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Abstract A class of spaces of multivariate polynomials, closed under differentiation, is studied and corresponding classes of well posed Hermite-type interpolation problems are presented. All Hermite-type problems are limits of well posed Lagrange problems.

The results are based on a duality between certain spaces of multivariate exponential-polynomials $\mathcal{H}$ and corresponding spaces of multivariate polynomials $\mathcal{P}$, used by Dyn and Ron (1988) to establish the approximation order of the span of translates of exponential box splines. In the interpolation theory $\mathcal{P}$ is the space of interpolating polynomials and $\mathcal{H}$ characterizes the interpolation points and the interpolation conditions, both spaces being defined in terms of a set of hyperplanes in $\mathbb{R}^s$.

This geometric approach extends the work of Chung and Yao (1977) on Lagrange interpolation, and also a subset of the Hermite-type problems considered via the Newton scheme, by several authors (see Gasca and Maetzu (1982) and references therein). For a different approach to the interpolation problem see Chui and Lai (1988).

It is the systematic and unified analysis of a wide class of interpolation problems which is the main contribution of this paper to the study of multivariate polynomial interpolation.

Keywords: Multivariate interpolation, multivariate polynomials, Hermite-type interpolation.

1. The Interpolating Polynomial Spaces

The spaces of interpolating polynomials we consider here are more general than the total degree polynomials π_m (polynomials of degree $\leq m$), but as the latter are closed under differentiation.

Given a set of directions $A = \{\mathbf{a}^1, \ldots, \mathbf{a}^n\} \subset \mathbb{R}^s$, with the property span $A = \mathbb{R}^s$, consider the space of polynomials

$$\mathcal{P}(A) = \operatorname{span}\left\{ \prod_{i \in I} (\mathbf{a}^i \cdot \mathbf{x}) \;\middle|\; I \in S(A) \right\} \tag{1}$$

where $S(A)$ consists of index sets corresponding to "small enough" subsets of A, namely

$$S(A) = \left\{ I \subset \{1, \ldots, n\} \;\middle|\; \operatorname{span}\{\mathbf{a}^i \mid i \notin I\} = \mathbb{R}^s \right\} . \tag{2}$$

By choosing $I \in S(A)$ such that $\{1, \ldots, n\} \backslash I$ is a basis of $\mathbb{R}^s$, we conclude that

$$\mathcal{P}(A) \subset \pi_{n-s} . \tag{3}$$

To see that $\mathcal{P}(A)$ is closed under differentiation, observe that

$$\frac{\partial}{\partial x_j} \prod_{i \in I} (\mathbf{a}^i \cdot \mathbf{x}) = \sum_{\ell \in I} a_j^\ell \prod_{\substack{i \in I \\ i \neq \ell}} (\mathbf{a}^i \cdot \mathbf{x}) , \tag{4}$$

and that if $I \in S(A)$ then any subset of I is in $S(A)$.

A more involved analysis is required in order to show the following two properties of $\mathcal{P}(A)$, demonstrated in Dyn and Ron (1988):

(a) Let $d = d(A) = \min \left\{ |I| \;\middle|\; I \subset \{1, \ldots, n\} \,,\; I \notin S(A) \right\}$. Then $\pi_{d-1} \subset \mathcal{P}(A)$.

(b) The dimension of $\mathcal{P}(A)$ equals the number of bases that can be formed from A.

Combining (a) and (3) we conclude that

$$\pi_{d-1} \subset \mathcal{P}(A) \subset \pi_{n-s} . \tag{5}$$

If $\mathbf{a}^1, \ldots, \mathbf{a}^n$ are in "general position", namely any s vectors among $\mathbf{a}^1, \ldots, \mathbf{a}^n$ form a basis of $\mathbb{R}^s$, then it is easy to see that $d = n - s + 1$. Hence $\mathcal{P}(A) = \pi_{n-s}$.

