MULTIGRID AND MGR[ν] METHODS

FOR DIFFUSION EQUATIONS

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ABSTRACT

The MGR[ν] algorithm of Ries, Trottenberg and Winter with $\nu = 0$ and the Algorithm 2.1 of Braess are essentially the same multigrid algorithm for the discrete Poisson equation: $-\Delta_h U = f$. In this report we consider the extension to the general diffusion equation $-\nabla \cdot p\nabla u = f$, $p = p(x,y) \geq p_0 > 0$. In particular, we indicate the proof of the basic result $\rho \leq \frac{1}{2}(1+Kh)$, thus extending the results of Braess and Ries, Trottenberg and Winter. In addition to this theoretical result we present computational results which indicate that other constant coefficient estimates carry over to this case.

MULTIGRID AND MGR[ ] METHODS FOR DIFFUSION EQUATIONS*

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Multigrid methods are proving themselves to be successful tools for the solution of the algebraic equations associated with the discretization of elliptic boundary-value problems. Nevertheless, it seems we are just beginning to understand this powerful idea. Hence there is a need for continued probing, experimentation and new proofs - less for the sake of proof and more for the sake of insight.

Let X_n be a finite dimensional vector space of dimension n. Let A_n be a non-singular linear operator mapping X_n onto X_n. We are concerned with the problem

$$A_n U = f . \tag{1}$$

Multigrid methods for the solution of (1) are based on the following set of ideas. Suppose that (1) arises from the discretization of an elliptic boundary value problem. Then U is an approximation to a "smooth function" $U(x,y)$. Moreover $U(x,y)$ can also be approximated by other approximants $\{U_m\} \in \{X_m\}$ - with X_m a finite dimensional vector space of dimension m. Thus U can be approximated by such a U_m with $m < n$. At the same time, most of the classical iterative methods for the solution of (1) converge very

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slowly. For these methods the spectral radius of the iteration matrix is of the form

(2) $$p \sim 1 - c/n .$$

Indeed, ADI and SOR methods are considered exceptionally good because

(3) $$p \sim 1 - c/\sqrt{n} .$$

The same analysis which yields (2) also shows that the eigenvectors associated with this slow rate of convergence are converging (as $n \to \infty$) to a very smooth function. That is, these (not <u>all</u>) classical iterative schemes have the effect of "smoothing" the error.

A multigrid method for the solution of (1) is based on the following entities

(a) <u>a smoothing operator</u> $\mathcal{S} : X_n \to X_n$

$\mathcal{S}$ is an affine operator of the form

(4) $$\mathcal{S}v = Gv + Kf$$

where G and K are linear operators. And, if u is the unique solution of (1) then u is a fixed point of $\mathcal{S}$, i.e.

(5) $$\mathcal{S}u = Gu + Kf = u .$$

(b) <u>A subspace</u> X_m with

(6) $$\dim X_m = m << \dim X_n = n .$$

(c) <u>two linear "communication" operators</u>:

(7) $$I_n^m : X_n \to X_m$$

(8) $$I_m^n : X_m \to X_n \ .$$

(d) <u>A course gird operator</u>: a nonsingular operator A_m,

(9) $$A_m : X_m \to X_m \ .$$

Having listed these ingredients let us describe the multigrid iterative scheme for the solution of (1).

<u>Step 1</u>. Let u^0 be a first guess.

<u>Step 2</u>. $\mathcal{S}u^0 = \tilde{u}$, $r = f - A\tilde{u}$.

<u>Step 3</u>. $r_m = I_n^m r$.

<u>Step 4</u>. Solve

$$A_m \hat{u} = r_m \ .$$

<u>Step 5</u>. $u^1 = \tilde{u} + I_m^n \hat{u}$.

<u>Remark</u>: It might appear that we have (merely) described a "two grid" iterative method. However, true "multigrid" iterative schemes are described by this outline. The operator A_m may require the use of other spaces $X_{m'}$.

In our discussion of these methods we follow a basic observation of S. McCormick and J. Ruge [2]: <u>we should focus our attention on the two basic spaces</u>

(10) $$\mathcal{R} := \text{Range } I_m^n \ ,$$

(11) $$N := \text{Nullspace } I_n^m A_n \ .$$

A basic result is

Theorem 1: Suppose $X_n = \mathcal{R} \oplus N$ and

$$A_m = \hat{A}_m := I_n^m A_n I_m^n . \tag{12}$$

Suppose $\hat{A}_m$ is nonsingular, and

$$\tilde{\varepsilon} := U - \tilde{u} = \eta + I_m^n w \tag{13}$$

where

$$\eta \in N, \quad w \in X_m . \tag{14}$$

Then

$$\varepsilon^1 := U - u^1 = \eta . \tag{15}$$

With this theorem we see the "right way" to view (i) the smoothing operator S and (ii) the coarse grid operator A_m. That is, S should make η "small" while A_m^{-1} should be a "good approximation" to $\hat{A}_m^{-1}$.

