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Report No. 117

Tolerance Analysis Considering Manufacturing Variability and the Cost of Deviating from the Nominal

Spencer Graves

May 1994

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Tolerance Analysis Considering Manufacturing Variability and the Costs of Deviating from the Nominal

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Center for Quality and Productivity
Improvement

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ABSTRACT

A number of different formulae for tolerance analysis and synthesis have appeared over the years. This article discusses the interrelationships between alternative formulae, showing how each is best for a specific set of assumptions regarding the cost of deviating from the nominal and the distributions of dimensions of parts. To increase the use of appropriate statistical tolerancing, a procedure is outlined for converting process capability studies into a simple formula tailored to a given manufacturing organization.

KEYWORDS: *Statistical decision theory; Statistical tolerancing.*

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Tolerance Analysis Considering Manufacturing Variability and the Costs of Deviating from the Nominal

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A number of different formulae for tolerance analysis and synthesis have appeared over the years. This article discusses the interrelationships between alternative formulae, showing how each is best for a specific set of assumptions regarding the cost of deviating from the nominal and the distributions of dimensions of parts. To increase the use of appropriate statistical tolerancing, a procedure is outlined for converting process capability studies into a simple formula tailored to a given manufacturing organization.

INTRODUCTION

Table 1 summarizes four different tolerance analyses formulae with loss functions and manufacturing distributions required to make each formula the optimal. These and other formulae are reviewed and compared. They are then related to costs and manufacturing distributions providing detail to support Table 1. The formulae include traditional linear and statistical tolerancing [expressions (1) and (2) below and cases 1 and 4 in Table 1] previously published modifications [(5)–(8a) below and case 2] and a formula related to mean square error [(12) below and case 3].

This article makes four specific contributions to the literature on engineering tolerances:

1. it describes the loss function and manufacturing distribution required to make each of several tolerance analysis formulae optimal;
2. it introduces a formula for tolerance analysis based on process capability, not 100% inspection as implied by other formulae;
3. it outlines a methodology for converting process capability studies into simple rules for tolerance analysis tailored to a given organization; and
4. it illustrates the utility for tolerancing of statistical decision theory, which uses loss functions and probability distributions to select from alternative solutions to problems. Tipnis (1990, 1992a; see also Srinivasan and Voelcker 1993; Voelcker 1993) has identified the need for a unifying mathematical foundation for engineering tolerances; statistical decision theory can provide an important part of the required foundation (see also Bisgaard and Graves, 1994).

The incorporation of these results into Computer-Aided Design / Computer-Integrated Manufacturing systems will, in the author's opinion, contribute to improving tolerancing practices in ways that simultaneously reduce costs while increasing quality.

THE HISTORY OF TOLERANCE ANALYSIS

Tolerancing is a necessary part of modern manufacturing. This section discusses the pre-history of tolerancing in the development of interchangeable parts before discussing traditional linear and statistical tolerancing and modifications.

TOLERANCING AND INTERCHANGEABLE PARTS

Formal tolerancing as used today seems to have appeared relatively late in the development of interchangeable parts. Christopher Polhem was manufacturing interchangeable gears for clocks and machinery in Sweden in the 1720s (Woodbury 1960, 243). The "uniformity principle" was advocated by the French General Gribeauval in 1765 (Hounshell, 1984, p. 25); it visibly contributed to the mobility of French light artillery that supported the colonists in the American Revolution (Smith 1981, p. 67). Shortly thereafter, US and British leaders began to advocate interchangeable parts. The Springfield (Massachusetts) Armory achieved international acclaim in the 1850s by demonstrating disassembly

Table 1. *Tolerance analysis formulae suggested by alternative loss functions and manufacturing distributions.*

CASE #	*MANUFACTURING DISTRIBUTION*	*LOSS FUNCTION*	*E(LOSS)*	*TOLERANCE ANALYSIS FORMULA SUGGESTED BY THE MODEL*	*FORMULA IN TEXT*
1	Minimax over distributions concentrated near the specification limits. (Burr 1963)	0 if assembly within specifications k_1 if not	0 if $T_a = \sum \lvert a_i \rvert T_i$	Traditional linear tolerancing	(1), (3a)
2	Minimax varying bias $m_i = 2(\theta_i - \mu_i)/T_i$ in $N(\mu_i, \sigma_i^2)$, $\sigma_i = \gamma_i(1 - m_i)T_i$	"	$k_1 \left\{ \Phi\left(\frac{L_a - \sum a_i \mu_i}{\sqrt{\sum (a_i \sigma_i)^2}} \right) + \left[1 - \Phi\left(\frac{U_a - \sum a_i \mu_i}{\sqrt{\sum (a_i \sigma_i)^2}} \right) \right] \right\}$	Ettinger-Bartky-Gilson (EBG) / Monsoor-Greenwood-Chase (MGC): $T_a = \sum \lvert a_i \rvert m_i T_i + 2Z\left[\sum a_i^2 \gamma_i^2 (1 - m_i)^2 T_i^2 \right]^{1/2}$	(9)
3	"	$k_0 (X_a - \theta_a)^2$	$k_0 \left\{ \left[\theta_a - \sum a_i \mu_i \right]^2 + \sum (a_i \sigma_i)^2 \right\}$	(bias)^2 + variance	(12)
4	[case 2 or 3 with $m_i = 0$, $\gamma_i = 1/(2Z)$]	(c3ase 2 or 3)	[case 2 or 3 with $m_i = 0$, $\gamma_i = 1/(2Z)$]	Traditional statistical tolerancing	(2), (4b)

a_i = error transmission rate for dimension i

k_0, k_1 = cost coefficients

L_a, U_a, L_i, U_i = lower and upper specification limits for the assembly and for dimension i

m_i = bias = deviation of the process mean from the nominal, relative to T_i for dimension i

T_a, T_i = tolerance range for the assembly ($U_a - L_a$) and for dimension i ($U_i - L_i$)

X_a, X_i = actual dimension for the assembly and for dimension i

Z, Z_i = number of standard deviations of the assembly from the average to the nearest limit

$\gamma_i = 1/(2Z_i)$

μ_i, σ_i = mean and standard deviation for dimension i

θ_a, θ_i = nominal for the assembly and for dimension i

$\Phi(.)$ = cumulative distribution function for the standard normal

and random reassembly of their muskets (Hounshell 1984).

