \end{cases}$$

where

$$N_1 = (1+\lambda' T')(1-\hat{\lambda} X^* e^{-\hat{\lambda} X'}) . \tag{15}$$

The improvement is given by:

$$I_1 = (1-S) - \int_0^{X'} \lambda'(T'-x/D) P'_W(x)\,dx \tag{16}$$
$$= (1-S) - \frac{\hat{\lambda}}{\omega} \int_0^{X'} (X'-x) P'_W(x)\,dx$$
$$= (1-S) - \left[\frac{e^{-\hat{\lambda} X} + \hat{\lambda} X' - 1 + (\hat{\lambda}^2 X'^2 - \hat{\lambda}^2 X' X + \hat{\lambda} X^*) e^{-\hat{\lambda} X'}}{\omega N_1}\right]$$
$$= \frac{1 - \hat{\lambda} X^* e^{-\hat{\lambda} X'} - e^{-\hat{\lambda} X}}{\omega(1+\lambda' T')(1-\hat{\lambda} X^* e^{-\hat{\lambda} X'})} \ .$$

In general, given the probability density function when $X/(m+1) \le X' \le X/m$, we can obtain the probability density function for $X/(m+2) \le X' \le X/(m+1)$ by using the relations:

$$P(X) = e^{-\hat{\lambda} X}/N_{m+1}$$

$$P'_W(x) = \begin{cases} f_1(x)/N_{m+1} & X-X' < x < X \\ f_2(x)/N_{m+1} & X-2X' \le x < X < X' \\ \vdots & \\ f_{m+1}(x)/N_{m+1} & X-(m+1)X \le x \le X-mX' \\ f_{m+2}(x)/N_{m+1} & 0 \le x \le X-(m+1)X' \end{cases}$$

where $f_1(x)\ldots,f_{m+1}(x)$ are taken from the definition of $P_W'(x)$ in the case $X/(m+1) \le X' \le X/m$ and $f_{m+2}(x)$ is obtained by solving

$$f'_{m+2}(x) = -\lambda f_{m+2}(x) + \lambda f_{m+1}(x+X')$$

with the initial condition $f_{m+2}(X-(m+1)X') = f_{m+1}(X-(m+1)X')$ ($f_{m+2}(x)$ may be determined in closed form by using the variation of parameters formula for linear differential equations) and N_{m+1} is chosen so as to satisfy

$$P(X) + \int_0^X P_W'(x)dx = S.$$

It may be verified that the recursion relations and side condition can be satisfied by letting $X' = T'D$ and $y_i = X - iX'$, and defining

$$f_1(x) \equiv \hat{\lambda}e^{-\hat{\lambda}x}, \tag{17}$$

$$f_2(x) \equiv (1-e^{-\hat{\lambda}X'})f_1(x) + \hat{\lambda}(x-(X-X'))f_1(x+X'),$$

$$f_{j+1}(x) \equiv f_j(x) + \left\{\frac{[\hat{\lambda}(x-y_j)]^j}{j!} - \frac{[\hat{\lambda}(x-y_j)]^{j-1}}{(j-1)!}\right\} f_1(x+jX')$$

$(j = 2,3,\ldots)$

so that

$$f_{j+1}(x) = f_1(x) + \sum_{i=1}^{j}\left\{\frac{[\hat{\lambda}(x-y_i)]^i}{i!} - \frac{[\hat{\lambda}(x-y_i)]^{i-1}}{(i-1)!}\right\} f_1(x+iX') \tag{18}$$

$(j = 1,2,3,\ldots)$

and

$$N_{m+1} \equiv (1+\lambda'T')\left(\sum_{i=0}^{m+1}(-1)^i \frac{y_i^{\,i}}{i!}e^{-\hat{\lambda}iX'}\right). \tag{19}$$