To introduce a basis of $\mathcal{P}(A)$, consider n hyperplanes

$$H_i = \{x \in \mathbb{R}^s \mid \mathbf{a}^i \cdot \mathbf{x} = \gamma_i\} , \qquad i = 1, \ldots, n , \tag{6}$$

determined by $\Gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$. For each $\mathbf{v} \in \mathbb{R}^s$ define

$$I_{\mathbf{v}} = \left\{ i \in \{1, \ldots, n\} \;\middle|\; \mathbf{v} \in H_i \right\} . \tag{7}$$

and consider the set of intersection points of $H_1, \dots, H_n$,

$$V(A,\Gamma) = \{\mathbf{v} \in I\!R^s \mid |I_{\mathbf{v}}| \geq s\} , \tag{8}$$

where $|I_{\mathbf{v}}|$ denotes the cardinality of $I_{\mathbf{v}}$. Choosing Γ so that $|I_{\mathbf{v}}| = s$ for $\mathbf{v} \in V(A,\Gamma)$, we conclude from (b) that

$$\dim \mathcal{P}(A) = |V(A,\Gamma)| . \tag{9}$$

Furthermore, the following polynomials

$$p_{\mathbf{v}}(\mathbf{x}) = \prod_{i \notin I_{\mathbf{v}}} \frac{(\mathbf{a}^i \cdot \mathbf{x} - \gamma_i)}{(\mathbf{a}^i \cdot \mathbf{v} - \gamma_i)} , \qquad \mathbf{v} \in V(A,\Gamma) , \tag{10}$$

are linearly independent, since

$$p_{\mathbf{v}}(\mathbf{u}) = \begin{cases} 0 & \mathbf{u} \neq \mathbf{v} , \\ 1 & \mathbf{u} = \mathbf{v} , \end{cases} \qquad \mathbf{v}, \mathbf{u} \in V(A,\Gamma) , \tag{11}$$

and hence constitute a basis of $\mathcal{P}(A)$.

The pair (A,Γ) is termed "simple" (for simple intersection points as opposed to multiple ones) if $|I_{\mathbf{v}}| = s$ for all $\mathbf{v} \in V(A,\Gamma)$.

Remark 1 It is shown by Ron (1988) that for fixed A the set of all $\Gamma \in I\!R^n$ such that (A,Γ) is simple, is dense in $I\!R^n$.

The explicit form (10) of a basis of $\mathcal{P}(A)$ indicates that the following result holds.

Proposition 1 $\mathcal{P}(A)$ consists of polynomials of degree $\leq n-s$, which are of degree $\leq n - s - \big|\{i \in \{1,\dots,n\} \mid \mathbf{a}^i \in \operatorname{span}\{\mathbf{y}\}\}\big| + 1$ along hyperplanes of the form $\mathbf{y} \cdot \mathbf{x} = \lambda$, $\mathbf{y} \in A$, $\lambda \in I\!R$. Furthermore, let $Y = \{\mathbf{y}^1, \dots, \mathbf{y}^k\}$, be $k < s$ pairwise distinct directions in A. Then the degree of any $p \in \mathcal{P}(A)$ along the intersection of k hyperplanes of the form

$$\mathbf{y}^j \cdot \mathbf{x} = \mu_j \quad , \quad j = 1, \dots, k , \qquad \mu_1, \dots, \mu_k \in R , \tag{12}$$

is at most

$$n - s - \big|\{i \in \{1,\dots,n\} \mid \mathbf{a}^i \in \langle Y \rangle\}\big| + \dim \langle Y \rangle , \tag{13}$$

where $\langle Y \rangle = \operatorname{span} Y$.

Proof Since for (A,Γ) simple, and $\mathbf{v} \in V(A,\Gamma)$, $\{\mathbf{a}^i \mid i \in I_\mathbf{v}\}$ is a basis of $\mathbb{R}^s$, each $p_\mathbf{v}$ in (10) consists of at least $\big|\{i \in \{1,\ldots,n\} \mid \mathbf{a}^i \in \operatorname{span}\{\mathbf{y}\}\}\big| - 1$ factors which are constant along $\mathbf{y}\cdot\mathbf{x} = \lambda$, $\lambda \in \mathbb{R}$. Similarly, one can count the constant factors in $p_\mathbf{v}$ of (10) along the intersection of the hyperplanes (12), to conclude (13).

Remark 2 The space $\mathcal{P}(A)$ consists of all polynomials over $\mathbb{R}^s$ with the properties stated in Proposition 1. This will be shown elsewhere.

