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Report No. 67

Sequential Methods in Statistical Process Monitoring

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This report is chapters 4 and 5 of a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Statistics) at the University of Wisconsin–Madison (1989). Thesis adviser: George Box.

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Sequential Methods in Statistical Process Monitoring

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ABSTRACT

Statistical models are usually fitted to the data with the aim of summarizing the information contained in it and describing the relationships between a set of variables. Once they been adequately identified, fitted, and checked, they can be used for modelling dependence and for prediction purposes. The question arises as to how long we can use expect to use the model before it becomes necessary to update the parameters in it. This report introduces a cumulative score – CUSCORE – that can be used to monitor the parameters in a model. CUSUM for location is a particular case of the CUSCORE.

KEYWORDS: ARL, CUSUM, CUSCORE, Fisher score, models, IMA

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CHAPTER 4

SEQUENTIAL MONITORING OF MODELS

Statistical models are usually fitted to the data with the aim of summarizing the information contained in it and to describe the relationship between a set of variables. After a model has been adequately identified, fitted and checked, it can be used for modelling the dependence of a response variable on some explanatory variables and for predicting future values of the response.

The question arises, see Box and Jenkins (1966), as to how long we can expect to use the model before it becomes neccesary to update the parameters in it. It is important then to be able to detect changes in the parameters as quickly as possible. Failure to do so can lead to wrong conclusions that are based on the forecasts produced by the model.

As new observations are available, the devised model provides us with estimated residuals. These residuals are obtained in a sequential manner that makes them suitable for the use of cumulative sum procedures.

4.1 One Parameter Case

Consider a model supposed to be exact $a_t(\theta) = f(y_t, x_t, \theta)$; where the a_t's are independent and identically distributed $N(0, \sigma_a^2)$, the y_t's are the observed values and x_t is some explanatory variable.

Let θ_0 be some value, possibly different from the true value θ of the parameter. The sequential probability ratio test for θ_0 against some other value θ_1 has likelihood ratio

$$LR_k = \prod_{t=1}^{k} \exp\left[\frac{1}{2\sigma_a^2} \left[a_t^2(\theta_0) - a_t^2(\theta_1) \right] \right] .$$

After taking logs this likelihood ratio leads to the cumulative sum

$$\begin{aligned} S_k &= \frac{1}{2\sigma_a^2} \sum_{t=1}^{k} [\, a_t^2(\theta_0) - a_t^2(\theta_1)] \\ &= \frac{1}{2\sigma_a^2} \sum_{t=1}^{k} [\, f^2(y_t,x_t,\theta_0) - f^2(y_t,x_t,\theta_1)] \, . \end{aligned} \tag{4.1}$$

Expanding $a_t^2(\theta) = f^2(y_t,x_t,\theta)$ around θ_0 and letting $\delta = (\theta_1 - \theta_0)$, and $d_t = -\dfrac{\partial a_t}{\partial \theta}$ we have

$$\begin{aligned} S_k &\approx \frac{1}{2\sigma_a^2} \sum_{t=1}^{k} [\, 2\delta a_t(\theta_0) d_t(\theta_0) - \delta^2 d_t^2(\theta_0)] \, , \\ &= \frac{\delta}{\sigma_a^2} \sum_{t=1}^{k} [\, a_t(\theta_0) d_t(\theta_0) - \frac{\delta}{2} d_t^2(\theta_0)] \, , \\ &= \frac{\delta}{\sigma_a^2} \sum_{t=1}^{k} r_t \, . \end{aligned} \tag{4.2}$$

The quantity $a_t(\theta_0)d_t(\theta_0)$ is equal to the measure for model discrepancies from the value θ_0 taken in the current model, given by Box (1980). This measure is a dimensionless function of the data alone and is given by Fisher's score function

$$g_\theta(a_t) = \left.\frac{\partial \ln p(a_t \mid \theta)}{\partial \theta}\right|_{\theta=\theta_0} \tag{4.3}$$

Where $p(a_t \mid \theta)$ is the predictive distribution of a_t conditional on some choice of θ.

The quantity $\frac{\delta}{2} d_t^2(\theta_0)$, can be viewed as a reference value, see for example equation (2.5), around which $a_t(\theta_0)d_t(\theta_0)$ is expected to vary if the parameter does not change.

Expanding $a_t(\theta_0)$ around the "true value" θ gives $a_t(\theta_0) \simeq [\, a_t(\theta) + (\theta - \theta_0)d_t(\theta_0)]$; and therefore the expected value of $a_t(\theta_0)d_t(\theta_0)$

$$\begin{aligned} E_\theta[\, a_t(\theta_0)d_t(\theta_0)] &= E_\theta[\, a_t(\theta)d_t(\theta_0) + (\theta - \theta_0)d_t^2(\theta_0)] \\ &= (\theta - \theta_0)\, d_t^2(\theta_0) \end{aligned} \tag{4.4}$$

The expected value of the increments r_t in the cumulative sum (4.2) is then

$$\begin{aligned} E_\theta[\, r_t\,] &= E_\theta[\, a_t(\theta_0)d_t(\theta_0) - \frac{\delta}{2} d_t^2(\theta_0)] \\ &= [\, (\theta - \theta_0) - \frac{\delta}{2}\,]\, d_t^2(\theta_0) \end{aligned} \tag{4.5}$$

We see that for $\theta = \theta_0$, the expected value is equal to $-\frac{\delta}{2} d_t^2(\theta_0)$ and therefore, if $\delta > 0$; i.e. $\theta_1 > \theta_0$, the cumulative sum will slope downwards. On the other hand, if $\theta = \theta_1$ the expected value is equal to $\frac{\delta}{2} d_t^2(\theta_0)$ and the cumulative sum will slope upwards. The converse holds true if $\delta < 0$; i.e. $\theta_0 > \theta_1$. Therefore, the cumulative sum given by (4.2) can be used, as in the case of monitoring the mean or the variability of a process, to monitor changes in the parameter θ.

Following Page (1954), see also Van Dobben de Bruyn (1968), we accumulate $r_t = [a_t(\theta_0)d_t(\theta_0) - \frac{\delta}{2}d_t^2(\theta_0)]$ only when it is relevant for taking a decision that the parameter has changed. In other words, if we are interested in monitoring increases in the parameter θ_0, we plot or compute, the cumulative sum only when it is positive and will reset it to zero whenever it becomes negative. Since the r_t's are a function of Fisher's score function, we shall call the test procedure cumulative score or *cuscore*.

Let CS_k denote the value of the cumulative score procedure plotted at time k; i.e. after observation k has been recorded, and let the superscripts + and − denote positive and negative biases in the parameter respectively. The operating formulas for the procedure can be written as

$$\begin{aligned} CS_k^+ &= \max(0, CS_{k-1}^+ + r_k), \\ CS_k^- &= \min(0, CS_{k-1}^- + r_k), \\ CS_0^+ &= CS_0^- = 0 . \end{aligned} \tag{4.6}$$

In chapter 2 we showed that this is equivalent to a succesion of Wald sequential test with boundaries 0 and h, where h is the decision interval. An approximation to the decision interval h can be obtained as a function of the type I and II errors, α and β, the magnitude of the change in the parameter, $\delta = (\theta_1 - \theta_0)$ and the variance of the a_t's.

$$h = \frac{\sigma_a^2 \log((1-\beta)/\alpha)}{\delta}$$

Note however, that in the cumulative sum procedure the null hypothesis is never accepted, therefore the type II error β has no practical meaning. So the approximation for h can be reduced to, see Johnson (1961),

$$h = \frac{\sigma_a^2 log\,(1/\alpha)}{\delta} \tag{4.7}$$

We will compute the cumulative sum score, *cuscore*, CS_k as described in equation (4.6). Signals are triggered either because $CS_k^+ \geq h$ or $CS_k^- \leq -h$.

4.2 Applications of the CUSCORE

4.2.1 Monitoring the mean value

Consider the case where we are interested in monitoring changes in the mean of a process from a level θ_0 to a level θ_1 say. Suppose that the observations satisfy $y_t = \theta + a_t$, where a_t is distributed as $N(0,1)$. Then $f\,(y_t, x_t, \theta) = y_t - \theta$, and $d_t = 1$. The cumulative score r_k becomes

$$r_k = [(y_k - \theta_0) - \frac{(\theta_1 - \theta_0)}{2}] = [y_k - \frac{(\theta_0 + \theta_1)}{2}]\,.$$

The cumulative sum procedure given by equation (4.6) with the above r_k's, is nothing more that the cumulative sum scheme for monitoring the mean level of a normal process with reference value $(\theta_0 + \theta_1)/2$ (Barnard 1946; Page 1957, 1961; Johnson 1961; Woodward and Goldsmith 1964; Goel and Wu 1971; Lucas 1976;

etc.).

4.2.2 Monitoring the slope of a line through the origin

Suppose that the observations from a process satisfy $y_t = \theta x_t + a_t$, for some explanatory variable x_t. In this case $f(y_t, x_t, \theta) = y_t - \theta x_t$; giving $d_t = x_t$, scores r_k

$$r_k = [(y_k - \theta_0 x_k)x_k - \frac{\delta}{2}x_k^2] = [y_k - \frac{(\theta_0 + \theta_1)}{2}x_k]x_k ,$$

and cumulative scores as in equation (4.6).

This is equivalent to example 4.2.1 if we think of monitoring changes in the expected value of the observations $E(y_t) = \theta x_t$. The reference value is again between the two expected values $\theta_0 x_t$ and $\theta_1 x_t$. The general linear model case will be consider in section 4.4.

4.2.3 Monitoring the mean of a process with sinusoidal noise

Let the observations of a process satisfy the equation $a_t = f(y_t, x_t, \theta)$ with $x_t = sin(\frac{\pi t}{6})$. In other words, $y_t = \theta\, sin(\frac{\pi t}{6}) + a_t$. In this case $a_t(\theta_0) = y_t - \theta_0 \sin(\frac{\pi t}{6})$, and $d_t = \sin(\frac{\pi t}{6})$.

The scores r_k for monitoring a change in the parameter θ are given by

$$r_k = [y_k sin(\frac{\pi k}{6}) - \frac{(\theta_0 + \theta_1)}{2}\sin^2(\frac{\pi k}{6})]$$

As an example 144 normal deviates a_t were generated from a $N(0,1)$, and the observations y_t were obtained by adding to the a_t's the corresponding sine wave. The first and last 48 observations have $\theta_0 = 0$ and the remaining ones have $\theta_1 = 0.5$ as shown in figure 4.1.

Figure 4.2 shows the Shewhart chart of the a_t's with warning limits at ± 2, the cumulative sum of the observations y_t and the cuscore CS_k^+. Note that the Shewhart chart shows all the points well within the 3-σ limits. Similarly there is no indication of a departure from the null hypothesis in chart of the observations which is esentially horizontal. The *cuscore* on the other hand clearly shows two changes of slope, upward and downward, corresponding to the two changes in θ. A decision limit can be obtained by using equation (4.7) with $\alpha = 0.05$ and $\delta = 0.5$. This gives $h = 6$ which would indicate a lack of control at observation 65.

4.2.4 Monitoring the smoothing parameter of a IMA process

Let z_t denote the outcomes of an integrated moving average process IMA(0,1,1), see for example Box and Jenkins (1970). If B denotes the backshift operator, $Bz_t = z_{t-1}$, the nonstationary model can be written as

$$(1 - B)z_t = (1 - \theta B)a_t \qquad -1 < \theta < 1 \tag{4.8}$$

In this case the function $f(z_t, x_t, \theta)$ satisfies

$$a_t(\theta) = f(z_t, x_t, \theta) = \frac{(1-B)}{1-\theta B} z_t = \left[1 - \frac{B(1-\theta)}{1-\theta B}\right] z_t$$

Figure 4.1 144 normal deviates from $N(0,1)$, sine wave with amplitude 0.5 and the corresponding observations y_t from the model $y_t = \theta \, sin\,(\frac{\pi t}{6}) + a_t$.