With this insight we study the MGR[ν] multigrid methods. These methods, the MultiGrid Reduction methods were developed independently by Braess [1], and Ries, Trottenberg and Winter [3] for the Poisson Equation.

In [1] Braess proposed and analyzed a class of multi-grid methods. In particular, he considered a particular algorithm for the Poisson Equation - "Algorithm 2.1" of [1]. He shows that the contraction number ρ for a two-grid method is given by

$$\rho \leq \frac{1}{2} \; ! \tag{16}$$

This result is valid in any polygonal region Ω provided that its corners belong to the coarsest grid, and the corners are "even" points. In [3] Ries, Trottenberg and Winter discuss the class of MGR[ν] methods for the Poisson Equation in a square. Using Fourier Analysis they obtain an explicit formula for the corresponding contraction numbers $\rho[\nu]$. In particular, they obtain

$$\rho[0] = \frac{1}{2}, \quad \rho[1] = \frac{2}{27} . \tag{17}$$

As it happens MGR[0] is the same as the "Algorithm 2.1" and the results of [1] and [3] are consistent. The results of [3] are more precise for more restricted problems.

In this report we consider the problem

$$-\nabla \cdot p(x,y)\nabla u = f \quad \text{in} \quad \Omega , \qquad p(x,y) \geq p_0 > 0 , \tag{18.a}$$

$$u = 0 \quad \text{on} \quad \partial\Omega , \tag{18.b}$$

and its standard finite difference analog (see section 2). We consider a class of multi-grid methods which generalize the MGR[ν] methods. In particular, when $p(x,y) \equiv 1$ these methods include the MGR[ν] methods. Our basic result is the following: Consider the two-grid method. Then

$$\rho \leq \frac{1}{2} + Kh \tag{19}$$

where the constant K is determined by the $C^1(\Omega)$ norm of the "diffusion coefficient" p. Moreover, the proof of (19) indicates why one should expect great improvement when more "smoothing" is introduced.

In section 2 we describe the basic discrete (finite-difference equations) problem when Ω is the unit square. In section 3 we "analyze" the multigrid algorithm developed in section 2. However, in fact, we do not provide a correct analysis. Rather, we give a heuristic argument which is "almost" right and is the basis of the correct, and very technical argument which will be presented in [4]. Finally, in section 4 we present some computational results. These computations were carried out on the CRAY I at the Los Alamos National Laboratory, Los Alamos, New Mexico, U.S.A.

2. THE PROBLEM

For the purposes of expository simplicity we choose Ω to be the unit square

(2.1) $$\Omega \equiv \{(x,y),\ 0 < x,y < 1\} .$$

Let

(2.2) $$h = \frac{1}{N+1} = \Delta x = \Delta y .$$

The function $p(x,y)$ [of (18.a)] is to be smooth and satisfy

(2.3) $$p(x,y) \geq p_0 > 0 .$$

Consider the difference scheme: for $0 < k,j < N$

(2.4a) $$\frac{1}{h^2}\Big[p_{k+\frac{1}{2},j}(U_{k+1,j}-U_{k,j}) - p_{k-\frac{1}{2},j}(U_{k,j}-U_{k-1,j})\Big] +$$

$$\frac{1}{h^2}\Big[p_{k,j+\frac{1}{2}}(U_{k,j+1}-U_{k,j}) - p_{k,j-\frac{1}{2}}(U_{k,j}-U_{k,j-1})\Big] = -f_{kj} ,$$

(2.4b) $$U_{kj} = 0 \quad \text{if} \quad k \quad \text{or} \quad j \quad \text{is} \quad 0 \quad \text{or} \quad N+1 ,$$

and

(2.5) $$p_{k+\frac{1}{2},j} = p((k+\tfrac{1}{2})h,jh) , \quad f_{kj} = f(kh,jh), \quad \text{etc.}$$

We rewrite (2.5) as

(2.6) $$[L_hU]_{kj} = f_{kj} .$$

We now turn to the question of the solution of these linear algebraic equations, via a "two-grid method". Let

(2.7a) $$\Omega_h \equiv \{(kh,jh): 0 < k, j < N\}$$

(2.7b) $$\Omega_E \equiv \{(kh,jh) \in \Omega_h: k+j \equiv 0 \pmod 2\}$$

(2.7c) $$\Omega_0 = \{(kh,jh) \in \Omega_h: k+j \equiv 1 \pmod 2\} .$$

Our two grids are Ω_h and Ω_E. Let S_h and S_E be the spaces of grid-function defined on Ω_h and Ω_E respectively. In both cases we assume the functions vanish on the boundaries, i.e.