2. The Interpolation Problems

In this section we present a class of interpolation problems which are unisolvent in $\mathcal{P}(A)$ for fixed A. The interpolation points and the data at each point, which is of Hermite type, are determined by the choice of $\Gamma = (\gamma_1,\ldots,\gamma_n) \in \mathbb{R}^n$. The set of interpolation points consists of all points of intersection of at least s of the hyperplanes (6), namely, it is the set denoted by $V(A,\Gamma)$. To define the interpolation conditions at each $\mathbf{v} \in V(A,\Gamma)$, we consider the set of directions related to $\mathbf{v}$

$$A_\mathbf{v} = \{\mathbf{a}^i \mid i \in I_\mathbf{v}\}\ , \tag{14}$$

and a corresponding polynomial space defined by

$$\mathcal{K}(A_\mathbf{v}) = \{p \in \pi \mid [\prod_{i\in I}(\mathbf{a}^i\cdot D)]p \equiv 0\ ,\ I \notin S(A_\mathbf{v})\}\ , \tag{15}$$

where $D = \left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_s}\right)$. Since each I in (15) satisfies $|I| \geq d_\mathbf{v} = d(A_\mathbf{v})$, it is clear that $\pi_{d_\mathbf{v}-1} \subset \mathcal{K}(A_\mathbf{v})$. The space $\mathcal{K}(A_\mathbf{v})$ is closed under differentiation since $D^\mathbf{m}$ commutes with any polynomial in D. In terms of $\mathcal{K}(A_\mathbf{v})$ the interpolation conditions at $\mathbf{v}$ are

$$\big[q(D)p\big](\mathbf{v}) = \big[q(D)f\big](\mathbf{v})\ , \qquad q \in \mathcal{K}(A_\mathbf{v})\ , \tag{16}$$

where f is smooth enough, and $q(D)$ is obtained from the polynomial $q(\mathbf{x})$ by replacing the vector $\mathbf{x}$ by the vector D. With these definitions we can introduce the interpolation problem determined by A and Γ:

$$\text{Find } p \in \mathcal{P}(A) \text{ satisfying (16) for all } \mathbf{v} \in V(A,\Gamma). \tag{17}$$

The solvability of (17) is due to the following result from Dyn and Ron (1988):

Theorem 1 The spaces $\mathcal{P}(A)$ and the space

$$\mathcal{H}(A,\Gamma) = \bigoplus_{\mathbf{v}\in V(A,\Gamma)} \{e^{\mathbf{v}\cdot\mathbf{x}}q(\mathbf{x}) \mid q \in \mathcal{K}(A_{\mathbf{v}})\} , \tag{18}$$

are dual to each other under the pairing

$$[p(D)h](0) = [q(D)p](\mathbf{v}) , \quad p \in \mathcal{P}(A) , \quad h(\mathbf{x}) = e^{\mathbf{v}\cdot\mathbf{x}}q(\mathbf{x}) \in \mathcal{H}(A,\Gamma) . \tag{19}$$

Corollary 1 There exists a unique $p \in \mathcal{P}(A)$ solving the interpolation problem (17).

It follows from Theorem 1 that $\mathcal{K}(A_{\mathbf{v}})$ is dual to $\mathcal{P}(A_{\mathbf{v}})$ in the sense of (19), and by (b) $\dim \mathcal{K}(A_{\mathbf{v}}) = \#$ of bases in $A_{\mathbf{v}}$. Furthermore, since $\mathcal{P}(A_{\mathbf{v}}) \subset \pi_{|A_{\mathbf{v}}|-s}$ we conclude that $\mathcal{K}(A_{\mathbf{v}}) \subset \pi_{|A_{\mathbf{v}}|-s}$. Hence

$$\pi_{d_{\mathbf{v}}-1} \subset \mathcal{K}(A_{\mathbf{v}}) \subset \pi_{|A_{\mathbf{v}}|-s} , \tag{20}$$

in analogy to (5). Moreover, if the directions in $A_{\mathbf{v}}$ are in general position then $\mathcal{K}(A_{\mathbf{v}}) = \pi_{|A_{\mathbf{v}}|-s}$.