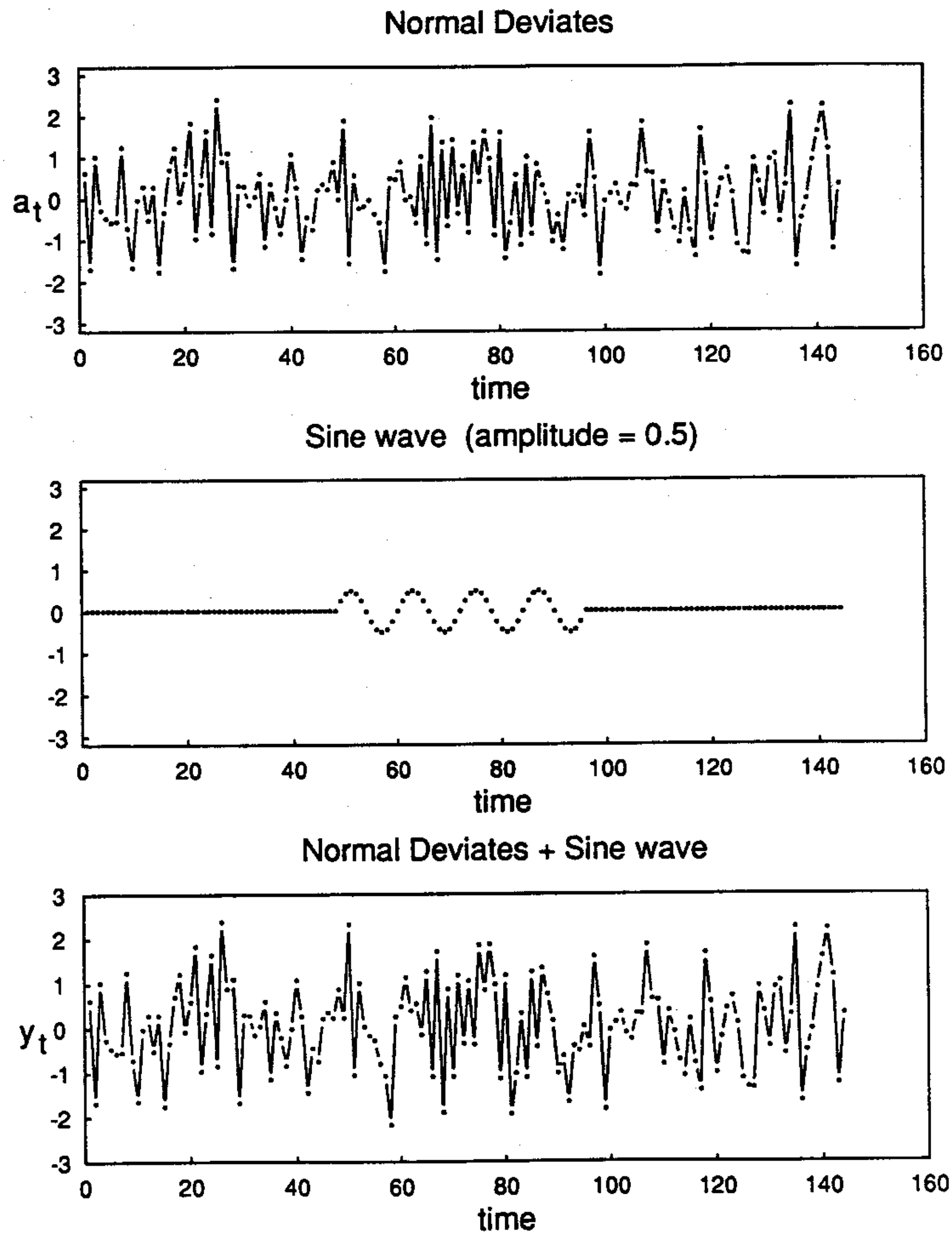
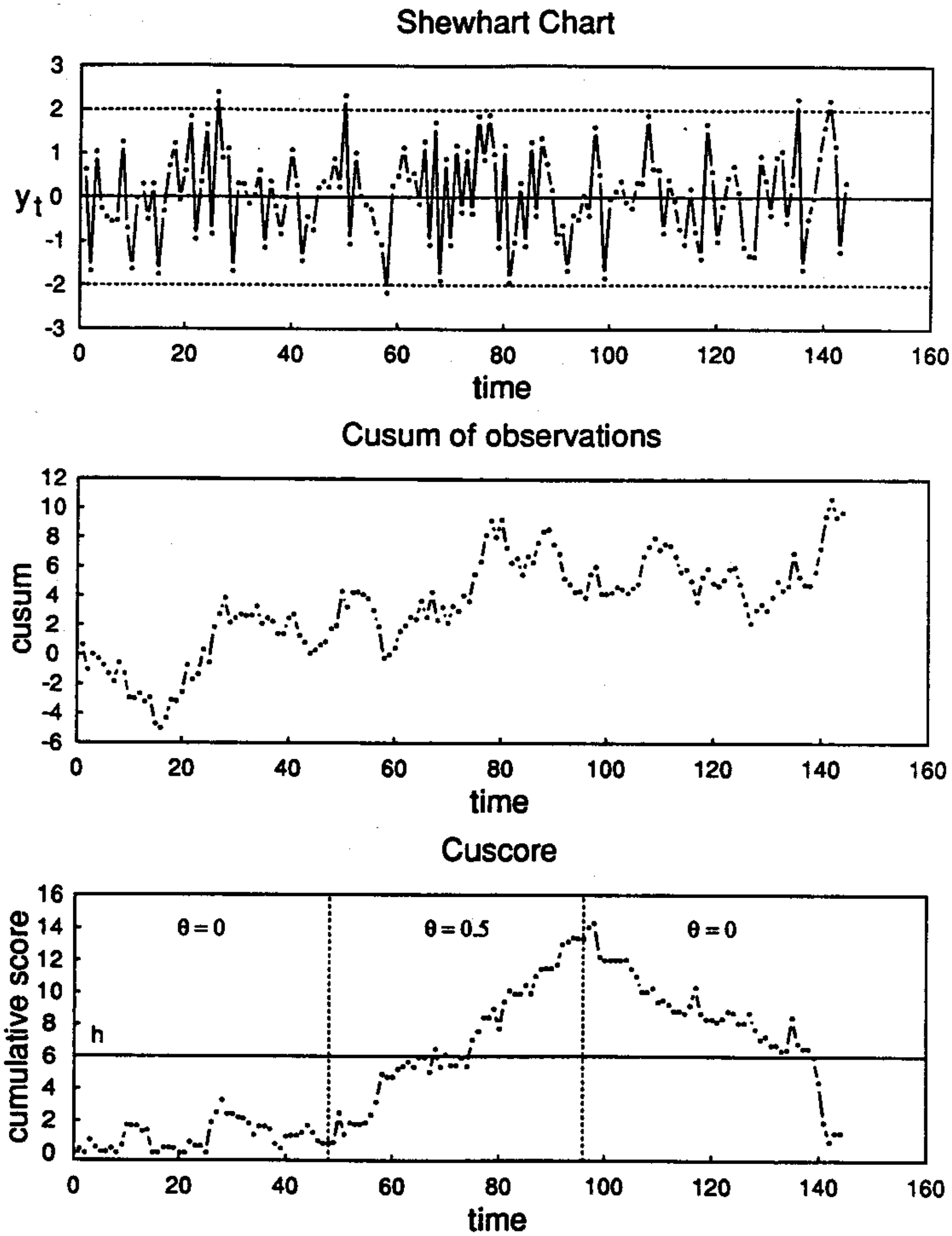


Figure 4.2 Shewhart chart with 2σ warning limits, cumulative sum chart and cuscore chart for the observations, $y_t = \theta \, sin\,(\frac{\pi t}{6}) + a_t$.



Hence $a_t(\theta) = z_t - \bar{z}_{t-1}$, where $\bar{z}_{t-1}$ is the exponentially weighted moving average with parameter $\lambda = 1 - \theta$.

In order to obtain the scores r_k we need to compute $d_t(\theta)$, the derivative of $a_t(\theta)$ with respect to θ

$$
\begin{aligned}
d_t(\theta_0) &= \left.\frac{\partial a_t(\theta)}{\partial \theta}\right|_{\theta=\theta_0} = -\frac{B(1-B)}{(1-\theta_0 B)^2} z_t = -\frac{(1-B)}{(1-\theta_0 B)^2} z_{t-1} \\
&= -\frac{a_{t-1}(\theta_0)}{1-\theta_0 B} = -\frac{\bar{a}_{t-1}(\lambda_0)}{1-\theta_0} \qquad (4.9) \\
&= -\sum_{j=1}^{\infty} \theta_0^{j-1}\, a_{t-j}(\theta_0)
\end{aligned}
$$

The score r_k is then a function of $a_t(\theta_0)$, the innovation occurring after time $t-1$, and $d_t(\theta_0)$, the exponentially weighted moving average of the residuals up to time $t-1$ with weights θ_0^{j-1}. In other words a_t and d_t are uncorrelated.

Monitoring increases or decreases in the parameter can be accomplished by means of CS_k^+ or CS_k^-, respectively. For this we compute the scores r_k

$$
r_k = -a_k(\theta_0) \sum_{j=1}^{\infty} \theta_0^{j-1}\, a_{k-j}(\theta_0) - \frac{\delta}{2}\left[\sum_{j=1}^{\infty} \theta_0^{j-1}\, a_{k-j}(\theta_0)\right]^2 \qquad (4.10)
$$

The cumulative sum score procedure is given by equation (4.6).

Three examples will help to clarify the use of CS_k^+ and CS_k^- to monitor changes in the smoothing parameter of an IMA process.

Example 1:

To see how the cuscore will perform we generated 200 normal deviates a_t from a $N(0,1)$, and the IMA process z_t with the following characteristics. For the first 50 observation we set $\theta_0 = 0.3$, for the next 50 observations we put $\theta_1 = 0.1$, then we increase the parameter to $\theta_1{}' = 0.5$, and for the last 50 observations we put again $\theta_0 = 0.3$. The z_t's are given by

$$z_t = \begin{cases} z_{t-1} + (1-0.3B)a_t & t = 1,\dots,50 \\ z_{t-1} + (1-0.1B)a_t & t = 51,\dots,100 \\ z_{t-1} + (1-0.5B)a_t & t = 101,\dots,150 \\ z_{t-1} + (1-0.3B)a_t & t = 151,\dots,200 \end{cases}$$

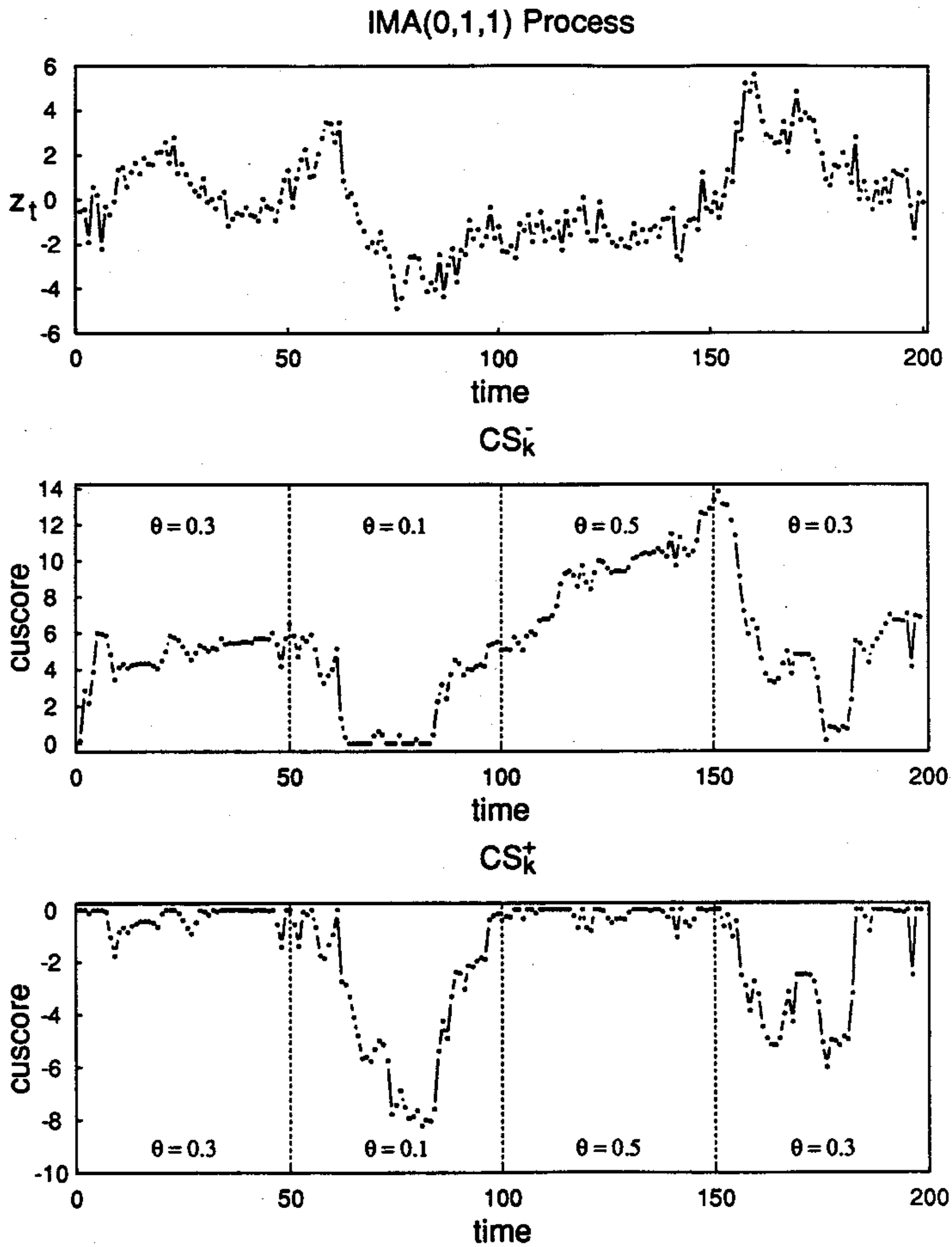
The cumulative sums for monitoring an increase or a decrease are obtained by using formulas (4.10) and (4.6), with $\delta = -0.2$ ($\theta_0 = 0.3$ and $\theta_1 = 0.1$), $\delta = 0.2$ ($\theta_0 = 0.3$ and $\theta_1' = 0.5$), and scores r_k

$$r_k^+ = -a_k(0.3)\sum_{j=1}^{\infty} 0.3^{j-1}\, a_{k-j}(0.3) - 0.1\left[\sum_{j=1}^{\infty} 0.3^{j-1}\, a_{k-j}(0.3)\right]^2$$

$$r_k^- = -a_k(0.3)\sum_{j=1}^{\infty} 0.3^{j-1}\, a_{k-j}(0.3) + 0.1\left[\sum_{j=1}^{\infty} 0.3^{j-1}\, a_{k-j}(0.3)\right]^2$$