Letting I_m denote the improvement when $X/(m+1) \le X' \le X/m$ and m is an integer, we have

$$\omega N_m I_m = (1-S)N_m + \int_0^{X'}\hat{\lambda}(x-X')f_{m+1}(x)dx +$$

$$\int_{y_m}^{X'}\hat{\lambda}(x-X')\hat{\lambda}(f_m(x) - f_{m+1}(x))dx$$

and

$$\omega N_{m+1}I_{m+1} = (1-S)N_{m+1} + \int_0^{X'}\hat{\lambda}(x-X')\; f_{m+1}(x)dx$$

$$+ \int_0^{y_{m+1}}\hat{\lambda}(x-X')(f_{m+2}(x) - f_{m+1}(x))dx\,.$$

Substituting the expression for $\omega N_m I_m$, we have

$$\omega N_{m+1}I_{m+1} = \omega N_m I_m + \frac{\hat{\lambda}X'(-\hat{\lambda}y_{m+1})^{m+1}e^{-(m+1)\hat{\lambda}X'}}{(m+1)!}$$

$$- \int_{y_m}^{X'}\hat{\lambda}(x-X')(f_m(x) - f_{m+1}(x))dx +$$

$$\int_0^{y_{m+1}}\hat{\lambda}(x-X')(f_{m+2}(x) - f_{m+1}(x))dx\,.$$

By successive integration by parts it is easily verified that

$$\int_{y_m}^{X'}\hat{\lambda}(x-X')(f_m(x) - f_{m+1}(x))dx = e^{-\hat{\lambda}(m+1)X'}$$

$$= \sum_{i=0}^{m}(-\lambda y_{m+1})^i/i! - e^{-\hat{\lambda}x}$$

and

$$\int_0^{y_{m+1}} \hat{\lambda}(x-X')(f_{m+2}(x) - f_{m+1}(x))dx =$$

$$= \frac{-\hat{\lambda}X'(-\hat{\lambda}y_{m+1})^{m+1}e^{-(m+1)\hat{\lambda}X'}}{(m+1)!}$$

$$+ e^{-\hat{\lambda}(m+1)X'} \sum_{i=0}^{m+1} (-\hat{\lambda}y_{m+1})^i/i! - e^{-\hat{\lambda}X}.$$

Substituting the last two relations into the expression for $\omega N_{m+1} I_{m+1}$ and canceling wherever possible yields

$$\omega N_{m+1} I_{m+1} = \omega N_m I_m + e^{-\hat{\lambda}(m+1)X'}(-\hat{\lambda}y_{m+1})^{m+1}/(m+1)!$$

so that

$$I_{m+1} = \frac{\omega N_m I_m + e^{-\hat{\lambda}(m+1)X'}(-\hat{\lambda}y_{m+1})^{m+1}/(m+1)!}{\omega N_{m+1}} \qquad (20)$$

$$= \frac{\sum_{i=0}^{m+1} (-\hat{\lambda}y_i e^{-\hat{\lambda}X'})^i/i! - e^{-\hat{\lambda}X}}{\omega(1+\lambda'T')(\sum_{i=0}^{m+1} (-\hat{\lambda}y_i e^{-\hat{\lambda}X'})^i/i!)}$$

and

$$I_1 = \frac{1 - \hat{\lambda}y_1 e^{-\hat{\lambda}X'} - e^{\hat{\lambda}X}}{\omega(1+\lambda'T')(1-\hat{\lambda}y_1 e^{-\hat{\lambda}X'})}.$$

By using the expression for $P_W'(x)$, the probability density distribution for the failure mode may be computed by using the relation

$$P_F'(x) = \int_x^{\min\{(x+X'),X\}} \frac{\lambda' P_W'(x)}{\omega} dx,$$

where the integral is to be interpreted as a Stieltjes integral if $\min\{(x+X'),X\} = X$ or $x = 0$.