(2.8) $$U_{kj} = 0 \text{ if } k \text{ or } j = 0 \text{ or } N+1 .$$

Our first step is to set-up "communication" between these two spaces. To be specific, we construct linear "interpolation" and "projection" operators I_h^E, I_E^h so that

(2.9a) $$I_h^E: S_h \to S_E , \qquad \text{(Projection)},$$

(2.9b) $$I_E^h: S_E \to S_h . \qquad \text{(Interpolation)}.$$

We define the interpolation operator I_E^h as follows

(i) if $k + j \equiv 0 \pmod 2$, then $(U \in S_E)$

(2.10a) $$[I_E^h U]_{kj} = U_{kj}$$

(ii) if $k + j \equiv 1 \pmod 2$, then

$$[I_E^h U]_{kj} = \frac{1}{C_{kj}} \Big[p_{k-\frac{1}{2}j} U_{k-1,j} + p_{k+\frac{1}{2},j} U_{k+1,j} + p_{k,j-\frac{1}{2}} U_{k,j-1} + \tag{2.10b}$$
$$p_{k,j+\frac{1}{2}} U_{k,j+1}\Big] ,$$

where

$$C_{kj} = [p_{k-\frac{1}{2},j} + p_{k+\frac{1}{2},j} + p_{k,j+\frac{1}{2}} + p_{k,j-\frac{1}{2}}] . \tag{2.10c}$$

The projection operator I_h^E is defined by: if $k + j \equiv 0 \pmod 2$ then ($U \in S_h$)

$$[I_h^E U]_{kj} = \frac{1}{2C_{kj}} \Big[p_{k-\frac{1}{2},j} U_{k-1,j} + p_{k+\frac{1}{2},j} U_{k+1,j} + p_{k,j-\frac{1}{2}} U_{k,j-1} +$$
$$p_{k,j+\frac{1}{2}} U_{k,j+1} + C_{kj} U_{kj}\Big] .$$

Remark: We note that

$$I_h^E = \frac{1}{2}(I_E^h)^T .$$

Let

$$\mathcal{R} := \text{Range } I_E^h . \tag{2.12}$$

The choice of interpolation operator I_E^h enables us to characterize the range $\mathcal{R}$ of I_E^h as follows:

<u>Lemma 2.1</u>: Let I_E^h be defined by (2.10a), (2.10b). Then, a function $U \equiv U(h) \in S_h$ is in $\mathcal{R}$ if and only if

$$[L_h U]_{kj} = 0 \quad \forall (k,j) \ni k + j \equiv 1 \pmod 2 \ . \tag{2.13}$$

<u>Corollary</u>: I_E^h is of full rank, i.e.

$$\dim \mathcal{R} = \dim S_E \ .$$

We are now ready to describe the class of two-grid methods under discussion. Let G be a "smoother". That is, given $u^0 \in S_h$ we construct $\tilde{u}$ via the formula

$$\tilde{u} = Gu^0 = u^0 + B(f - L_h)u^0 \tag{2.14}$$

where B is a fixed, given matrix. The two-grid iteration procedure (based on G) is given by:

<u>Algorithm</u>:

<u>Step 1</u>: Given $u^0 \in S_h$ form

$$\tilde{u} = Gu^0 \ . \tag{2.15}$$

<u>Step 2</u>: From the function $\hat{u}$ given by: for $k + j \equiv 0 \pmod 2$

$$\hat{u}_{kj} = \hat{u}_{kj} \tag{2.16a}$$

for $k + j \equiv 1 \pmod 2$ solve for $\hat{u}_{kj}$ from the equation

$$[L_h \hat{u}]_{kj} = f_{kj} \ . \tag{2.16b}$$

Note: in other words we "relax" the equations on the "odd" points

Step 3: Form

$$r = f - L_h \hat{u} \tag{2.17a}$$

and

$$r_E = I_h^E r \; . \tag{2.17b}$$

Step 4: Find the function $\phi \in S_E$ which satisfies

$$L_E^{(1)} \phi = r_E \tag{2.18a}$$

where $L_E^{(1)}$ is the difference operator described by:

For $k + j \equiv 0 \pmod 2$

$$\begin{cases} [L_E^{(1)}\phi]_{kj} = -a_{k-\frac{1}{2},j-\frac{1}{2}}\phi_{k-1,j-1} - a_{k+\frac{1}{2},j+\frac{1}{2}}\phi_{k+1,j+1} \\ + \gamma_{kj}\phi_{kj} - b_{k-\frac{1}{2},j+\frac{1}{2}}\phi_{k-1,j+1} - b_{k+\frac{1}{2},j-\frac{1}{2}}\phi_{k+1,j-1} \end{cases} \tag{2.18b}$$

where

$$a_{k-\frac{1}{2},j-\frac{1}{2}} = \frac{1}{h^2}\left[\frac{p_{k-\frac{1}{2},j}\,p_{k-1,j-\frac{1}{2}}}{c_{k-1,j}} + \frac{p_{k,j-\frac{1}{2}}\,p_{k-\frac{1}{2},j-1}}{c_{k,j-1}}\right] \tag{2.18c}$$

$$b_{k-\frac{1}{2},j+\frac{1}{2}} = \frac{1}{h^2}\left[\frac{p_{k-\frac{1}{2},j}\,p_{k-1,j+\frac{1}{2}}}{c_{k-1,j}} + \frac{p_{k,j+\frac{1}{2}}\,p_{k-\frac{1}{2},j+1}}{c_{k,j+1}}\right] \tag{2.18d}$$

$$\gamma_{kj} = a_{k-\frac{1}{2},j-\frac{1}{2}} + b_{k-\frac{1}{2},j+\frac{1}{2}}, + b_{k+\frac{1}{2},j-\frac{1}{2}} + a_{k+\frac{1}{2},j+\frac{1}{2}} \; . \tag{2.18e}$$

Step 5: Set

$$u^1 = \hat{u} + I_E^h \phi$$

Step 6: Set

$$u^1 \to u^0 \quad \text{and return to step 1.}$$

The operator $L_E^{(1)}$ chosen is Step 4, i.e., in (2.18a) is the easier operator to analyze. However, it is not the right operator to use in practical problems. It is more convenient to use the natural "skewed" 5-point difference operator on the even grid, that is if $k+j \equiv 0 \pmod 2$ then

$$(2.19a) \quad [L_E^{(2)}U]_{kj} = \frac{1}{2h^2}\{-p_{k+\frac{1}{2},j+\frac{1}{2}}U_{k+1,j+1} - p_{k+\frac{1}{2},j-\frac{1}{2}}U_{k+1,j-1}$$

$$-p_{k-\frac{1}{2},j-\frac{1}{2}}U_{k-1,j-1} - p_{k-\frac{1}{2},j+\frac{1}{2}}U_{k-1,j+1} + S_{kj}U_{kj}\}$$

where

$$(2.19b) \quad S_{kj} = \{p_{k+\frac{1}{2},j+\frac{1}{2}} + p_{k+\frac{1}{2},j-\frac{1}{2}} + p_{k-\frac{1}{2},j+\frac{1}{2}} + p_{k-\frac{1}{2},j-\frac{1}{2}}\} .$$

Fortunately, the basic result (19) holds with this choice $L_E^{(2)}$ because of the basic estimate

$$(2.20) \quad (1-Kh)\langle L_E^{(1)}\psi,\psi\rangle \le \langle L_E^{(2)}\psi,\psi\rangle \le (1+Kh)\langle L_E^{(1)}\psi,\psi\rangle .$$

3. ANALYSIS OF THE ALGORITHM

We begin our analysis with an observation which is essentially the restatement of Theorem 1 (of the introduction) in our setup. Let

(3.1) $$\hat{L}_E := I_h^E L_h I_E^h .$$

Consider Steps 4-5 of the two-grid iteration. Suppose we replace L_E by $\hat{L}_E$, i.e. suppose we find the function ψ which satisfies

$$\hat{L}_E \psi = r_E ,$$

and set

$$u^1 = \hat{u} + I_E^h \psi .$$

We claim that

$$L_h u^1 = f ,$$

i.e. u^1 is the desired solution! To see this we set

(3.2) $$\tilde{\varepsilon} = U - \hat{u}$$

and observe that Step 2 implies that if $k + j \equiv 1 \pmod 2$, then

$$(L_h \tilde{\varepsilon})_{kj} = (L_h U - L_h \hat{u})_{kj} = (f - L_h \hat{u})_{kj} = 0 .$$