Figure 4.3 shows the plot of the z_t's, CS_k^+ and CS_k^-. Note that from the CS_k^- plot, we observe that a decrease in the parameter occurred between observations 50 and 100, and again between observations 150 and 200, corresponding to the two changes in θ. The CS_k^+ plot, on the other hand, shows a change of slope right

Figure 4.3 200 observations from an IMA(0,1,1) process described in example 1. and the cuscore charts CS_k^+ and CS_k^-.



around observation 100, which corresponds to the change in θ from 0.1 to 0.5.

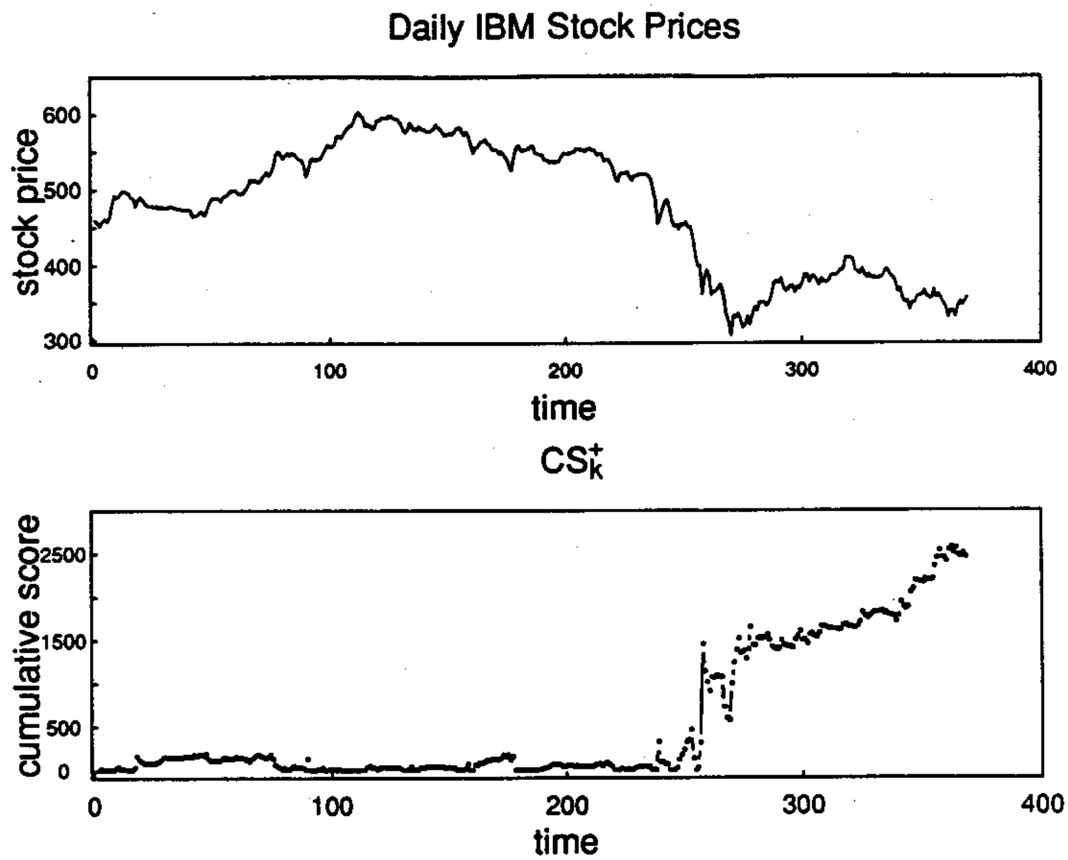
Example 2: IBM Stock Prices Data

Box and Jenkins (1970), fitted an IMA(0,1,1) model to the series of IBM closing stock prices. Later they argued that the parameter θ in the model might have changed over time making the fitted model inadequate. They divided the series and fitted IMA(0,1,1) processes separately to the first and second halves of the series, obtaining $\theta = -0.29$ for the first half and $\theta = -0.03$ for the second half.

We can use the cuscore CS_k to monitor parameter change using the whole series with $\theta_0 = -0.29$ and $\theta_1 = 0$. Using the approximation given in (4.7) we obtain $h = 260$ and the procedure signals an increase in the parameter at observation 239, see figure 4.4.

The test given here is equivalent to the one in Bagshaw and Johnson (1977). These authors use the fact that the cumulative sums, $\sum_{t=1}^{k} [a_t^2(\theta_1) - a_t^2(\theta_0)]$, converge in distribution to a Wiener process when the a_t are normally distributed to approximate the properties of the test.

Figure 4.4 Daily IBM stock prices and cuscore CS_k^+ for detecting a change in the parameter from $\theta_0 = -0.29$ to $\theta_1 = 0$.



4.3. Multiparameter Case

We now consider the case in which the number of parameters, in the model that best describes the observations y_t's, is greater than one. In other words, the residuals a_t's satisfy $a_t(\Theta)=f(y_t,\mathbf{x_t},\Theta)$, where $\Theta' = (\theta_1,\theta_2,\cdots,\theta_p)$ is a $p \times 1$ vector of parameters and $\mathbf{x}_t' = (x_{1t},x_{2t},\cdots,x_{pt})$ is a $p \times 1$ vector of explanatory variables.

Let Θ_0 be the current value of the vector of parameters Θ in this model and let Θ_1 be some other value of the vector of parameters. Following the derivation of equation (4.1) and after taking logs, the sequential probability ratio test for Θ_0 against Θ_1 gives the cumulative sum, equation (4.2),

$$S_k = \frac{1}{2\sigma_a^2}\sum_{t=1}^{k}\left[a_t^2(\Theta_0) - a_t^2(\Theta_1)\right] .$$

Proceeding as in section 4.1 and taking the multivariate Taylor expansion of $a_t^2(\Theta)$ around $\Theta = \Theta_0$ we get

$$a_t^2(\Theta) \approx a_t^2(\Theta_0) + 2a_t(\Theta_0)\Delta\Theta'\nabla a_t(\Theta_0) + [\Delta\Theta'\nabla a_t(\Theta_0)]^2 .$$

Evaluating this expression at the point $\Theta = \Theta_1$, we see that the difference in the cumulative sum, $a_t^2(\Theta_0) - a_t^2(\Theta_1)$, can be written as

$$a_t^2(\Theta_0) - a_t^2(\Theta_1) \approx 2a_t(\Theta_0)\Delta\Theta'\mathbf{d_t}(\Theta_0) - [\Delta\Theta'\mathbf{d_t}(\Theta_0)]^2 .$$

Where $\Delta\Theta = \Theta_1 - \Theta_0$ is the change in the parameter Θ, and $\mathbf{d_t}(\Theta_0) = -\nabla a_t(\Theta_0)$ is the vector of partial derivatives of a_t with respect to Θ evaluated at Θ_0.

The cumulative sum S_k becomes

$$
\begin{aligned}
S_k &= \frac{1}{\sigma_a^2}\sum_{t=1}^{k}\left[a_t(\Theta_0) - \frac{\Delta\Theta'\mathbf{d}_t(\Theta_0)}{2}\right]\Delta\Theta'\mathbf{d}_t(\Theta_0) \\
&= \frac{1}{\sigma_a^2}\sum_{t=1}^{k} r_t
\end{aligned}
\tag{4.11}
$$

The quantity r_t is the score to be used in the sequential procedure for monitoring changes in the vector Θ from the value Θ_0 to some other value Θ_1.

If we apply the procedures given in equation (4.6), i.e. CS_k^+ or CS_k^-, the cuscore will signal a change in the vector of parameters but it will not tell us which of the parameters has changed. Also some of the components in Θ may be decreasing while others may be increasing, which may result in r_t being close to zero. To overcome this problem it is better to keep a cuscore for each of the elements of the vector Θ applying CS_k^+ or CS_k^- depending on which change we want to detect.

The score r_t can be written as a sum of p components, each one corresponding to one of the elements of the vector $\Theta' = (\theta_1, \theta_2, \cdots, \theta_p)$. For the m^{th} element, θ_m, of Θ we have

$$
\begin{aligned}
r'_{mt} &= [a_t(\Theta_0) - \frac{1}{2}\delta_m d_{mt}]\,\delta_m d_{mt} - \frac{1}{2}\sum_{j\neq m}\delta_m\delta_j d_{mt} d_{jt} \\
&= r_{mt} - cp_{mt}
\end{aligned}
\tag{4.12}
$$

Where δ_m and d_{mt} are the m^{th} components of the vectors $\Delta\Theta' = (\delta_1, \delta_2, \cdots, \delta_p)$ and $\mathbf{d}_t' = (d_{1t}, d_{2t}, \cdots, d_{pt})$ respectively.

The component r_{mt} corresponds to the score obtained by applying the sequential procedure, described in section 4.1, to the single parameter θ_m. The term cp_{mt} is half the sum of all the products of the change in the parameter θ_m, measured by $\delta_m d_{mt}$, with the change in all the other parameters. In other words we substract from the score for θ_m, r_{mt}, the contribution from all the other parameters present in the model. In this way each r'_{mt} will measure the change in the parameter θ_m independently of the other parameters in the model.

Individual charts can be now kept for each parameter θ_m, by applying the sequential procedures CS_k^+ or CS_k^-, to the scores r'_{mt}.

4.3.1. Case p=2

To clarify these ideas two examples for the case p=2 are presented. In the first example both parameters in the model are increasing; while in the second example one of the parameters increases and the other one decreases.

Setting p=2 in equation (4.12) we see that the score r_t takes the form

$$\begin{aligned} r_t &= [a_t(\Theta_0) - \tfrac{1}{2}(\delta_1,\delta_2)\cdot(d_{1t},d_{2t})]\,[(\delta_1,\delta_2)\cdot(d_{1t},d_{2t})] \\ &= [a_t(\Theta_0) - \tfrac{1}{2}\delta_1 d_{1t}]\delta_1 d_{1t} + [a_t(\Theta_0) - \delta_2 d_{2t}]\delta_2 d_{2t} - \delta_1\delta_2 d_{1t} d_{2t} \\ &= r_{1t} + r_{2t} - \delta_1\delta_2 d_{1t} d_{2t}\,. \end{aligned}$$

As we have seen r_{1t} and r_{2t} correspond to the scores obtained by applying the sequential procedure to the parameters θ_1 and θ_2 respectively. The term $\delta_1\delta_2 d_{1t} d_{2t}$ is the joint contribution of the change in the parameters θ_1 and θ_2 to the score r_t. The individual scores for the parameters θ_1 and θ_2 are given by

$$\begin{aligned} r'_{1t} &= r_{1t} - \tfrac{1}{2}\delta_1\delta_2 d_{1t} d_{2t} \\ r'_{2t} &= r_{2t} - \tfrac{1}{2}\delta_1\delta_2 d_{1t} d_{2t} \,. \end{aligned} \tag{4.13}$$

Computing the cumulative scores CS_k^+ or CS_k^- using r'_{1t} and r'_{2t}, we are able to monitor changes in both parameters simultaneously.