Corollary 2 Let Γ be such that for each $\mathbf{v} \in V(A,\Gamma)$ the directions in $A_{\mathbf{v}}$ are in general position. Then the interpolation conditions in (16) are pure Hermite of the form

$$D^{\mathbf{m}}p(\mathbf{v}) = D^{\mathbf{m}}f(\mathbf{v}) , \qquad |\mathbf{m}| = \sum_{i=1}^{s} m_i \le |A_{\mathbf{v}}| - s \ , \ m_i \ge 0 \ , \ i = 1,\ldots,s . \tag{21}$$

In case A consists of directions in general position, then so does each $A_{\mathbf{v}}$, $\mathbf{v} \in V(A,\Gamma)$, and the interpolation problem becomes: Find $p \in \pi_{n-s}$ satisfying (21) for each $\mathbf{v} \in V(A,\Gamma)$. In R^2 the conditions on Γ in Corollary 2 are satisfied if $\gamma_i \neq \lambda\gamma_j$ whenever $\mathbf{a}^i = \lambda\mathbf{a}^j$, $\lambda \in I\!R$, $i \neq j$, $i,j \in \{1,\ldots,n\}$, namely if the hyperplanes $H_1,\ldots,H_n$ in (6) are pairwise disjoint.

An especially interesting interpolation problem is the Lagrange interpolation, obtained when (A,Γ) is simple. In this case $|A_{\mathbf{v}}| = s$, $\mathcal{K}(A_{\mathbf{v}}) = \pi_0$, and p satisfies $p(\mathbf{v}) = f(\mathbf{v})$, $\mathbf{v} \in V(A,\Gamma)$. The solution is given explicitly, in terms of the basis (10), as

$$p(x) = \sum_{\mathbf{v}\in V(A,\Gamma)} f(\mathbf{v})p_{\mathbf{v}}(x) . \tag{22}$$

This together with Remark 1 implies that the interpolation problem (17) is a limit of a sequence of Lagrange interpolation problems.

For general (A,Γ) the interpolation conditions (16) at $\mathbf{v} \in V(A,\Gamma)$ are determined by the structure of a chosen basis of $\mathcal{K}(A_{\mathbf{v}})$. The construction of such bases is discussed by Dahmen (this volume) and by deBoor and Ron (1988).

3. Examples

The first two examples are in R^2 and can be displayed graphically. We consider two Lagrange interpolation problems, for the same set of directions A, and then two Hermite-type problems, obtained as limits of the Lagrange problems.

Example 1 Let $A = \{\mathbf{a}^1, \ldots, \mathbf{a}^6\}$ with $\mathbf{a}^1 = \mathbf{a}^4 = (1,0)$, $\mathbf{a}^2 = \mathbf{a}^5 = (0,1)$, $\mathbf{a}^3 = \mathbf{a}^6 = (1,1)$, and let $\Gamma = \left(0,0,1,\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon,\frac{1}{2}\right)$ for $\varepsilon > 0$. The space $\mathcal{P}(A)$ is of dimension 12 and consists of quartic polynomials which reduce to cubics along hyperplanes of the form $\mathbf{a}^i \cdot \mathbf{x} = \text{const}$. The hyperplanes $\mathbf{a}^i \cdot \mathbf{x} = \gamma_i$, $i = 1, \ldots, 6$ are depicted in Figure 1, together with the twelve interpolation points. Since each interpolation point belongs to exactly two hyperplanes, (A,Γ) is simple, and the data at each point is just the function value.

For $\varepsilon = 0$ the three interpolation points $\mathbf{v}^1 = (1,0)$, $\mathbf{v}^2 = (0,1)$, $\mathbf{v}^3 = (0,0)$ remain unchanged together with the corresponding $A_{\mathbf{v}^i}$. Hence also in this problem only function values are required at $\mathbf{v}^i$, $i = 1,2,3$. Each of the other three interpolation points $\mathbf{v}^4 = \left(\frac{1}{2},0\right)$, $\mathbf{v}^5 = \left(0,\frac{1}{2}\right)$, $\mathbf{v}^6 = \left(\frac{1}{2},\frac{1}{2}\right)$ is the limit of three interpolation points in the case $\varepsilon > 0$, with $A_{\mathbf{v}^i} = \{\mathbf{a}^1,\mathbf{a}^2,\mathbf{a}^3\}$, $i = 4,5,6$. Thus $\mathcal{K}(A_{\mathbf{v}^i}) = \pi_1$, $i = 4,5,6$, and the Hermite conditions are of the form

$$(f-p)(\mathbf{v}^i) = 0 \quad , \quad \frac{\partial}{\partial x_1}(f-p)(\mathbf{v}^i) = 0 \quad , \quad \frac{\partial}{\partial x_2}(f-p)(\mathbf{v}^i) = 0 \quad , \quad i = 4,5,6 \; .$$