TABLE 1

FRACTION OF TIME PRODUCT IS AVAILABLE

OMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
20.00	0.125	0.262	0.523	0.723	0.819	0.911	0.956	0.974	0.991	B
		0.262	0.523	0.722	0.812	0.910	0.956	0.973	0.991	A
20.00	0.250	0.262	0.525	0.731	0.829	0.919	0.961	0.977	0.992	B
		0.262	0.525	0.730	0.827	0.818	0.960	0.976	0.992	A
20.00	0.500	0.263	0.525	0.735	0.837	0.930	0.969	0.983	0.995	B
		0.263	0.525	0.734	0.836	0.927	0.967	0.981	0.994	A
20.00	1.000	0.263	0.525	0.735	0.840	0.940	0.981	0.992	0.999	B
		0.263	0.525	0.735	0.839	0.937	0.975	0.986	0.996	A
20.00	2.000	0.263	0.525	0.735	0.840	0.944	0.968	0.997	1.000	B
		0.263	0.525	0.735	0.840	0.943	0.983	0.993	0.998	A
20.00	4.000	0.263	0.525	0.735	0.840	0.945	0.993	1.000	1.000	B
		0.263	0.525	0.735	0.840	0.945	0.989	0.997	1.000	A
20.00	3.000	0.263	0.525	0.735	0.840	0.945	0.996	1.000	1.000	B
		0.263	0.525	0.735	0.340	0.945	0.993	0.999	1.000	A
20.00	16.000	0.263	0.525	0.735	0.840	0.945	0.997	1.000	1.000	B
		0.263	0.525	0.735	0.840	0.945	0.996	1.000	1.000	A
10.00	0.125	0.274	0.536	0.729	0.821	0.912	0.956	0.974	0.991	B
		0.274	0.535	0.728	0.820	0.911	0.956	0.973	0.991	A
10.00	0.250	0.275	0.546	0.746	0.837	0.922	0.962	0.977	0.992	B
		0.275	0.545	0.744	0.335	0.920	0.960	0.976	0.992	A
10.00	0.500	0.275	0.550	0.762	0.857	0.938	0.972	0.984	0.995	B
		0.275	0.549	0.759	0.852	0.933	0.968	0.981	0.994	A
10.00	1.000	0.275	0.550	0.769	0.873	0.960	0.989	0.996	1.000	B
		0.275	0.550	0.768	0.868	0.949	0.978	0.988	0.996	A
10.00	2.000	0.275	0.550	0.770	0.879	0.975	0.997	0.999	1.000	B
		0.275	0.550	0.770	0.378	0.964	0.983	0.994	0.998	A
10.00	4.000	0.275	0.550	0.770	0.830	0.883	1.000	1.000	1.000	B
		0.275	0.550	0.770	0.880	0.976	0.996	0.999	1.000	A
10.000	8.000	0.275	0.550	0.770	0.880	0.988	1.000	1.000	1.000	B
		0.275	0.550	0.770	0.880	0.984	0.999	1.000	1.000	A
10.000	16.000	0.275	0.550	0.770	0.880	0.990	1.000	1.000	1.000	B
		0.275	0.550	0.770	0.880	0.988	1.000	1.000	1.000	A

TABLE 1 (continued)