Example 1:

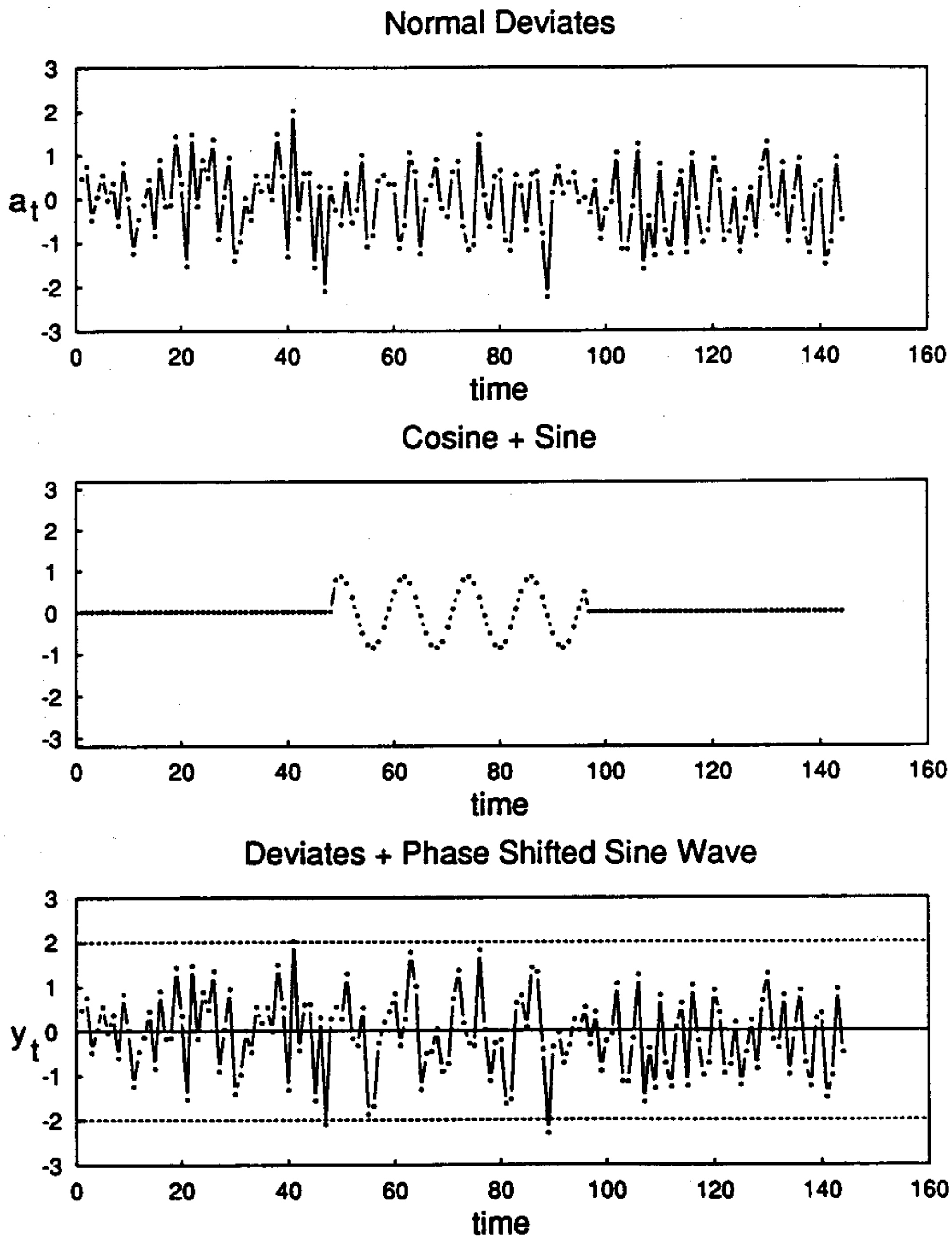
Consider the function $y_t = A\sin(\omega x_t + \phi) + a_t$, which is the harmonic of amplitude $|A|$, frequency ω and initial phase ϕ and added noise a_t. The periodic part of this function can be represented in the form $\theta_1\cos\omega x_t + \theta_2\sin\omega x_t$. Letting $\omega = \frac{\pi}{6}$ and $x_t = t$ we have

$$a_t(\theta_1,\theta_2) = y_t - \theta_1\sin(\frac{\pi}{6}t) - \theta_2\cos(\frac{\pi}{6}t)\,.$$

For this example 144 normal deviates a_t were generated as shown in figure 4.6a. For the first and last 48 observations we set $\Theta_0 = (0,0)$; while for observations 49 to 96 we have $\Theta_1 = (0.5,0.7)$.

The component $0.5\sin(\frac{\pi}{6}t) + 0.7\cos(\frac{\pi}{6}t)$ and the observations y_t are shown in figures 4.6b and 4.6c respectively. Note that all the observations in figure 4.6c are well within the Shewhart "3σ limits" indicating no lack of control.

Figure 4.5 144 normal deviates from $N(0,1)$, $0.5cos(\pi t/6) + 0.7sin(\pi t/6)$, and the corresponding observations $y_t = \theta_1 cos(\pi t/6) + \theta_2 sin(\pi t/6) + a_t$.



The combined score r_t for monitoring an increase in the parameters θ_1 and θ_2 is equal to

$$r_t = [y_t - (0.25,0.35)\cdot(d_{1t},d_{2t})](0.5,0.7)\cdot(d_{1t},d_{2t});$$

where $d_t = (cos(\frac{\pi}{6}t), sin(\frac{\pi}{6}t))$. The individual scores for θ_1 and θ_2 are given by

$$r'_{1t} = [y_t - .25cos(\frac{\pi}{6}t)]0.5cos(\frac{\pi}{6}t) - .175cos(\frac{\pi}{6}t)sin(\frac{\pi}{6}t)$$

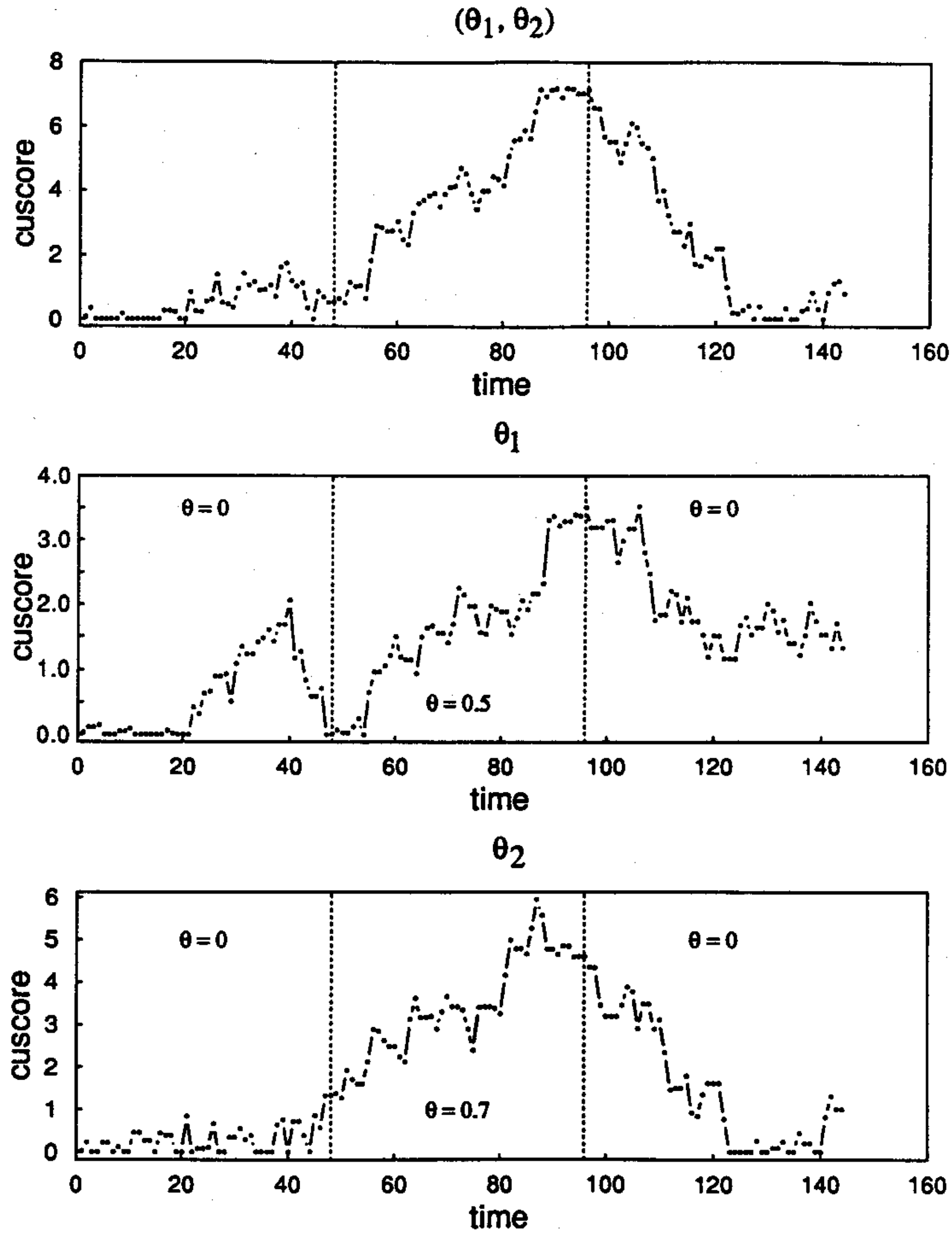
$$r'_{2t} = [y_t - .35sin(\frac{\pi}{6}t)]0.7sin(\frac{\pi}{6}t) - .175cos(\frac{\pi}{6}t)sin(\frac{\pi}{6}t).$$

Figure 4.7 shows the plot of the cuscore CS_k^+ using r_t that for monitoring increases in $\Theta = (\theta_1, \theta_2)$, and the cuscores using r'_{1t} and r'_{2t} that monitor θ_1 and θ_2 respectively. Note that all the plots show a change of slope around observation 50 indicating the change that has occurred in the parameter values.

With $\alpha = 0.05$ equation (4.7), putting $\delta = 1$, gives a decision limit $h = 3$, and the combined chart indicates a change in the vector Θ at observation 60. The chart for θ_1 signals a change at observation 89, while the chart for θ_2 signals a change at observation 63.

Since both parameters are changing in the same direction the detection with the combined chart is faster. Note however that from the combined chart is impossible to tell which of the parameters is changing. The individual charts on the other hand, clearly show the changes, if any, in each of the parameters.

Figure 4.6 Cuscores CS_k^+ for the vector of parameters $\Theta = (\theta_1, \theta_2)$, the parameter θ_1, and the parameter θ_2.



Example 2:

As a second example 100 observations from an IMA(0,2,2) process were generated according to the following model

$$\nabla^2 z_t = \begin{cases} a_t & t = 1,\ldots,50 \\ (1 - 0.6B + 0.3B^2)\, a_t & t = 51,\ldots,100 \end{cases}$$

For the first 50 observations $\Theta = (0,0)$ after which θ_1 increased to 0.6, while θ_2 decreased to -0.3.

For an IMA(0,2,2) process we can write the residuals as

$$a_t = \frac{(1-B)^2}{(1-\theta_1 B - \theta_2 B^2)}\, z_t$$

The vector of derivatives of a_t with respect to Θ, is then given by

$$\mathbf{d}_t' = -\nabla a_t(\Theta_0) = \begin{bmatrix} \dfrac{-B(1-B)^2}{(1-\theta_1 B - \theta_2 B^2)^2} \\ \dfrac{-B^2(1-B)^2}{(1-\theta_1 B - \theta_2 B^2)^2} \end{bmatrix} z_t = \begin{bmatrix} \dfrac{-B}{(1-\theta_1 B - \theta_2 B^2)} \\ \dfrac{-B^2}{(1-\theta_1 B - \theta_2 B^2)} \end{bmatrix} a_t$$

$$= \begin{bmatrix} \dfrac{-a_{t-1}}{(1-\theta_1 B - \theta_2 B^2)} \\ \dfrac{-a_{t-2}}{(1-\theta_1 B - \theta_2 B^2)} \end{bmatrix} = \begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} \tag{4.14}$$

Starting with values of zero the residuals a_t's and the derivatives x_t's, can be calculated recursively (see Box and Jenkins (1970) page 238), by means of the following formulas

$$a_t = \nabla^2 z_t + \theta_{10} a_{t-1}(\Theta_0) + \theta_{20} a_{t-2}(\Theta_0)$$

$$x_t = \theta_{10} x_{t-1} + \theta_{20} x_t^2 - a_t(\Theta_0) ,$$

where the subindex 0 denotes the initial value of the parameters.