Hence Lemma 2.1 asserts that there is a function $V \in S_E$ and

$$\tilde{\varepsilon} = I_E^h V .$$

We now verify that

$$\hat{L}_E V = I_h^E(L_h I_E^h V) = I_h^E L_h \tilde{\varepsilon} = r_E$$

Hence,

$$\psi = V$$

and

$$\hat{u} - I_E^h \psi = \hat{u} - \tilde{\varepsilon} = U!! \tag{3.2*}$$

Unfortunately we have chosen Step 4 with $L_E^{(1)}$ and <u>not</u> $\hat{L}_E$. This choice was not merely pique on our part (or the part of Braes and Ries, Trottenberg and Winter). The point is -- having chosen $L_E^{(1)}$ as a five point star we can now proceed to replace Step 4 with a new two grid step -- i.e. we can build a true multi-grid.

In any case, the problem of <u>Step 4</u> is seen to be

$$L_E^{(K)} \phi = \hat{L}_E \psi , \qquad K = 1,2 . \tag{3.3}$$

We now turn to a complete description of the operator $\hat{L}_E$.

<u>Definition</u>: Let $\tilde{L}_E$ be the difference operator defined on S_E by the formula $[k + j \equiv 0 \pmod 2]$

$$[\tilde{L}_E V]_{kj} = -A_{k-1,j} U_{k-2,j} - A_{k+1,j} U_{k+2,j} - B_{k,j-1} U_{k,j-2} - B_{k,j+1} U_{k,j+2} + D_{kj} U_{kj} , \tag{3.4}$$

where

$$A_{k+1,j} = \frac{p_{k+\frac{1}{2}j}\,p_{k+3/2j}}{c_{k+1,j}} \tag{3.5a}$$

$$B_{k,j+1} = \frac{p_{k,j+\frac{1}{2}}\,p_{k,j+3/2}}{c_{k,j+1}} \tag{3.5b}$$

$$D_{k,j} = [A_{k+1,j}+A_{k-1,j}+B_{k,j+1}+B_{k,j-1}] \tag{3,5c}$$

Lemma 3.1: For "interior" points, (x_k,y_j) with $2 \leq k,j \leq N - 1$, we have the identity

$$\hat{L}_E = \frac{1}{2} L_E^{(1)} + \frac{1}{2} \tilde{L}_E \; . \tag{3.6}$$

Proof: Direct Computation.

Unfortunately, (3.6) does not hold on the points (x_k,y_j) with $k = 1$ or N and $j = 1$ or N. The argument in [4] holds in very general domains. But, as you an imagine, it is technically complicated. So, we shall simply assume that (3.6) holds throughout Ω.

Having (3.3) and (3.6) we obtain

$$L_E^{(1)}\phi = (\frac{1}{2} L_E^{(1)} + \frac{1}{2} \tilde{L}_E)\psi \tag{3.7}$$

Thus

$$u^1 = \tilde{u} + I_E^h\phi = (\tilde{u} + I_E^h\psi) + I_E^h(\phi-\psi) \; .$$

Using (3.2*) we see that

$$\varepsilon^1 = u^1 - U = I_E^h(\phi-\psi) \tag{3.8}$$

and, we recall that

$$\tilde{\varepsilon} = \tilde{u} - U = I_E^h(\psi) \; . \tag{3.9}$$

We now turn to an estimate which is an extension of the basic result of Braess [1](see [4] also).

Theorem 3.1: Assume (3.6) holds throughout Ω_h. Let the "smoother" G of Step 1 in the MGR Algorithm satisfy $\|I-BL_h\| \le 1$. Let ρ denote the spectral radius of this two grid iteration scheme $(h \to \sqrt{2}h)$. Then, there is a constant K_0 depending only on $\|\nabla p\|_\infty$, the maximum norm of the first derivatives of the "diffusion coefficient" $p(x,y)$, such that

$$\rho \le \frac{1}{2} + K_0 h \; . \tag{3.10}$$

Proof: From (3.7) we see that

$$\phi = \frac{1}{2}\psi + \frac{1}{2}L_E^{-1}\tilde{L}_E\psi$$

$$\psi - \phi = \frac{1}{2}[I - L_E^{-1}\tilde{L}_E]\psi \; .$$

Thus, we turn to the spectrum of

$$T := \frac{1}{2}(I - L_E^{-1}\tilde{L}_E) \; .$$

Let $\langle \lambda, V \rangle$ be an eigenpair of T. Then an elementary computation yields

(3.11a) $$(1-2\lambda)L_E V = \tilde{L}_E V .$$

Hence

(3.11b) $$(1-2\lambda)(V^T L_E V) = (V^T \tilde{L}_E V) .$$

Since both L_E and $\tilde{L}_E$ are symmetric positive definite operators (see Lemma 3.1)

(3.11c) $$1 - 2\lambda > 0 \quad \text{and} \quad \lambda < \frac{1}{2} .$$

The proof of the theorem now follows from the following basic, but elementary, lemma.