FRACTION OF TIME PRODUCT IS AVAILABLE

OMEGA	X/X'	STREAM FACTOR								CASE
		0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	
4.00	0.125	0.299	0.549	0.734	0.824	0.912	0.956	0.974	0.991	B
		0.298	0.547	0.732	0.822	0.911	0.956	0.974	0.991	A
4.00	0.250	0.309	0.579	0.761	0.844	0.924	0.962	0.977	0.992	B
		0.309	0.575	0.756	0.840	0.921	0.961	0.977	0.992	A
4.00	0.500	0.312	0.608	0.801	0.379	0.945	0.974	0.985	0.995	B
		0.312	0.603	0.789	0.867	0.937	0.969	0.982	0.994	A
4.00	1.000	0.312	0.623	0.843	0.926	0.981	0.995	0.998	1.000	B
		0.312	0.620	0.825	0.900	0.957	0.980	0.988	0.996	A
4.00	2.000	0.312	0.625	0.867	0.957	0.99	1.000	1.000	1.000	B
		0.312	0.625	0.855	0.933	0.979	0.991	0.995	0.999	A
4.00	4.000	0.313	0.625	0.874	0.977	1.000	1.000	1.000	1.000	B
		0.313	0.625	0.871	0.960	0.994	0.998	0.999	1.000	A
4.00	8.000	0.313	0.625	0.875	0.988	1.000	1.000	1.000	1.000	B
		0.313	0.625	0.875	0.978	0.999	1.000	1.000	1.000	A
4.00	16.000	0.313	0.625	0.875	0.994	1.000	1.000	1.000	1.000	B
		0.313	0.625	0.875	0.988	1.000	1.000	1.000	1.000	A
2.00	0.125	0.316	0.555	0.736	0.824	0.912	0.956	0.974	0.991	B
		0.314	0.553	0.734	0.823	0.912	0.956	0.974	0.991	A
2.00	0.250	0.347	0.598	0.768	0.847	0.924	0.962	0.977	0.992	B
		0.344	0.591	0.761	0.842	0.922	0.961	0.977	0.992	A
2.00	0.500	0.369	0.658	0.822	0.888	0.947	0.974	0.985	0.995	B
		0.366	0.641	0.802	0.872	0.938	0.969	0.982	0.994	A
2.00	1.000	0.375	0.716	0.901	0.957	0.990	0.997	0.999	1.000	B
		0.374	0.694	0.855	0.913	0.960	0.981	0.989	0.996	A
2.00	2.000	0.375	0.744	0.951	0.989	0.999	1.000	1.000	1.000	B
		0.375	0.732	0.910	0.955	0.983	0.992	0.996	0.999	A
2.00	4.000	0.375	0.750	0.981	0.999	1.000	1.000	1.000	1.000	B
		0.375	0.748	0.953	0.985	0.997	0.999	0.999	1.000	A
2.00	8.000	0.375	0.750	0.996	1.000	1.000	1.000	1.000	1.000	B
		0.375	0.750	0.981	0.998	1.000	1.000	1.000	1.000	A
2.00	16.000	0.375	0.750	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.375	0.750	0.995	1.000	1.000	1.000	1.000	1.000	A

TABLE 1 (continued)

FRACTION OF TIME PRODUCT IS AVAILABLE

OMEGA	X/X'	STREAM FACTOR								CASE
		0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	
1.00	0.125	0.328	0.559	0.737	0.825	0.912	0.956	0.974	0.991	B
		0.325	0.556	0.734	0.823	0.912	0.956	0.974	0.991	A
1.00	0.250	0.382	0.611	0.771	0.848	0.925	0.962	0.977	0.992	B
		0.373	0.600	0.764	0.843	0.922	0.961	0.977	0.992	A
1.00	0.500	0.444	0.697	0.835	0.894	0.949	0.975	0.985	0.995	B
		0.430	0.667	0.810	0.876	0.939	0.970	0.982	0.994	A
1.00	1.000	0.488	0.816	0.944	0.977	0.995	0.999	1.000	1.000	B
		0.476	0.750	0.872	0.920	0.962	0.981	0.989	0.996	A
1.00	2.000	0.499	0.893	0.988	0.997	1.000	1.000	1.000	1.000	B
		0.497	0.833	0.937	0.965	0.985	0.993	0.996	0.999	A
1.00	4.000	0.500	0.942	0.999	1.000	1.000	1.000	1.000	1.000	B
		0.500	0.900	0.982	0.992	0.997	0.999	0.999	1.000	A
1.00	8.000	0.500	0.970	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.500	0.944	0.998	1.000	1.000	1.000	1.000	1.000	A
1.00	16.00	0.500	0.985	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.500	0.971	1.000	1.000	1.000	1.000	1.000	1.000	A
0.50	0.125	0.335	0.561	0.737	0.825	0.912	0.956	0.974	0.991	B
		0.331	0.557	0.735	0.823	0.912	0.956	0.974	0.991	A
0.50	0.250	0.406	0.618	0.773	0.849	0.925	0.962	0.977	0.992	B
		0.393	0.605	0.765	0.844	0.922	0.961	0.977	0.992	A
0.50	0.500	0.514	0.721	0.842	0.897	0.949	0.975	0.985	0.995	B
		0.480	0.681	0.814	0.877	0.939	0.970	0.982	0.994	A
0.50	1.000	0.638	0.893	0.970	0.988	0.997	0.999	1.000	1.000	B
		0.580	0.782	0.881	0.923	0.962	0.981	0.989	0.996	A
0.50	2.000	0.713	0.972	0.997	0.999	1.000	1.000	1.000	1.000	B
		0.669	0.887	0.949	0.969	0.986	0.993	0.996	0.999	A
0.50	4.000	0.744	0.998	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.725	0.964	0.990	0.995	0.998	0.999	0.999	1.000	A
0.50	8.000	0.750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.747	0.995	1.000	1.000	1.000	1.000	1.000	1.000	A
0.50	16.000	0.750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.750	1.000	1.000	1.000	1.000	1.000	1.000	1.000	A