The scores r'_{1t} and r'_{2t} are calculated using equation (4.13) with $\mathbf{d}'_t$ given by (4.14) and $\Delta\Theta = (0.6, -0.3)$. Figure 4.8 shows the cumulative scores CS_k^+ and CS_k^- for monitoring increases in the parameter θ_1 and decreases in the parameter θ_2 respectively. Note the change of slope after observation 50 signaling the parameter changes.

4.4. General Linear Model

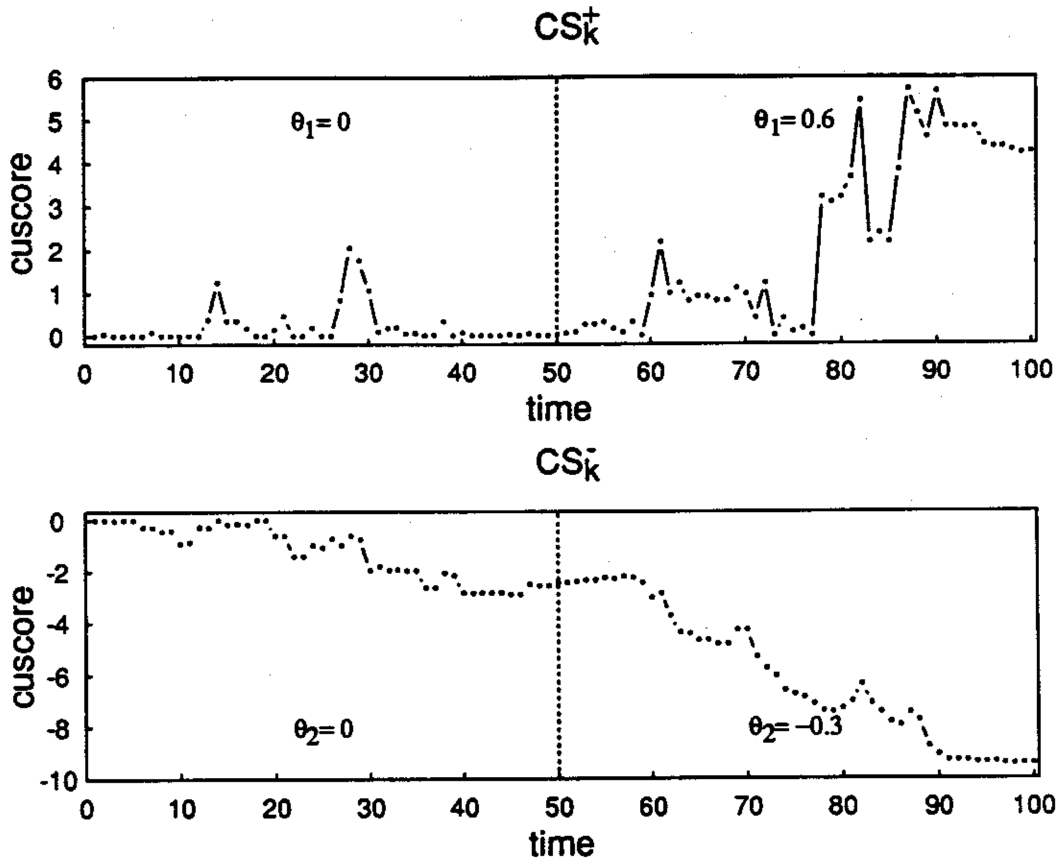
Consider the process that generates observations y_t over time and that follow the linear model

$$y_t = \mathbf{x_t}'\Theta + a_t \ ,$$

where the a_t are independent and identically distributed as $N(0, \sigma_a^2)$, Θ is a $p \times 1$ vector of parameters and $\mathbf{x_t}$ is a $p \times 1$ vector of the values of p explanatory variables; i.e.,

$$\Theta' = (\theta_1, \theta_2, \cdots, \theta_p) ,$$
$$\mathbf{x_t}' = (x_{1t}, x_{2t}, \cdots, x_{pt}) .$$

Figure 4.7 Cuscores CS_k^+ for the parameter θ_1, and CS_k^- for the parameter θ_2 for the IMA(0,2,2) process described in example 2.



Where the first explanatory variable x_{1t} will be equal to one if the model contains a constant term.

Regression analysis of time-series data, is usually based on the assumption that the regression relationship is constant over time. We want to be able to detect changes in the parameter Θ that may have occurred over time, especially if the model is going to be used for forecasting. This parameter change could correspond to a change in the structure of the economy at an unknow point in time. The cuscore procedures described in section 4.3 can be used to sequentially monitor changes in the parameters.

Following the derivation of equation (4.11) with $\mathbf{d_t} = -\mathbf{x_t}$, the sequential probability ratio test for testing $\Theta = \Theta_0$ versus $\Theta = \Theta_1$ leads to a score r_t

$$r_t = [y_t - \mathbf{x_t}'\frac{(\Theta_0 + \Theta_1)}{2}]\, \mathbf{x_t}'(\Theta_1 - \Theta_0)\,. \tag{4.15}$$

By letting $\overline{\Theta} = \dfrac{(\Theta_0 + \Theta_1)}{2}$ and $\Delta\Theta = (\Theta_1 - \Theta_0)$, be the average and the difference of the two vectors of parameters respectively, the score r_t can be written as

$$r_t = a_k(\overline{\Theta})\mathbf{d_t'}\Delta\Theta \ . \tag{4.16}$$

Now under Θ_1 the expectation of y_t is

$$E_{\Theta_1}(y_t) = \mathbf{x_t'}\Theta_1 = \mathbf{x_t'}\Theta_0 + \mathbf{x_t'}(\Theta_1 - \Theta_0) = E_{\Theta_0}(y_t) + \mathbf{d_t'}\Delta\Theta\,.$$

Therefore, the quantity $\mathbf{d_t'}\Delta\Theta = E_{\Theta_1}(y_t) - E_{\Theta_0}(y_t)$, is a measure of the amount of change in the expectation of y_t.

The score r_t is then a function of the residuals, $a_k(\overline{\Theta})$, of a model which is "between" the models given by Θ_0 and Θ_1, in the sense that we are using the parameter $\overline{\Theta}$ which is between Θ_0 and Θ_1, and the amount of change in the expectation of y_t.

As an example consider again the case where $p=2$, i.e., $\Theta=(\theta_1,\theta_2)$, $a_t(\Theta)=y_t-\theta_1-\theta_2 x_t$ and $\mathbf{d}_t'=(1,x_t)$. The score for monitoring changes in Θ from $\Theta_0=(\theta_{10},\theta_{20})$ to $\Theta_1=(\theta_{11},\theta_{21})$, is given by

$$r_t=\left[y_t-\frac{(\theta_{10}+\theta_{11})+(\theta_{20}+\theta_{21})x_t}{2}\right][(\theta_{11}-\theta_{10})+(\theta_{21}-\theta_{20})x_t]\,.$$

Note that if both slopes are zero; i.e, $\theta_{20}=\theta_{21}=0$ the score becomes

$$r_t=\left[y_t-\frac{(\theta_{10}+\theta_{11})}{2}\right](\theta_{11}-\theta_{10})\,.$$

Which corresponds to cumulative sum chart for monitoring the mean. On the other hand if both lines pass through the origin;i.e., $\theta_{10}=\theta_{11}=0$ then

$$r_t=\left[y_t-\frac{(\theta_{20}+\theta_{21})x_t}{2}\right](\theta_{21}-\theta_{20})x_t\,.$$

This corresponds to the example given in section 4.2.2 of monitoring the slope of a line through the origin.

If both the slope and the intersect are different from zero, they can be monitored using the scores r_{1t}' and r_{2t}' as given by equation (4.13)

$$r'_{1t} = \left[y_t - \frac{(\theta_{10} + \theta_{11})}{2} \right] (\theta_{11} - \theta_{10}) - \frac{1}{2}(\theta_{11} - \theta_{10})(\theta_{21} - \theta_{20})x_t \; .$$

$$r'_{2t} = \left[y_t - \frac{(\theta_{21} + \theta_{20})x_t}{2} \right] (\theta_{21} - \theta_{20})x_t - \frac{1}{2}(\theta_{11} - \theta_{10})(\theta_{21} - \theta_{20})x_t \; .$$

As an example 30 observations y_t were generated according to the model $y_t = \theta_0 + \theta_1 x_t + a_t$, where the a_t's are distributed as $N(0,1)$ and the x_t's are equally spaced in the interval [2,7]. For the first 15 observations $\Theta_0 = (4,6)$ while for the remaining 15 the vector of parameters changed to $\Theta_1 = (4.1, 6.3)$.

Figure 4.9a shows a plot of y_t against x_t. The plot does not indicate anything unusual and it is hard to tell if there has been a change of slope or intercept. Figure 4.9b is a plot of CS_k^+ for the vector of parameters Θ with score given by

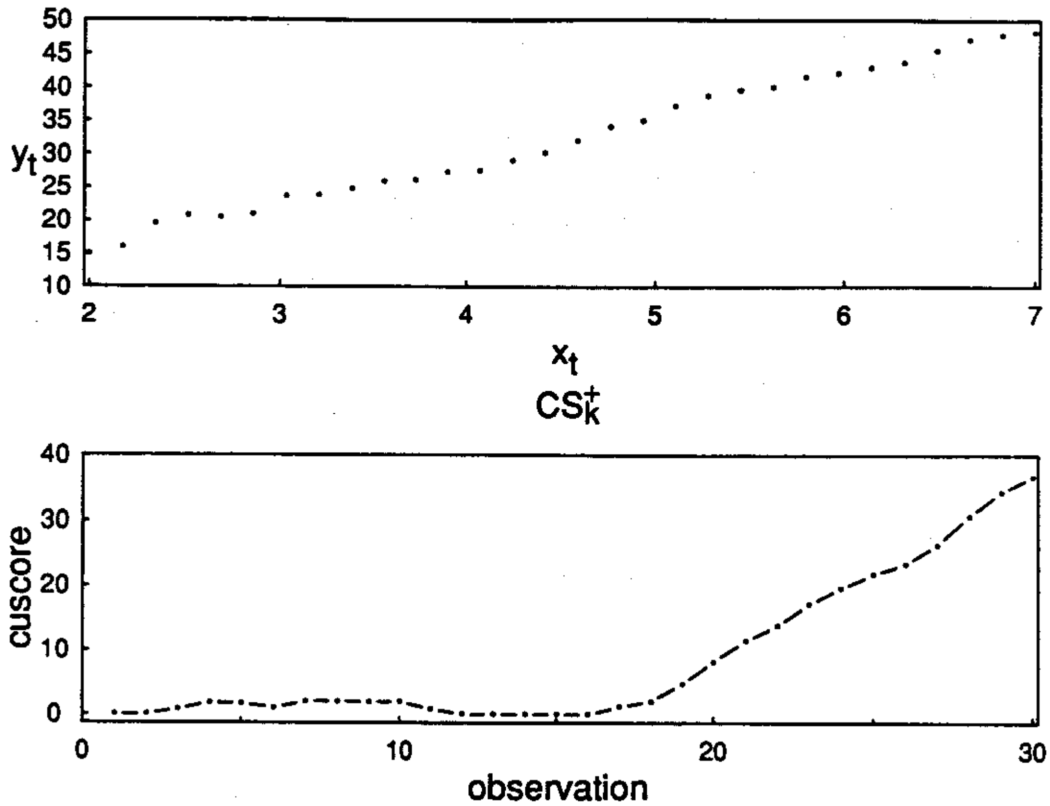
$$r_t = [y_t - (4.05 + 6.15x_t)][0.1 + 0.3x_t] \; .$$

Note the change of slope around observation 16 indicating a change in the vector of parameters. This shows the great sensitivity of the cuscore procedure in detecting small changes in the parameter values.

4.4.1 Recursive Residuals

Techniques for detecting departures from constancy of regression relationships have been considered by Brown, Durbin and Evans (1975). In that paper the authors describe how cumulative sums of recursive residuals can be used for this purpose. We show that the score r_t is a function of the squared recursive residuals

Figure 4.8 Plot of y_t versus x_t with Θ changing after observation 15 from $\Theta_0 = (4,6)$ to $\Theta_1 = (4.1, 6.3)$. Cuscore CS_k^+ signaling the change around observation 17.



when applied to a set of n observations.