Example 2 Let A be as in Example 1 and let $\Gamma = (0,0,1,\varepsilon,\varepsilon,1-\varepsilon)$ for $\varepsilon > 0$. The space $\mathcal{P}(A)$ is as in Example 1. The hyperplanes $\mathbf{a}^i \cdot \mathbf{x} = \gamma_i$, $i = 1, \ldots, 6$ are depicted in Figure 2. These hyperplanes have twelve intersection points, each belonging to exactly two hyperplanes. Thus for $\varepsilon > 0$, (A,Γ) is simple and the interpolation is of Lagrange type.

In the limit $\varepsilon \to 0$, there are only three interpolation points: $\mathbf{v}^1 = (1,0)$, $\mathbf{v}^2 = (0,1)$, $\mathbf{v}^3 = (0,0)$, each being the limit of four interpolation points in the case $\varepsilon > 0$. The

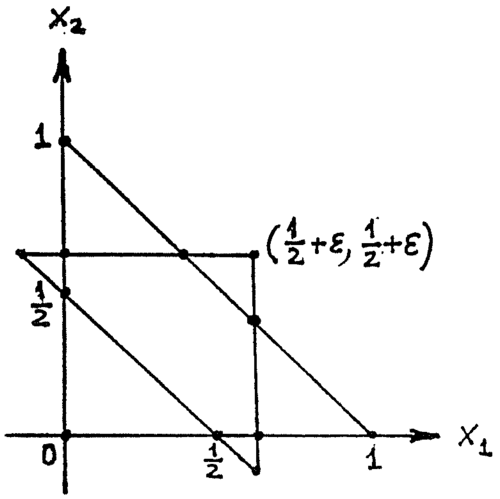

Figure 1

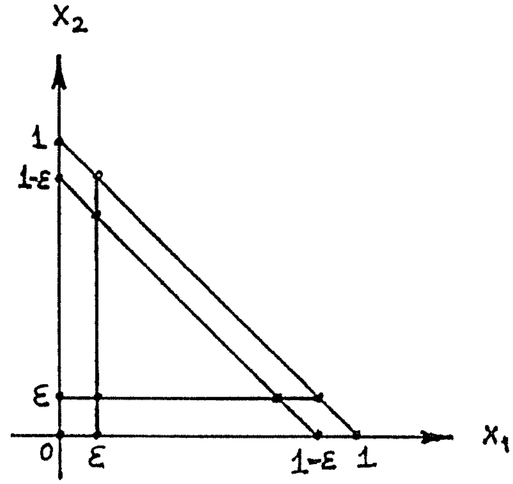

Figure 2

interpolation conditions at v^i are determined by $\mathcal{K}(A_{\mathrm{v}^i})$ where

$$A_{\mathrm{v}^i} = \{\mathrm{a}^j, \mathrm{a}^j \mid j \neq i \ , \ j = 1,2,3\} \ .$$

By (15)

$$\begin{aligned}\mathcal{K}(A_{\mathrm{v}}) &= \{p \in \pi \mid (\mathrm{a}^j \cdot D)^2 p = 0 \ , \ j \neq i \ , \ j = 1,2,3\} \\ &= \mathrm{span}\{1, x_1, x_2, \prod_{\substack{j=1 \\ j \neq i}}^{3} (\mathrm{n}^j \cdot \mathrm{x})\} \ ,\end{aligned}$$

where $\mathrm{n}^i \cdot \mathrm{a}^i = 0$, $i = 1,2,3$. Hence the interpolation conditions are

$$(f-p)(\mathrm{v}^i) = 0 \ , \ \frac{\partial}{\partial x_1}(f-p)(\mathrm{v}^i) = 0 \ , \ \frac{\partial}{\partial x_2}(f-p)(\mathrm{v}^i) = 0 \ , \quad i = 1,2,3$$