TABLE 1 (continued)

FRACTION OF TIME PRODUCT IS AVAILABLE

OMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
0.20	0.125	0.340	0.562	0.737	0.825	0.912	0.956	0.974	0.991	B
		0.335	0.558	0.735	0.823	0.912	0.956	0.974	0.991	A
0.20	0.250	0.424	0.622	0.774	0.850	0.925	0.962	0.977	0.992	B
		0.406	0.608	0.766	0.844	0.922	0.961	0.977	0.992	A
0.20	0.500	0.574	0.738	0.847	0.899	0.950	0.975	0.985	0.995	B
		0.517	0.690	0.816	0.878	0.939	0.970	0.982	0.994	A
0.20	1.000	0.814	0.953	0.988	0.995	0.999	1.000	1.000	1.000	B
		0.664	0.803	0.886	0.925	0.963	0.982	0.989	0.996	A
0.20	2.000	0.939	0.996	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.815	0.916	0.955	0.971	0.986	0.993	0.996	0.999	A
0.20	4.000	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.931	0.984	0.993	0.996	0.998	0.999	0.999	1.000	A
0.20	3.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.937	0.999	1.000	1.000	1.000	1.000	1.000	1.000	A
0.20	16.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	A
0.10	0.125	0.342	0.562	0.737	0.825	0.912	0.956	0.974	0.991	B
		0.337	0.558	0.735	01823	0.912	0.956	0.974	0.991	A
0.10	0.250	0.431	0.623	0.775	0.850	0.925	0.962	0.977	0.992	B
		0.411	0.609	0.766	0.844	0.922	0.961	0.977	0.992	A
0.10	0.500	0.598	0.744	0.848	0.899	0.950	0.975	0.985	0.995	B
		0.531	0.694	0.817	0.878	0.939	0.970	0.982	0.994	A
0.10	1.000	0.893	0.976	0.994	0.998	0.999	1.000	1.000	1.000	B
		0.694	0.809	0.888	0.926	0.963	0.982	0.989	0.996	A
0.10	2.000	0.986	0.999	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.860	0.924	0.957	0.972	0.986	0.993	0.996	0.999	A
0.10	4.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.967	0.988	0.994	0.996	0.998	0.999	0.999	1.000	A
0.10	8.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	A
0.10	16.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	B
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	A