Lemma 3.1: Let V be a grid-vector defined on the EVEN points, i.e.

$$V = \{V_{kj}\}, \quad k + j \equiv 0 \pmod 2), \quad 0 < k, j < N + 1 .$$

Let (see Fig. 1)

(3.12)
$$(V_\zeta)_{kj} = \frac{V_{k+1,j+1} - V_{k,j}}{\sqrt{2}h} ,$$

$$(V_\eta)_{kj} = \frac{V_{k-1,j+1} - V_{k,j}}{\sqrt{2}h} ,$$

$$(V_x)_{kj} = \frac{V_{k+2,j} - V_{k,j}}{2h} ,$$

$$(V_y)_{kj} = \frac{V_{k,j+2} - V_{k,j}}{2h} ,$$

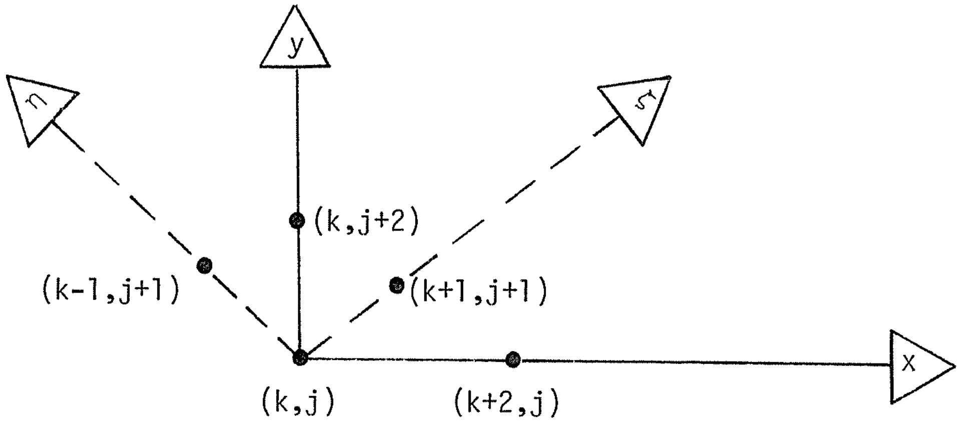

Figure 1

Then

(3.13a) $$V^T L_E V = (2h^2) \sum_{k,j} \left[a_{k+\frac{1}{2},j+\frac{1}{2}} (V_\zeta)^2_{kj} + b_{k-\frac{1}{2},j+\frac{1}{2}} (V_\eta)^2_{kj} \right],$$

(3.13b) $$V^T \tilde{L}_E V = (4h^2) \sum \left[A_{k+1,j} (V_x)^2_{kj} + B_{k,j+1} (V_y)^2_{kj} \right].$$

Moreover

(3.14a) $$V_x = \frac{1}{\sqrt{2}} \left[-(V_\eta)_{k+2,j} + (V_\zeta)_{k,j} \right]$$

(3.14b) $$V_y = \frac{1}{\sqrt{2}} \left[(V_\eta)_{k+1,j+1} + (V_\zeta)_{kj} \right].$$

<u>Proof</u>: The equalities (3.13a), (3.13b) follow from a direct computation using summation by parts. The equalities (3.14a), (3.14b) are an immediate result of the definitions (3.12).