The recursive residuals w_t are standarized residuals from the regression of each observation y_t, with the regression coefficients being calculated from the observations $y_1, \ldots, y_{t-1}$ for $t = p+1, \cdots, n$, where p is the number of regressors and n is the number of observations. This is a transformation of the least squared residuals to independent $N(0,\sigma^2)$ variables, see Schweder (1976). The recursive residuals allow to test for a model change over time, which is something that the least squared residuals are not well suited to do.

Assuming H_0 to be true let $\hat{\Theta}_t$ be the least squares estimate of Θ based on the first t observations, i.e. $\hat{\Theta}_t = (X_t' X_t)^{-1} X_t Y_t$, where $X_t' = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_t]$, $Y_t' = [y_1, y_2, \cdots, y_t]$ and $(X_t' X_t)^{-1}$ is non-singular. The recursive residuals, see Brown et al (1975), are defined as

$$w_t = \frac{y_t - \mathbf{x}_t' \hat{\Theta}_{t-1}}{\sqrt{1 + \mathbf{x}_t'(X_{t-1}' X_{t-1})^{-1} \mathbf{x}_t}} \qquad t = p+1, \cdots, n . \qquad (4.17)$$

Here we are using the first p data points as the base for the regression but this is not the only sensible choice. One can use the last p points or any other set of arbitrarily selected p points. See for example Galpin and Hawkins (1984).

To test the null hypothesis that the regression coefficients are stable; i.e, that $\Theta_0 = \Theta_1$, the cumulative sums $W_r = \frac{1}{s} \sum_{i=p+1}^{r} w_i$; where s denotes the estimated standard deviation, $s^2 = \sum_{i=p+1}^{n} w_i^2 / (n-p)$, and the squared recursive residuals,

$S_r = \frac{1}{(n-p)s^2} \sum_{i=p+1}^{r} w_i^2$, are plotted and monitored for possible departures from zero. Garbade (1977) has shown that the test based on S_r is more powerful than the one based on W_r. We will show that the squared recursive residual w_t^2 is a function of the score defined by equation (4.15).

The cuscore, as defined by equation (4.6) with r_t as in equation (4.15), is a sequential procedure for monitoring changes in the parameters from Θ_0 to Θ_1 in which the sample size is not fixed. We can however, by considering the elements of the sample as a sequence, compute the score by comparing, at each point t, the least squares estimate $\hat{\Theta}_t$ with the one obtained by using the first $t-1$ observations $\hat{\Theta}_{t-1}$. Using equation (4.15) we have that the score r_t can be written as

$$r_t = [y_t - \mathbf{x}_t' \frac{(\hat{\Theta}_{t-1} + \hat{\Theta}_t)}{2}]\mathbf{x}_t'(\hat{\Theta}_t - \hat{\Theta}_{t-1}) \tag{4.18}$$

Now from equation (4) in Brown et al (1975), and the discussion by Anderson

$$\hat{\Theta}_t - \hat{\Theta}_{t-1} = (X_t'X_t)^{-1}\mathbf{x}_t(y_t - \mathbf{x}_t'\hat{\Theta}_{t-1})$$

$$\hat{\Theta}_t - \hat{\Theta}_{t-1} = w_t(X_t'X_t)^{-1}\mathbf{x}_t\sqrt{1 + \mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t}$$

Hence the recursive residuals are a function of the changes in the parameter estimates , and equation (4.18) can be written as

$$r_t = [(y_t - \mathbf{x}_t'\hat{\Theta}_{t-1}) - \frac{1}{2}\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t(y_t - \mathbf{x}_t'\hat{\Theta}_{t-1})]\,[\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,(y_t - \mathbf{x}_t'\hat{\Theta}_{t-1})$$

$$= [1 - \frac{1}{2}\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,[\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,(y_t - \mathbf{x}_t'\hat{\Theta}_{t-1})^2 \tag{4.19}$$

$$= [1 - \frac{1}{2}\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,[\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,[1 + \mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t]\,w_t^2$$

Now the matrix $(X_t'X_t)^{-1}$ can be written in terms of the matrix $(X_{t-1}'X_{t-1})^{-1}$ of the first $r-1$ observations, see equation (3) of Brown et al (1975),

$$(X_t'X_t)^{-1} = \frac{(X_{t-1}'X_{t-1})^{-1}[1 + \mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t] - (X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t\mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}}{1 + \mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t}$$

Substituting this expression for the middle term of equation (4.19) above we have

$$\begin{aligned} r_t &= [1 - \frac{1}{2}\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t] \times \\ &\qquad [\mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}[1 + \mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t - \mathbf{x}_t\mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}]\mathbf{x}_t]\,w_t^2 \end{aligned} \tag{4.20}$$

$$r_t = [1 - \frac{1}{2}\mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t]\,[\mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t]\,w_t^2\,.$$

Which shows that the squared recursive residual w_t^2 is a function of the score r_t.

For the t observations $Y_t' = [y_1, y_2, \cdots, y_t]$ the matrix $H_t = (X_t'X_t)^{-1}$ is the hat matrix; i.e, the projection matrix that maps the Y_t' into the fitted values $\hat{Y}_t' = [\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_t]$. The quantity $h_t = \mathbf{x}_t'(X_t'X_t)^{-1}\mathbf{x}_t$ is the tth element in the diagonal of $H_t = X_t(X_t'X_t)^{-1}X_t'$; and is the amount of leverage or influence exerted on $\hat{y}_t$ by y_t.

The influence measure h_t does not depend on the actual value of y_t since H_t depends only on X_t. It is for this reason, see Weisberg (1985), that Cook and Weisberg prefer to call the h_t *potential* since its importance is uncertain depending on the value of y_t. Good measures of influence are therefore those that take into account both the value of h_t and the value of the observation y_t. We will show that the score r_t is a function of h_t and w_t; the latter being a function of y_t.

If A is a non-singular matrix and u and v are two columns vectors the following result holds, see Rao (1973) page 33

$$(A + u'v)^{-1} = A^{-1} - \frac{(A^{-1}u)(v'A^{-1})}{1 + v'A^{-1}u} .$$

Then $\mathbf{x}_t'(X_{t-1}'X_{t-1})^{-1}\mathbf{x}_t = h_t/(1-h_t)$, and formula (4.20) becomes

$$r_t = [1 - \frac{1}{2}h_t]\frac{h_t}{1-h_t} w_t^2 \qquad (4.21)$$

The size of r_t is determined by two different sources: the potential h_t, which measures the location of $\mathbf{x}_t$ relative to $\bar{\mathbf{x}}$, and the size of the recursive residual w_t^2 which is the standardized prediction error of y_t when predicted from $y_1, \cdots, y_{t-1}$.

CHAPTER 5

SUMMARY AND FUTURE RESEARCH

5.1 Sequential Methods for Variance

We have discussed how the Wald-Barnard sequential ratio test leads to the use of the cumulative sums $\sum_{i=1}^{k}(x_i-\mu)^2-s^2$ for monitoring variances. This CUSUM procedure can quickly detect specific changes, as for example detection of small shifts.

A contour nomogram have been developed to aid in the design of the charts. The nomogram has contour lines of the average run length at the acceptable quality, σ_a, and the rejectable quality level, σ_r, in the plane (h/σ_a^2 , σ_r/σ_a). The action limit h and the reference value s^2, can be chosen from the nomogram to yield the desired ARL values.

Approximate formulas for the ARL are given to give an idea of what the performance of a particular chart will be when detecting a shift of magnitude σ_r/σ_a and using an action limit h. Also adjustment factors were obtained for the case in which the mean is replaced by the sample average $\bar{X}$.

The effect of kurtosis in the ARL values was studied using the exponential family of distributions. It was shown that for platykurtic distributions the assumption of normality overestimates the ARL; while for leptokurtic distributions the

ARL values are underestimated.

Following the approach of Box and Tiao (1968), we showed how the CUSUM procedure can be used for outlier detection. By assuming that "good" observations are distributed about a fixed mean μ with variance σ^2, with a prior probability α, and that a "bad" observation is distributed about the same mean but with variance $k\sigma^2$, with prior probability $1-\alpha$, the problem reduce to that of detecting changes in variability of magnitude k. The sequential procedure leads to a CUSUM that takes into account the prior probability α. In particular the reference value $s^2(\alpha)$ depends on this prior probability and coincides with s^2 when $\alpha = ½$.

There are several ways in which this research might be extended

Unstable μ

The CUSUM procedures presented in chapters 2 and 3 assume that the mean of the quality characteristic is stable and known. In section 3.4 we discussed the effect of using the sample average as an estimate of μ when the mean is not known. It is desirable then to study the effects of unstable means on the design of the charts. The results can then be applied to CUSUM charts for means for which, see Gibra (1975), no attempts have been made to study the effect of unstable variances.

Trend in σ

For control charts for the mean some studies have been made, Bissell (1984), Davies and Woodall (1988), on the performance of the charts under linear trend.

Linear trends can easily occur in processes as for example tool wear. For obtaining ARL values, one assumes that the shift in process variabilty occurs pointwise. The performance of the CUSUM chart, which is based on the ARL values, under linear drift in the process variance should be investigated.

Comparisons with other Charts

Comparisons need to be made between the CUSUM procedures presented here and the one suggested by Page (1963) of using cumulative sums of ranges, and the one proposed by Hawkins (1981) of using cumulative sums of $|X_i/\sigma|^{1/2}$, where X_i is distributed as $N(0,\sigma)$.

Effect of Non-normality

The effect of non-normality was studied using the exponential power family of distributions. For this family the distributions are symmetric and departures from normality are measured in terms of a parameter which is a function of the kurtosis. Departures from normality in terms of symmetry can be studied by using, for example, the Gamma distribution.

5.2 Sequential Methods for Models

In chapter 4 we showed how a sensitive test for shifts of the parameters of any model can be derived using a cumulative score technique that we denoted CUSCORE. Once a model has been adequatly identified, fitted and checked, the residuals obtained from such a model can be used to monitor parameter changes.

Several examples were given that illustrate how the CUSCORE can detect patterns that even the cumulative sum of the observations can not detect. In particular it was shown how the CUSCORE can be used to detect a sine wave buried in random noise.

One difficulty with this approach is that the distribution of the increments in the cumulative sums, neccesary to obtain ARL values, is model dependent making it hard to evaluate the performance of the CUSCORE in terms of ARL.

APPENDIX

TABLES OF ARL VALUES FOR THE EXPONENTIAL POWER DISTRIBUTION

In this appendix we present tables for the average run length values at the acceptable quality level σ_a, denoted by L_a, and at the rejectable quality level σ_r, denoted by L_r, for different values of the parameter γ, measuring the amount of kurtosis, in the exponential power family of distributions.

The parameter values in the tables cover shifts in the standard deviation from 20% up to 200%; i.e, from $\sigma_r/\sigma_a = 1.2$ to $\sigma_r/\sigma_a = 3.0$, and five different values of the decision interval h. Ten values of the parameter were chosen to cover the interval $(-1, 1]$.

The ARL values were obtained by solving the integral equations for $P(z)$ and $N(z)$; given by equations (2.6) and (2.7) with density function $f(x)$ given by (2.19), and substituting these values in equation (2.9).

There are twenty tables corresponding to the ten values of γ and the two levels L_a and L_r. Once the parameter γ is determined, the appropiate table can be used to choose the value of h according to the pre-specified L_a and L_r. By fixing the value of σ_r/σ_a, five different CUSUM schemes are available, corresponding to the five different values of h; one can then decide which is the more reasonable (L_a, L_r) combination.

Table 1a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 1$.

Average Run Length at σ_a for $\gamma = 1$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	110.80	240.43	630.13	1094.94	2514.16
1.3	136.17	325.53	1031.83	2075.81	5652.49
1.4	165.14	438.74	1638.56	3829.42	12486.47
1.5	198.17	578.26	2545.87	6710.32	27557.20
1.6	234.99	749.74	3884.96	11178.27	56004.24
1.7	275.71	957.58	5708.01	18131.68	105461.10
1.8	320.10	1198.61	8087.87	28632.89	148488.64
1.9	368.25	1480.76	11205.58	43231.29	343764.22
2.0	419.66	1804.07	15264.16	62814.12	508754.61
2.1	474.71	2165.30	20005.43	88650.87	806997.89
2.2	532.72	2568.06	26237.94	121064.12	1202211.70
2.3	594.23	3018.39	32669.35	164560.80	1830181.33
2.4	658.40	3507.38	41728.94	217485.30	2503325.70
2.5	725.19	4034.55	51653.75	282529.29	2851623.11
2.6	795.05	4608.87	62973.07	358365.28	7684960.28
2.7	867.02	5226.49	75687.14	444872.15	5346543.68
2.8	941.49	5880.67	90270.10	562442.01	8346669.69
2.9	1018.44	6572.13	105699.11	673476.15	$> 10^7$
3.0	1097.14	7307.23	123659.87	810304.93	$> 10^7$

Table 1b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 1$.