TABLE 2

FRACTION OF TIME TANK IS FULL

OMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
20.00	0.125	0.000	0.041	0.240	0.428	0.682	0.833	0.898	0.965	B
		0.000	0.044	0.251	0.442	0.692	0.839	0.902	0.667	A
20.00	0.250	0.000	0.003	0.082	0.229	0.516	0.730	0.831	0.941	B
		0.000	0.004	0.095	0.254	0.546	0.751	0.845	0.947	A
20.00	0.500	0.000	0.000	0.010	0.066	0.206	0.561	0.712	0.895	B
		0.000	0.000	0.014	0.089	0.355	0.620	0.757	0.914	A
20.00	1.000	0.000	0.000	0.000	0.005	0.098	0.332	0.523	0.809	B
		0.000	0.000	0.000	0.012	0.168	0.457	0.641	0.869	A
20.00	2.000	0.000	0.000	0.000	0.000	0.014	0.183	0.422	0.792	B
		0.000	0.000	0.000	0.000	0.045	0.295	0.520	0.824	A
20.00	4.000	0.000	0.000	0.000	0.000	0.000	0.038	0.378	0.790	B
		0.000	0.000	0.000	0.000	0.004	0.167	0.428	0.797	A
20.00	8.000	0.000	0.000	0.000	0.000	0.000	0.037	0.370	0.790	B
		0.000	0.000	0.000	0.000	0.000	0.083	0.381	0.790	A
20.00	16.000	0.000	0.000	0.000	0.000	0.000	0.011	0.370	0.790	B
		0.000	0.000	0.000	0.000	0.000	0.035	0.371	0.790	A
10.00	0.125	0.006	0.143	0.410	0.585	0.783	0.890	0.933	0.978	B
		0.006	0.151	0.421	0.595	0.790	0.893	0.935	0.978	A
10.00	0.250	0.000	0.041	0.240	0.423	0.682	0.833	0.898	0.965	B
		0.000	0.048	0.263	0.455	0.702	0.845	0.906	0.968	A
10.00	0.500	0.000	0.003	0.082	0.229	0.516	0.730	0.831	0.941	B
		0.000	0.005	0.109	0.280	0.573	0.770	0.858	0.951	A
10.00	1.000	0.000	0.000	0.010	0.066	0.296	0.561	0.712	0.895	B
		0.000	0.000	0.020	0.118	0.414	0.669	0.793	0.928	A
10.00	2.000	0.000	0.000	0.000	0.007	0.154	0.481	0.676	0.890	B
		0.000	0.000	0.001	0.024	0.258	0.565	0.726	0.905	A
10.00	4.000	0.000	0.000	0.000	0.000	0.065	0.453	0.670	0.890	B
		0.000	0.000	0.000	0.001	0.136	0.489	0.683	0.892	A
10.00	8.000	0.000	0.000	0.000	0.000	0;020	0.450	0.670	0.890	B
		0.000	0.000	0.000	0.000	0.059	0.455	0.671	0.890	A
10.00	16.000	0.000	0.000	0.000	0.000	0.003	0.450	0.670	0.890	B
		0.000	0.000	0.000	0.000	0.018	0.450	0.670	0.890	A

TABLE 2 (continued)

FRACTION OF TIME TANK IS FULL

OMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
4.00	0.125	0.056	0.303	0.565	0.706	0.851	0.925	0.955	0.985	B
		0.059	0.311	0.572	0.711	0.854	0.927	0.956	0.985	A
4.00	0.250	0.012	0.184	0.456	0.623	0.805	0.901	0.940	0.980	B
		0.015	0.201	0.476	0.640	0.815	0.907	0.944	0.981	A
4.00	0.500	0.001	0.068	0.297	0.485	0.721	0.855	0.912	0.970	B
		0.001	0.089	0.345	0.533	0.754	0.874	0.924	0.974	A
4.00	1.000	0.000	0.009	0.126	0.294	0.577	0.770	0.857	0.951	B
		0.000	0.019	0.200	0.400	0.671	0.829	0.896	0.965	A
4.00	2.00	0.000	0.000	0.033	0.171	0.517	0.752	0.850	0.950	B
		0.000	0.001	0.031	0.267	0.586	0.784	0.869	0.956	A
4.00	4.000	0.000	0.000	0.003	0.092	0.501	0.750	0.850	0.950	B
		0.000	0.000	0.017	0.160	0.525	0.757	0.853	0.951	A
4.00	8.000	0.000	0.000	0.000	0.048	0.500	0.750	0.850	0.950	B
		0.000	0.000	0.001	0.089	0.503	0.750	0.850	0.950	A
4.00	16.000	0.000	0.000	0.000	0.024	0.500	0.750	0.850	0.950	B
		0.000	0.000	0.000	0.047	0.500	0.750	0.850	0.950	A
2.00	0.125	0.118	0.389	0.629	0.752	0.875	0.938	0.963	0.988	B
		0.122	0.395	0.633	0.754	0.877	0.938	0.963	0.988	A
2.00	0.250	0.056	0.303	0.565	0.706	0.851	0.925	0.955	0.985	B
		0.063	0.319	0.578	0.716	0.857	0.928	0.957	0.986	A
2.00	0.500	0.012	0.184	0.456	0.623	0.805	0.901	0.940	0.980	B
		0.017	0.218	0.495	0.655	0.824	0.911	0.947	0.982	A
2.00	1.000	0.001	0.068	0.297	0.485	0.721	0.855	0.912	0.970	B
		0.001	0.113	0.389	0.574	0.780	0.888	0.933	0.977	A
2.00	2.000	0.000	0.013	0.198	0.422	0.702	0.850	0.910	0.970	B
		0.000	0.036	0.281	0.490	0.734	0.865	0.919	0.973	A
2.00	4.000	0.000	0.001	0.137	0.402	0.700	0.850	0.910	0.970	B
		0.000	0.005	0.194	0.429	0.707	0.853	0.911	0.970	A
2.00	8.000	0.000	0.000	0.109	0.400	0.700	01850	0.910	0.970	B
		0.000	0.000	0.138	0.404	0.700	0.850	0.910	0.970	A
2.00	16.000	0.000	0.000	0.101	0.400	0.700	0.850	0.910	0.970	B
		0.000	0.000	0.110	0.400	0.700	0.850	0.910	0.970	A