Proof of the Theorem: We observe that there is a constant K depending only on $\|p\|_{C^2}$ and p_0 such that

$$(3.15a)\qquad |h^2 a_{k\pm\frac{1}{2},j\pm\frac{1}{2}} - \frac{1}{2} p_{kj}| \le Kh ,$$

and
$$|h^2 b_{k\pm\frac{1}{2},j\mp\frac{1}{2}} - \frac{1}{2} p_{kj}| \le Kh ,$$

$$(3.15b)\qquad h^2 a_{k\pm\frac{1}{2},j\pm\frac{1}{2}} = \frac{1}{2} p_{kj} + O(h) ,$$

and
$$h^2 b_{k\pm\frac{1}{2},j\mp\frac{1}{2}} = \frac{1}{2} p_{kj} + O(h) ,$$

$$(3.16a)\qquad |h^2 A_{k\pm 1,j} - \frac{1}{4} p_{kj}| \le Kp_{kj} h ,$$

$$(3.16b)\qquad |h^2 B_{k,j\pm 1} - \frac{1}{4} p_{kj}| \le Kp_{kj} h$$

i.e.

$$(3.17a)\qquad h^2 A_{k\pm 1,j} = \frac{1}{4} p_{kj} + O(h)$$

$$(3.17b)\qquad h^2 B_{k,j\pm 1} = \frac{1}{4} p_{kj} + O(h) .$$

Using these results together with (3.13a), (3.13b), (3.14a), (3.14b) yields

$$4h^2 A_{k+1,j} (V_x)^2_{kj} = \frac{1}{2} h^2 \left[p_{kj} + O(h)\right] \left[(V_\eta)^2_{k+2,j} - 2(V_\eta)_{k+2,j}(V_\zeta)_{kj} + (V_\zeta)^2_{kj}\right]$$

$$4h^2 B_{k,j+1} (V_y)^2_{kj} = \frac{1}{2} h^2 \left[p_{kj} + O(h)\right] \left[(V_\eta)^2_{k+1,j+1} + 2(V_\eta)_{k+1,j+1}(V_\zeta)_{kj} + (V_\zeta)^2_{kj}\right] .$$

Thus

$$(3.18a)\quad 4h^2\left[A_{k+1,j}(V_x)^2_{kj} + B_{k,j+1}(V_y)^2_{kj}\right] = \frac{1}{2}\left[p_{k,j}+0(h)\right]\left[R_{k,j}+Q_{kj}\right]$$

where

$$(3.18b)\qquad R_{kj} = \left[(V_\eta)^2_{k+2,j} + (V_\eta)^2_{k+1,j+1} + 2(V_\zeta)^2_{kj}\right],$$

$$(3.18c)\qquad Q_{kj} = 2\left[(V_\eta)_{k+1,j+1} - (V_\eta)_{k+2,j}\right](V_\zeta)_{kj}.$$

Hence

$$(3.19)\quad 4h^2\left[A_{k+1,j}(V_x)^2_{kj} + B_{k,j+1}(V_y)^2_{k,j}\right] \le 2p_{kj}(1+Kh)(V_\zeta)^2_{kj} +$$

$$p_{k+1,j}(1+Kh)(V_\eta)^2_{k+2,j} + P_{k+1,j+1}(1+Kh)(V_\eta)^2_{k+1,j+1}.$$

Similarly

$$(3.20)\quad 2h^2\left[a_{k+\frac{1}{2},j+\frac{1}{2}}(V_\zeta)^2 + b_{k-\frac{1}{2},j+\frac{1}{2}}(V_\eta)^2\right] \ge p_{kj}(1-kh)\left[(V_\zeta)^2_{kj} + (V_\eta)^2_{kj}\right].$$

Finally, from these estimates and (3.13a) and (3.13b) we obtain

$$V^T\tilde{L}_E V \le 2\left[\sum p_{kj}(1+Kh)\left[(V_\zeta)^2_{kj} + (V_\eta)^2_{kj}\right]\right]$$

$$\le 2\left[\frac{(1+Kh)}{1-Kh}\right] V^T L_E V.$$

Therefore

$$1 - 2\lambda \leq 2\,\frac{1+Kh}{1-Kh} \leq 2 + K_0h$$

and

$$-\frac{(1+K_0h)}{2} \leq \lambda \ .$$

This estimate and (2.11c) prove the theorem.

It is of some interest to consider the role of "smoothing" before solving (2.19a). We have

$$L_E\phi = r_E = I_h^E\, L_h\tilde{\varepsilon} \ ,$$

or

$$L_E\phi = \hat{L}_E u \ .$$

If "smoothing" is applied either on S_h or on S_E we have

$$L_E\phi = \hat{L}_E G u$$

and we are concerned with

$$\frac{\|Gu-\phi\|}{\|u\|} = \frac{\|Gu-\phi\|}{\|Gu\|}\,\frac{\|Gu\|}{\|u\|} \ .$$

Therefore, smoothing can be advantageous either because

$$\frac{\|Gu\|}{\|u\|}$$

is small or because

$$\frac{\| Gu-\phi \|}{\| Gu \|} = \frac{1}{2} \, \| (I - L_E^{-1}\tilde{L}_E) Gu \| \Big/ \| Gu \|$$

is small. Quite clearly, this quantity is small when Gu is smooth.