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1 \, \partial x_2}\right)(f-p)(\mathrm{v}^1) = 0 \quad , \quad \left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1 \, \partial x_2}\right)(f-p)(\mathrm{v}^2) = 0 \ ,$$

$$\frac{\partial^2}{\partial x_1 \, \partial x_2}(f-p)(\mathrm{v}^3) = 0 \ .$$

This interpolation problem is a special case of the one solved by Gregory (1985), where the interpolation points are the vertices of a simplex in $I\!R^s$, and A consists of $s+1$ directions in general position each repeated $N \geq 2$ times. The next example deals with an extended version of this case in terms of our analysis.

Example 3 Let $B = \{\mathrm{b}^1, \ldots, \mathrm{b}^{s+1}\} \subset I\!R^s$ be in general position and let (B, Δ) be simple, with $\Delta = (\delta_1, \ldots, \delta_{s+1}) \in I\!R^{s+1}$. Given $s+1$ positive integers $m_1, \ldots, m_{s+1}$, $n = \sum_{i=1}^{s+1} m_i$, consider $A = \{\mathrm{a}^1, \ldots, \mathrm{a}^n\}$ consisting of b^i repeated m_i times, and $\Gamma =$

$(\gamma_1, \ldots, \gamma_n)$ consisting of δ_i repeated m_i times, $i = 1, \ldots, s+1$. The hyperplanes $H_i = \{\mathbf{x} \mid \mathbf{b}^i \cdot \mathbf{x} = \delta_i\}$, $i = 1, \ldots, s+1$, intersect at $s+1$ points $\mathbf{v}^1, \ldots, \mathbf{v}^{s+1}$, forming the vertices of a simplex. Let $\mathbf{v}^i$ denote the intersection of the s hyperplanes H_j, $j \neq i$, $j = 1, \ldots, s+1$. Then $A_{\mathbf{v}^i}$ consists of $\mathbf{b}^j$ repeated m_j times $j \neq i$, $j = 1, \ldots, s+1$, and by (15)

$$\mathcal{K}(A_{\mathbf{v}^i}) = \left\{ p \in \pi \;\middle|\; (\mathbf{b}^j \cdot D)^{m_j} p \equiv 0 \;,\; j \neq i \;,\; j = 1, \ldots, s+1 \right\} .$$

The dimension of $\mathcal{K}(A_{\mathbf{v}^i})$ is the number of bases in $A_{\mathbf{v}^i}$ given by $M_i = \prod_{j=1, j\neq i}^{s+1} m_j$.

Now the edge of the simplex connecting vertices $\mathbf{v}^i$ and $\mathbf{v}^\ell$ belongs to the intersection of the hyperplanes H_j, $j \neq i, \ell$, $j = 1, \ldots, s+1$. Hence $(\mathbf{v}^i - \mathbf{v}^\ell) \cdot \mathbf{b}^j = 0$, $j \neq i, \ell$, $j = 1, \ldots, s+1$, from which we conclude that any polynomial of the form

$$\prod_{\substack{\ell=1 \\ \ell \neq i}}^{s+1} \left[(\mathbf{v}^i - \mathbf{v}^\ell) \cdot \mathbf{x} \right]^{\alpha_\ell} , \qquad 0 \leq \alpha_\ell < m_\ell \;,\; \ell = 1, \ldots, s+1 \;,$$

is annihilated by $(\mathbf{b}^j \cdot D)^{m_j}$, $j \neq i$, and therefore belongs to $\mathcal{K}(A_{\mathbf{v}^i})$. The number of these polynomials is M_i and they are linearly independent, thus forming a basis of $\mathcal{K}(A_{\mathbf{v}})$. In terms of this basis the Hermite type conditions at $\mathbf{v}^i$ are

$$\prod_{\substack{j=1 \\ j \neq i}}^{s+1} \left[(\mathbf{v}^i - \mathbf{v}^j) \cdot D \right]^{\alpha_j} (f - p)(\mathbf{v}^i) = 0 \;, \quad 0 \leq \alpha_j < m_j \;,\; j \neq i \;,\; j = 1, \ldots, s+1 \;.$$

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