TABLE 2 (continued)

FRACTION OF TIME TANK IS FULL

OMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
1.00	0.125	0.172	0.441	0.663	0.775	0.888	0.944	0.966	0.989	B
		0.175	0.444	0.666	0.777	0.888	0.944	0.966	0.989	A
1.00	0.250	0.188	0.389	0.629	0.752	0.875	0.938	0.963	0.938	B
		0.127	0.400	0.636	0.757	0.878	0.939	0.963	0.938	A
1.00	0.500	0.056	0.303	0.565	0.706	0.851	0.925	0.955	0.985	B
		0.070	0.333	0.590	0.724	0.861	0.930	0.958	0.986	A
1.00	1.000	0.012	0.184	0.456	0.623	0.805	0.901	0.940	0.980	B
		0.024	0.250	0.528	0.680	0.838	0.919	0.951	0.984	A
1.00	2.000	0.001	0.107	0.412	0.603	0.800	0.900	0.940	0.980	B
		0.003	0.167	0.463	0.635	0.815	0.907	0.944	0.981	A
1.00	4.000	0.000	0.058	0.401	0.600	0.800	0.900	0.940	0.980	B
		0.000	0.100	0.418	0.608	0.803	0.901	0.941	0.980	A
1.00	8.000	0.000	0.030	0.400	0.600	0.800	0.900	0.940	0.980	B
		0.000	0.056	0.402	0.600	0.800	0.900	0.940	0.980	A
1.00	16.000	0.000	0.015	0.400	0.600	0.800	0.900	0.940	0.980	B
		0.000	0.029	0.400	0.600	0.800	0.900	0.940	0.980	A
0.50	0.125	0.207	0.470	0.681	0.788	0.894	0.947	0.968	0.989	B
		0.209	0.471	0.683	0.788	0.894	0.947	0.968	0.989	A
0.50	0.250	0.072	0.441	0.663	0.775	0.888	0.944	0.966	0.989	B
		0.179	0.447	0.668	0.778	0.889	0.944	0.967	0.989	A
0.50	0.500	0.118	0.389	0.629	0.752	0.875	0.938	0.963	0.988	B
		0.135	0.409	0.643	0.761	0.881	0.940	0.964	0.988	A
0.50	1.000	0.056	0.303	0.565	0.706	0.851	0.925	0.955	0.985	B
		0.085	0.359	0.610	0.738	0.869	0.934	0.961	0.987	A
0.50	2.000	0.019	0.264	0.551	0.700	0.850	0.925	0.955	0.985	B
		0.041	0.306	0.576	0.716	0.857	0.928	0.957	0.986	A
0.50	4.000	0.003	0.251	0.550	0.700	0.850	0.925	0.955	0.985	B
		0.012	0.268	0.555	0.703	0.851	0.925	0.955	0.985	A
0.50	8.000	0.000	0.250	0.550	0.700	0.850	0.925	0.955	0.985	B
		0.002	0.252	0.550	0.700	0.850	0.925	0.955	0.985	A
0.50	16.000	0.000	0.250	0.550	0.700	0.850	0.925	0.955	0.985	B
		0.000	0.250	0.550	0.700	0.850	0.925	0.966	0.985	A