4. COMPUTATIONAL RESULTS

The theoretical results of the preceeding section extend the work of Braess [1] for the MGR [0] iterative scheme and suggest the value of additional smoothing steps i.e. MGR[ν] with $\nu \geq 1$. We have undertaken some computational experiments which study this case and illustrate and document the theory.

The results of Ries, Trottenberg and Winter [3] for the case $p(x,y) = 1$ yield

$$(4.1) \qquad \rho[0] \nearrow \tfrac{1}{2}\,, \quad \rho[1] \nearrow {}^{2}/_{27}\,, \quad \rho[\nu] \nearrow \tfrac{1}{2}\,\frac{(2\nu)^{2\nu}}{(2\nu+1)^{2\nu+1}}\,.$$

The symbol $\nearrow$ means that the corresponding $\rho[\nu]$ increases to

$$\sigma(\nu) := \tfrac{1}{2}\,\frac{(2\nu)^{2\nu}}{(2\nu+1)^{2\nu+1}}$$

as $h \downarrow 0$.

Generally speaking the computational results indicate that (4.1) holds with a possible error of $O(h)$. We give four illustrative results.

In all cases Ω is the unit square,

$$(4.2) \qquad h = \frac{1}{N}\,.$$

<u>Case 1</u>: $\qquad p(x,y) = e^{-xy}$

$$u(x,y) = (1-e^{x})(x-1)y \cos \frac{\pi}{2} y$$

<u>Case 1.1</u>: $\qquad L_E = L_E^{(1)}$, see (2.18b).

N \ ν	0	1	2	3
15	.4857	.0797	.0482	.0351
31	.4842	.0739	.0431	.0312
63	.4836	.0714	.0399	.0278
$\sigma(\nu)$	.5000	.0741	.0410	.0283

Case 1.2: $L_E = L_E^{(2)}$, see (2.19a) .

N \ ν	0	1	2	3
15	.4853	.0650	.0347	.0207
31	.4841	.0697	.0376	.0253
63	.4835	.0708	.0386	.0263
$\sigma(\nu)$	.5000	.0741	.0410	.0283

Case 2:

$$p(x,y) = \left(\frac{1}{3-x}\right)\left(\frac{1}{3-y}\right) ,$$

$$u(x,y) = e^{xy} \sin \pi x \quad \sin \pi y$$

Case 2.1: $L_E = L_E^{(1)}$.

$N \backslash \nu$	0	1	2	3
15	.4879	.1084	.0727	.0565
31	.4854	.0901	.0582	.0442
63	.4851	.0784	.0473	.0350
$\sigma(\nu)$	.5000	.0741	.0410	.0283

Case 2.2: $$L_E = L_E^{(2)} .$$

$N \backslash \nu$	0	1	2	3
15	.4869	.0686	.0377	.0255
31	.4851	.0710	.0390	.0270
63	.4850	.0715	.0390	.0270
$\sigma(\nu)$	.5000	.0741	.0410	.0283

To compute the elements of $L_E^{(1)}$ for points on the boundary of Ω_h the following procedure was used. Use formula 2.18b to compute a_{kj}, referring to points inside Ω_h and then rather then setting d_{kj} to be the sum of the a_{kj}'s, set d_{kj} to be the average of d_{kj} at the two nearest interior points. At the corners of Ω_E, set d_{kj} to be d_{kj} from the entry of the nearest interior point. This approximation to $L_E^{(1)}$ differs from $L_E^{(1)}$ by no more than $O(h)$ and as can be seen from the computational results appears to work almost as well as $L_E^{(2)}$, which is the 'ideal' choice.

REFERENCES:

[1] D. Braess; The Contraction Number of a Multigrid Method for Solving the Poisson Equation, Numer. Math., 37, 387-404 (1981).

[2] S. F. McCormick and J. W. Ruge; Multigrid Methods for Variational Problems, SIAM J. Num. Anal., 19, 924-929 (1982).

[3] M. Ries, U. Trottenberg and G. Winter; A Note on MGR Methods, Linear Algebra Appl. 49, 1-26 (1983).

[4] D. Kamowitz and S. V. Parter; on MGR[ν] Multigrid Methods, University of Wisconsin Department of Computer Sciences Technical Report #575 (January 1985).