Average Run Length at σ_r for $\gamma = 1$					
	Decision Interval h				
$\frac{\sigma_r}{\sigma_a}$	5	7	10	12	15
1.2	38.42	60.02	95.66	119.88	160.23
1.3	30.70	46.21	71.22	88.05	113.44
1.4	25.40	37.40	56.01	68.82	87.14
1.5	21.61	31.22	45.95	55.80	70.92
1.6	18.77	26.73	38.92	46.98	59.40
1.7	16.57	23.35	33.63	40.43	50.89
1.8	14.84	20.69	29.54	35.50	44.38
1.9	13.43	18.57	26.33	31.56	39.27
2.0	12.27	16.85	23.76	28.29	35.20
2.1	11.30	15.41	21.62	25.74	31.95
2.2	10.47	14.20	19.83	23.55	29.21
2.3	9.76	13.18	18.30	21.73	26.88
2.4	9.15	12.29	17.00	20.15	24.89
2.5	8.61	11.51	15.88	18.79	23.15
2.6	8.14	10.84	14.89	17.59	21.64
2.7	7.72	10.24	14.02	16.54	20.30
2.8	7.35	9.70	13.24	15.60	19.13
2.9	7.01	9.23	12.55	14.77	18.10
3.0	6.71	8.80	11.93	14.03	17.17

Table 2a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0.7$.

Average Run Length at σ_a for $\gamma = 0.7$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	85.92	180.03	450.11	787.84	1607.82
1.3	103.27	235.92	698.15	1351.29	3564.76
1.4	122.84	307.12	1056.89	2263.16	7242.45
1.5	144.29	389.67	1534.64	3763.20	13630.14
1.6	167.40	489.80	2210.96	5883.40	24020.54
1.7	192.55	601.80	3068.43	8757.44	43081.08
1.8	219.18	731.51	4115.28	12852.23	70646.32
1.9	247.20	873.88	5441.97	18207.92	101246.83
2.0	276.81	1031.40	7030.85	24905.27	164749.28
2.1	307.35	1203.94	8861.12	33014.37	230108.35
2.2	339.48	1387.14	4887.96	42949.41	321608.72
2.3	372.27	1587.84	13424.12	55078.87	448328.38
2.4	406.42	1798.91	16165.35	68804.80	568176.60
2.5	441.07	2019.35	19165.97	84815.88	698039.96
2.6	476.97	2257.18	22422.46	102193.14	906151.52
2.7	513.25	2502.46	26154.35	122265.56	1047073.49
2.8	550.56	2756.32	30172.65	146553.04	1387798.96
2.9	588.27	3025.17	34412.88	171742.49	1649354.11
3.0	626.66	3300.67	38875.38	196592.63	1943814.53

Table 2b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15 . The non-normality parameter is $\gamma = 0.7$.

Average Run Length at σ_r for $\gamma = 0.7$					
	Decision Interval h				
$\frac{\sigma_r}{\sigma_a}$	5	7	10	12	15
1.2	30.51	47.07	74.33	94.27	122.50
1.3	24.24	36.08	55.14	68.13	88.37
1.4	20.00	29.07	43.28	52.70	67.58
1.5	16.96	24.15	35.25	42.88	54.06
1.6	14.68	20.63	29.72	35.83	44.74
1.7	12.94	17.93	25.57	30.61	38.34
1.8	11.56	15.86	22.36	26.73	33.33
1.9	10.44	14.19	19.87	23.68	29.38
2.0	9.53	12.84	17.86	21.20	26.20
2.1	8.76	11.73	16.19	19.16	23.60
2.2	8.11	10.78	14.77	17.48	21.52
2.3	7.56	9.99	13.64	16.08	19.74
2.4	7.08	9.30	12.64	14.87	18.21
2.5	6.66	8.70	11.77	13.82	16.89
2.6	6.30	8.18	11.01	12.90	15.74
2.7	5.97	7.73	10.36	12.11	14.73
2.8	5.68	7.32	9.77	11.40	13.85
2.9	5.43	6.95	9.25	10.78	13.07
3.0	5.19	6.63	8.78	10.22	12.37

Table 3a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0.5$.

Average Run Length at σ_a for $\gamma = 0.5$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	70.15	142.40	346.78	570.90	1222.84
1.3	83.03	182.14	505.99	961.78	2314.15
1.4	97.06	230.53	737.48	1523.67	4273.92
1.5	112.12	285.40	1030.10	2307.54	7905.19
1.6	128.29	348.22	1405.60	3484.33	13201.30
1.7	145.31	418.31	1873.38	4951.84	20582.82
1.8	163.05	494.52	2411.01	6727.07	31839.54
1.9	181.36	578.35	3075.14	9220.40	47097.89
2.0	200.46	666.99	3815.30	12031.92	66099.00
2.1	220.07	762.64	4635.62	15246.38	88095.06
2.2	240.02	862.45	5601.53	19173.96	116318.33
2.3	260.56	967.97	6632.81	23611.52	150049.75
2.4	281.43	1077.77	7732.23	28468.84	188149.51
2.5	302.53	1191.03	8961.34	33709.99	233922.94
2.6	324.11	1309.92	10263.28	39808.37	273515.66
2.7	345.59	1430.35	11397.74	46297.36	328064.47
2.8	367.86	1555.98	13070.24	53171.91	378684.27
2.9	390.02	1683.43	14628.79	60343.73	435071.48
3.0	412.47	1812.94	16223.22	68164.14	482935.39

Table 3b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15 . The non-normality parameter is $\gamma = 0.5$.

Average Run Length at σ_r for $\gamma = 0.5$					
	Decision Interval h				
$\frac{\sigma_r}{\sigma_a}$	5	7	10	12	15
1.2	25.42	38.71	60.97	75.94	101.58
1.3	20.14	29.60	44.75	55.50	71.26
1.4	16.56	23.77	35.09	42.75	54.06
1.5	14.01	19.71	28.52	34.38	43.54
1.6	12.12	16.77	23.91	28.75	35.98
1.7	10.67	14.57	20.52	24.50	30.43
1.8	9.52	12.85	17.89	21.24	26.35
1.9	8.60	11.49	15.86	18.79	23.19
2.0	7.84	10.38	14.22	16.78	20.63
2.1	7.21	9.47	12.86	15.13	18.54
2.2	6.68	8.70	11.75	13.78	16.81
2.3	6.23	8.05	10.81	12.64	15.39
2.4	5.83	7.50	10.00	11.67	14.17
2.5	5.49	7.02	9.30	10.82	13.12
2.6	5.20	6.60	8.70	10.10	12.20
2.7	4.93	6.23	8.17	9.47	11.40
2.8	4.70	5.90	7.70	8.91	10.71
2.9	4.49	5.61	7.29	8.41	10.09
3.0	4.30	5.35	6.92	7.97	9.54

Table 4a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0.3$.

Average Run Length at σ_a for $\gamma = 0.3$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	55.18	107.91	250.03	409.38	768.54
1.3	64.23	134.28	348.39	625.56	1461.37
1.4	73.82	163.89	480.88	938.69	2441.25
1.5	83.88	197.51	634.98	1321.69	3984.98
1.6	94.35	234.40	832.33	1878.64	6238.93
1.7	105.16	273.81	1047.43	2499.66	9034.83
1.8	116.27	315.22	1308.39	3298.64	12895.17
1.9	127.62	360.02	1584.23	4190.34	17727.59
2.0	139.17	406.31	1903.00	5170.13	23194.55
2.1	150.88	454.04	2235.93	6383.58	29104.85
2.2	162.82	504.16	2601.87	7639.87	37798.73
2.3	174.87	554.49	2989.73	9015.59	46251.52
2.4	187.03	607.59	3386.57	10556.37	55444.25
2.5	199.25	660.96	3826.85	12126.05	65781.53
2.6	211.54	715.57	4268.67	13751.71	76969.03
2.7	223.97	770.96	4728.83	15630.47	88963.30
2.8	236.44	827.11	5213.53	17492.92	99664.94
2.9	248.89	884.05	5700.57	19372.79	111863.66
3.0	261.49	941.60	6205.72	21436.00	126118.54

Table 4b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15 . The non-normality parameter is $\gamma = 0.3$.

Average Run Length at σ_r for $\gamma = 0.3$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	20.52	30.73	47.79	59.89	77.30
1.3	16.23	23.45	35.01	43.07	55.66
1.4	13.32	18.74	27.33	33.17	41.91
1.5	11.26	15.51	22.10	26.50	33.34
1.6	9.73	13.19	18.51	22.09	27.50
1.7	8.56	11.43	15.80	18.74	23.16
1.8	7.64	10.07	13.78	16.26	19.96
1.9	6.91	9.00	12.17	14.30	17.50
2.0	6.31	8.13	10.90	12.72	15.50
2.1	5.81	7.42	9.85	11.48	13.88
2.2	5.39	6.82	8.99	10.43	12.59
2.3	5.03	6.32	8.26	9.55	11.49
2.4	4.72	5.89	7.64	8.81	10.55
2.5	4.45	5.52	7.11	8.17	9.76
2.6	4.22	5.19	6.65	7.62	9.08
2.7	4.01	4.91	6.25	7.14	8.48
2.8	3.83	4.66	5.90	6.72	7.95
2.9	3.67	4.44	5.58	6.34	7.48
3.0	3.52	4.24	5.31	6.01	7.08

Table 5a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0.1$.

Average Run Length at σ_a for $\gamma = 0.1$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	41.58	76.97	122.42	263.33	484.41
1.3	47.39	92.68	222.17	378.18	803.18
1.4	53.45	109.73	285.13	513.44	1202.92
1.5	59.62	127.94	360.57	695.50	1813.49
1.6	66.01	147.13	445.92	901.68	2514.13
1.7	72.43	167.14	537.49	1138.28	3453.42
1.8	78.92	187.81	636.61	1406.38	4496.31
1.9	85.54	209.03	744.82	1692.25	5724.91
2.0	92.17	230.61	853.41	2012.92	7124.74
2.1	98.84	252.66	973.93	2342.51	8525.26
2.2	105.54	275.21	1095.74	2705.01	10302.47
2.3	112.27	298.00	1220.93	3073.55	12031.70
2.4	119.03	320.96	1351.30	3462.88	13785.70
2.5	125.81	343.86	1482.02	3864.47	15918.69
2.6	132.61	367.33	1618.36	4261.28	17866.32
2.7	139.43	382.06	1753.89	4702.28	19922.39
2.8	146.27	414.42	1894.22	5128.16	22206.73
2.9	153.12	438.06	2033.94	5576.70	24396.74
3.0	159.98	461.96	2177.10	6023.49	26529.05

Table 5b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0.1$.

Average Run Length at σ_r for $\gamma = 0.1$					
	Decision Interval h				
$\frac{\sigma_r}{\sigma_a}$	5	7	10	12	15
1.2	15.96	23.24	32.91	44.06	57.24
1.3	12.59	17.71	25.97	31.73	40.61
1.4	10.34	14.15	20.10	24.19	30.37
1.5	8.74	11.70	16.27	19.40	24.14
1.6	7.57	9.94	13.58	16.03	19.71
1.7	6.67	8.62	11.59	13.59	16.61
1.8	5.97	7.61	10.09	11.76	14.23
1.9	5.41	6.81	8.92	10.33	12.44
2.0	4.96	6.17	7.98	9.20	11.02
2.1	4.58	5.64	7.23	8.28	9.86
2.2	4.26	5.20	6.60	7.53	8.93
2.3	3.99	4.83	6.08	6.90	8.14
2.4	3.76	4.51	5.63	6.37	7.48
2.5	3.56	4.24	5.25	5.92	6.92
2.6	3.38	4.01	4.93	5.53	6.44
2.7	3.23	3.80	4.64	5.19	6.02
2.8	3.09	3.62	4.39	4.90	5.66
2.9	2.97	3.46	4.17	4.64	5.33
3.0	2.86	3.31	3.97	4.41	5.05

Table 6a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0$ (normal distribution).