TABLE 2 (continued)

FRACTION OF TIME TANK IS FULL

IMEGA	X/X'	STREAM FACTOR 0.25	0.50	0.70	0.80	0.90	0.95	0.97	0.99	CASE
0.20	0.125	0.232	0.488	0.693	0.795	0.898	0.949	0.969	0.990	B
		0.233	0.488	0.693	0.795	0.898	0.949	0.969	0.990	A
0.20	0.250	0.215	0.476	0.685	0.790	0.895	0.948	0.969	0.990	B
		0.219	0.478	0.687	0.791	0.896	0.948	0.969	0.990	A
0.20	0.500	0.185	0.452	0.671	0.780	0.890	0.945	0.967	0.989	B
		0.197	0.462	0.677	0.784	0.892	0.946	0.968	0.989	A
0.20	1.000	0.137	0.409	0.642	0.761	0.880	0.940	0.964	0.988	B
		0.167	0.439	0.663	0.775	0.887	0.944	0.966	0.989	A
0.20	2.000	0.112	0.401	0.640	0.760	0.880	0.940	0.964	0.988	B
		0.137	0.417	0.649	0.766	0.883	0.941	0.965	0.988	A
0.20	4.000	0.102	0.400	0.640	0.760	0.880	0.940	0.964	0.988	B
		0.114	0.403	0.641	0.761	0.880	0.940	0.964	0.988	A
0.20	8.000	0.100	0.400	0.640	0.760	0.880	0.940	0.964	0.988	B
		0.103	0.400	0.640	0.760	0.880	0.940	0.964	0.988	A
0.20	16.000	0.100	0.400	0.640	0.760	0.880	0.940	0.964	0.988	B
		0.100	0.400	0.640	0.760	0.880	0.940	0.964	0.988	A
0.10	0.125	0.241	0.494	0.696	0.798	0.899	0.949	0.970	0.990	B
		0.241	0.404	0.696	0.798	0.899	0.949	0.970	0.990	A
0.10	0.250	0.232	0.483	0.693	0.795	0.898	0.949	0.969	0.990	B
		0.234	0.489	0.693	0.796	0.898	0.949	0.969	0.990	A
0.10	0.500	0.215	0.476	0.685	0.790	0.895	0.948	0.969	0.990	B
		0.222	0.481	0.688	0.792	0.896	0.948	0.969	0.990	A
0.10	1.000	0.185	0.452	0.671	0.780	0.890	0.945	0.967	0.989	B
		0.206	0.469	0.681	0.787	0.894	0.947	0.968	0.989	A
0.10	2.000	0.176	0.450	0.670	0.780	0.890	0.945	0.967	0.989	B
		0.189	0.458	0.674	0.783	0.891	0.946	0.967	0.989	A
0.10	4.000	0.175	0.450	0.670	0.780	0.890	0.945	0.967	0.989	B
		0.078	0.451	0.671	0.780	0.890	0.945	0.967	0.989	A
0.10	8.000	0.175	0.450	0.670	0.780	0.890	0.945	0.967	0.989	B
		0.175	0.450	0.670	0.780	0.890	0.945	0.967	0.989	A
0.10	16.000	0.175	0.450	0.670	0.780	0.890	0.945	0.967	0.989	B
		0.175	0.450	0.670	0.780	0.890	0.945	0.967	0.989	A

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