Average Run Length at σ_a for $\gamma = 0$					
	Decision Interval h				
$\frac{\sigma_r}{\sigma_a}$	5	7	10	12	15
1.2	35.31	63.15	131.92	203.51	369.12
1.3	39.86	74.69	169.51	277.75	560.27
1.4	44.51	86.95	212.56	369.46	815.86
1.5	49.26	99.76	260.52	475.75	1139.50
1.6	54.06	113.06	312.74	596.48	1531.33
1.7	58.91	126.70	368.58	729.09	1988.70
1.8	63.80	140.65	427.26	873.89	2506.25
1.9	68.71	154.80	488.83	1028.31	3079.29
2.0	73.65	169.14	552.55	1190.94	3710.19
2.1	78.60	183.64	617.96	1360.94	4385.36
2.2	83.57	198.25	684.98	1537.32	5095.30
2.3	88.56	212.97	753.38	1719.15	5850.71
2.4	93.56	227.77	822.75	1905.70	6626.74
2.5	98.58	242.62	893.01	2095.97	7443.60
2.6	103.60	257.56	964.09	2289.27	8273.50
2.7	108.64	272.54	1036.04	2486.66	9132.09
2.8	113.69	287.56	1108.45	2686.32	10002.77
2.9	118.75	302.63	1181.31	2886.81	10889.99
3.0	123.81	317.73	1255.60	3090.96	11792.52

Table 6b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = 0$ (normal distribution).

Average Run Length at σ_r for $\gamma = 0$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	13.82	19.74	29.66	36.76	47.91
1.3	10.91	15.05	21.72	26.32	33.50
1.4	8.97	12.03	16.84	20.16	25.16
1.5	7.60	9.96	13.62	16.10	19.85
1.6	6.59	8.48	11.37	13.31	16.25
1.7	5.83	7.37	9.71	11.29	13.65
1.8	5.23	6.52	8.46	9.76	11.71
1.9	4.75	5.85	7.49	8.58	10.23
2.0	4.36	5.31	6.72	7.65	9.06
2.1	4.04	4.86	6.09	6.90	8.12
2.2	3.77	4.50	5.57	6.29	7.35
2.3	3.54	4.19	5.14	5.77	6.71
2.4	3.35	3.92	4.78	5.34	6.18
2.5	3.18	3.70	4.47	4.97	5.72
2.6	3.03	3.50	4.20	4.66	5.34
2.7	2.90	3.33	3.97	4.38	5.00
2.8	2.78	3.18	3.76	4.14	4.70
2.9	2.68	3.05	3.58	3.93	4.45
3.0	2.59	2.93	3.42	3.74	4.22

Table 7a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15 . The non-normality parameter is $\gamma=-0.1$.

Average Run Length at σ_a for $\gamma=-0.1$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	29.49	50.73	100.83	151.86	266.81
1.3	32.98	58.76	125.24	199.30	377.54
1.4	36.50	67.37	152.18	253.86	516.90
1.5	40.06	76.06	181.27	314.39	689.73
1.6	43.64	85.02	212.03	379.72	883.97
1.7	47.24	93.99	244.12	448.29	1089.20
1.8	50.87	103.22	277.31	523.46	1330.79
1.9	54.51	112.41	311.44	601.28	1576.22
2.0	58.16	121.64	346.12	681.06	1834.72
2.1	61.82	131.04	381.39	759.73	2115.63
2.2	65.50	140.40	417.06	843.84	2385.38
2.3	69.20	149.79	452.96	929.39	2684.22
2.4	72.90	159.21	489.07	1013.89	2961.29
2.5	76.62	168.66	524.95	1099.75	3286.34
2.6	80.34	178.14	561.72	1188.01	3579.52
2.7	84.08	187.63	598.68	1273.98	3897.03
2.8	87.83	197.17	635.46	1363.43	4199.25
2.9	91.60	206.72	672.03	1451.66	4526.64
3.0	95.37	216.28	709.39	1540.49	4832.78

Table 7b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15 . The non-normality parameter is $\gamma = -0.1$.

Average Run Length at σ_r for $\gamma = -0.1$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	11.81	16.48	24.28	29.72	38.54
1.3	9.35	12.59	17.77	21.43	26.96
1.4	7.71	10.08	13.82	16.37	20.24
1.5	6.55	8.37	11.19	13.10	15.96
1.6	5.70	7.14	9.35	10.84	13.06
1.7	5.06	6.23	8.01	9.20	10.98
1.8	4.56	5.53	7.00	7.97	9.43
1.9	4.16	4.98	6.21	7.03	8.25
2.0	3.83	4.54	5.59	6.29	7.32
2.1	3.56	4.18	5.09	5.69	6.58
2.2	3.33	3.88	4.67	5.19	5.97
2.3	3.14	3.62	4.33	4.79	5.47
2.4	2.98	3.41	4.03	4.44	5.05
2.5	2.84	3.22	3.78	4.15	4.69
2.6	2.71	3.06	3.57	3.90	4.39
2.7	2.60	2.92	3.38	3.68	4.13
2.8	2.51	2.80	3.22	3.49	3.90
2.9	2.42	2.69	3.08	3.33	3.70
3.0	2.35	2.59	2.95	3.18	3.52

Table 8a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.3$.

Average Run Length at σ_a for $\gamma = -0.3$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	19.51	30.13	53.04	74.17	117.41
1.3	21.44	33.83	61.86	88.76	148.21
1.4	23.38	37.54	70.84	104.81	181.31
1.5	25.35	41.26	80.02	120.62	217.96
1.6	27.33	45.02	89.38	137.47	255.30
1.7	29.33	48.79	98.81	153.89	293.07
1.8	31.36	52.60	108.19	170.96	331.71
1.9	33.40	56.42	117.62	187.74	370.67
2.0	35.47	60.25	127.15	204.57	410.02
2.1	37.55	64.12	136.58	221.82	447.27
2.2	39.66	68.01	146.17	238.82	489.49
2.3	41.78	71.91	155.63	255.83	529.19
2.4	43.93	75.85	165.35	272.32	569.81
2.5	46.09	79.80	174.95	290.29	610.21
2.6	48.27	83.77	184.54	307.53	650.46
2.7	50.46	87.77	194.27	324.75	690.78
2.8	52.67	91.78	203.97	342.00	730.98
2.9	54.90	95.83	213.66	359.30	770.12
3.0	57.14	99.89	223.42	376.60	811.33

Table 8b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.3$.

Average Run Length at σ_r for $\gamma = -0.3$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	8.28	10.81	14.94	17.87	22.53
1.3	6.63	8.35	11.07	12.96	15.87
1.4	5.54	6.78	8.70	10.01	11.99
1.5	4.77	5.71	7.13	8.09	9.53
1.6	4.21	4.94	6.04	6.77	7.86
1.7	3.78	4.37	5.24	5.82	6.68
1.8	3.44	3.93	4.64	5.11	5.80
1.9	3.18	3.58	4.17	4.56	5.13
2.0	2.96	3.31	3.81	4.13	4.61
2.1	2.78	3.08	3.51	3.78	4.19
2.2	2.63	2.89	3.26	3.50	3.85
2.3	2.50	2.73	3.06	3.27	3.57
2.4	2.39	2.60	2.88	3.07	3.34
2.5	2.30	2.48	2.74	2.90	3.14
2.6	2.21	2.38	2.61	2.76	2.97
2.7	2.14	2.29	2.50	2.63	2.82
2.8	2.07	2.21	2.40	2.52	2.70
2.9	2.02	2.14	2.32	2.43	2.59
3.0	1.96	2.08	2.24	2.34	2.49

Table 9a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.5$.

Average Run Length at σ_a for $\gamma = -0.5$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	12.18	16.16	23.56	29.59	40.58
1.3	13.32	17.75	26.17	33.17	46.21
1.4	14.51	19.39	28.84	36.86	52.17
1.5	15.73	21.07	31.53	40.55	57.94
1.6	17.00	22.79	34.27	44.25	63.80
1.7	18.30	24.56	37.05	48.02	69.68
1.8	19.64	26.37	39.88	51.82	75.59
1.9	21.02	28.21	42.74	55.67	81.53
2.0	22.42	30.09	45.65	59.55	87.49
2.1	23.85	32.00	48.60	63.48	93.54
2.2	25.31	33.94	51.59	67.44	99.58
2.3	26.79	35.91	54.62	71.46	105.68
2.4	28.30	37.91	57.67	75.51	111.84
2.5	29.83	39.94	60.76	78.59	118.02
2.6	31.38	41.98	63.88	83.71	124.21
2.7	32.94	44.06	67.03	87.86	130.51
2.8	34.53	46.15	70.20	92.00	136.82
2.9	36.13	48.26	73.40	96.27	143.16
3.0	37.74	50.40	76.63	100.51	149.54

Table 9b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.5$.

Average Run Length at σ_r for $\gamma = -0.5$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	5.61	6.62	8.19	9.28	10.95
1.3	4.62	5.29	6.31	6.99	8.03
1.4	3.96	4.44	5.15	5.61	6.31
1.5	3.50	3.85	4.37	4.70	5.20
1.6	3.15	3.42	3.82	4.07	4.44
1.7	2.89	3.10	3.41	3.61	3.90
1.8	2.68	2.85	3.10	3.26	3.49
1.9	2.51	2.65	2.86	2.99	3.18
2.0	2.37	2.49	2.67	2.78	2.93
2.1	2.26	2.36	2.51	2.60	2.73
2.2	2.16	2.25	2.38	2.46	2.57
2.3	2.08	2.15	2.26	2.33	2.43
2.4	2.00	2.07	2.17	2.23	2.32
2.5	1.94	2.00	2.09	2.14	2.22
2.6	1.88	1.94	2.02	2.07	2.13
2.7	1.83	1.88	1.95	2.00	2.06
2.8	1.79	1.83	1.90	1.94	1.99
2.9	1.75	1.79	1.85	1.89	1.94
3.0	1.71	1.75	1.81	1.84	1.89

Table 10a Average Run Length values (L_a) at the acceptable quality level σ_a for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.7$.

Average Run Length at σ_a for $\gamma = -0.7$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	8.44	9.23	10.48	11.35	12.73
1.3	9.57	10.43	11.78	12.74	14.25
1.4	10.80	11.72	13.19	14.23	15.88
1.5	12.10	13.10	14.68	15.81	17.60
1.6	13.46	14.54	16.25	17.47	19.41
1.7	14.88	16.03	17.88	19.19	21.28
1.8	16.34	17.58	19.57	20.97	23.22
1.9	17.85	19.17	21.30	22.81	25.22
2.0	19.39	20.81	23.07	24.68	27.26
2.1	20.96	22.47	24.88	26.60	29.34
2.2	22.57	24.17	26.73	28.55	31.46
2.3	24.20	25.89	28.60	30.53	33.62
2.4	25.86	27.65	30.51	32.54	35.80
2.5	27.54	29.42	32.44	34.58	38.02
2.6	29.24	31.22	34.39	36.65	40.26
2.7	30.97	33.04	36.37	38.74	42.53
2.8	32.71	34.88	38.37	40.85	44.82
2.9	34.47	36.75	40.39	42.98	47.14
3.0	36.25	38.63	42.43	45.14	49.48

Table 10b Average Run Length values (L_r) at the rejectable quality level σ_r for displacements of $\frac{\sigma_r}{\sigma_a}$ from 1.2 to 3.0, in increments of 0.1, and decision intervals 5, 7, 10, 12, and 15. The non-normality parameter is $\gamma = -0.7$.

Average Run Length at σ_r for $\gamma = -0.7$					
$\frac{\sigma_r}{\sigma_a}$	Decision Interval h				
	5	7	10	12	15
1.2	4.12	4.32	4.61	4.80	5.08
1.3	3.55	3.67	3.85	3.97	4.14
1.4	3.14	3.23	3.35	3.42	3.54
1.5	2.85	2.90	2.99	3.04	3.12
1.6	2.62	2.66	2.72	2.76	2.82
1.7	2.44	2.47	2.52	2.55	2.60
1.8	2.29	2.32	2.36	2.38	2.42
1.9	2.18	2.20	2.23	2.25	2.27
2.0	2.08	2.09	2.12	2.13	2.16
2.1	1.99	2.01	2.03	2.04	2.06
2.2	1.92	1.93	1.95	1.96	1.98
2.3	1.86	1.87	1.88	1.89	1.91
2.4	1.80	1.81	1.83	1.83	1.85
2.5	1.76	1.76	1.78	1.78	1.79
2.6	1.71	1.72	1.73	1.74	1.75
2.7	1.67	1.68	1.69	1.70	1.71
2.8	1.64	1.65	1.65	1.66	1.67
2.9	1.61	1.61	1.62	1.63	1.63
3.0	1.58	1.59	1.59	1.60	1.60

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