Interchangeability was initially achieved using locally produced gauges (Hounshell, 1984, p. 45, 55) because existing standards of measurement were not sufficiently accurate. Micrometer calipers and other innovations in standardized metrology were introduced into manufacturing in the late nineteenth century, permitting in some cases the interchange of parts made in different locations from written specifications (Uselding, 1981; Lea, 1946, p. 14-18; Miller, 1962, p. 76-77).

Tolerancing practices evolved as the number and complexity of manufactured parts increased. An early standard for limits and fits of shafts and holes was the Newall system, which appeared in 1902 (Miller, 1962, p. 162-170). The practice of specifying tolerances on drawings did not begin, at least in the US, until shortly after 1900 (Buckingham, 1954, p. 1; Uselding, 1981, p. 104).

Linear tolerancing was doubtless applied to analyze the clearance between a hole and a shaft in the middle 1800s. However, there seems to have been relatively little tolerance analysis until genuine interchangeability was established at the Ford Motor Company after 1906 and popularized in an article officially written by Henry Ford in 1925 (Hounshell, 1984, p. 1, 6, 219-221).

TRADITIONAL LINEAR AND STATISTICAL TOLERANCING

Buckingham (1921) described linear tolerancing without using algebra, recommending experiments to validate functionality at component tolerance limits (p. 42). Rüdenberg (1929) discussed both linear and statistical tolerancing,

$$T_a = \sum T_i \tag{1}$$

and

$$T_a = \sqrt{\sum T_i^2}\,, \tag{2}$$

where T_a is the width of the tolerance interval of an assembly or a sum dimension, T_i is the tolerance width for the ith dimension in the tolerance chain, and Σ in this article denotes the sum from $i = 1, ..., n$. (Some authors write T_i as *half* the width of the interval; the present definition seems more common.) Expression (2) later appeared in Brandenberger (1946, p. 72-77), Grant (1946, p. 326), Bruyevich (1946), Juran (1951, p. 75-77; 1962, p. 3-42, 50), and many more recent works.

Shewhart (1931, p. 256-257) discussed potentially non-linear assemblies in which a dimension $X_a = F(X_1, ..., X_n)$ can be approximated as follows:

$$X_a \cong \theta_a + \sum a_i(X_i - \theta_i), \tag{3}$$

where θ_a is the nominal for the assembly with X_a as its actual dimension, θ_i is the nominal for X_i and $a_i = \partial F/\partial X_i$ evaluated at θ_i. The a_i's can be estimated from experiments if existing theory is not adequate (Bisgaard 1993). With this, expression (1) has been generalized to

$$T_a = \sum |a_i| T_i\,. \tag{3a}$$

Shewhart's (1931, ch. 17) point in discussing linear tolerancing was to introduce statistical tolerancing in the form

$$\sigma_a = \sqrt{a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2} \tag{4}$$

where σ_a is the standard deviation of X_a, σ_i is the standard deviation of X_i, and the X_i's are assumed to be uncorrelated. If X_a is a non-linear function of the X_i's, (4) is only an approximation to the standard deviation of X_a. The approximation is accurate if the expected value of the residual from $(X_a - \theta_a)^2$ in (3) is negligible. If this residual is negligible with high probability, even if its expectation does not exist, the normal distribution using (4) can still be a reasonable approximation to the distribution of X_a provided the distributions of the X_i's satisfy certain regularity conditions (Graves, 1983, p. 226-255; Skovgaard, 1981).

The step from (4) to (2) assumes that T_a and each of the T_i's contain the same number of standard deviations and all processes are centered at the nominal. For more complicated fits, this can be generalized using (3) to

$$T_a = \sqrt{\sum a_i^2 T_i^2}\,. \tag{4a}$$

ETTINGER-BARTKY-GILSON (EBG) FORMULAE

While (1) requires T_i's that are tighter than generally needed to achieve a given T_a, it came to be recognized that (2) permits T_i's that are often too loose. Ettinger and Bartky (1936) wrote (in our notation)

$$\sigma_a = \sqrt{\sum \gamma_i^2 T_i^2} \tag{5}$$

where $\gamma_i = \sigma_i/T_i = 1/(2Z_i)$, with Z_i being the number of standard deviations of component i from its process average to the nearest tolerance limit, and processes are assumed to be centered at the nominal. Thus, $\gamma_i = 1/(2\sqrt{3})$ for the uniform distribution between $\pm T_i/2$, $1/(2\sqrt{6})$ for the triangular distribution between $\pm T_i/2$, and $1/(2\sqrt{9}) = 1/6$ for the normal distribution with $T_i = 6\sigma_i$. In terms of the now-popular process capability indices (Gitlow, et al., 1989, p. 241-257; Kotz and Johnson, 1993; Montgomery, 1991, ch. 9), $\gamma_i = 1/(6C_{pk})$ assuming processes centered at the nominal.

If the γ_i's do not vary with i, this implies

$$T_a = f\sqrt{\sum T_i^2} \tag{6}$$

where $f = 2Z\gamma_i = Z/Z_i$, and Z is the number of standard deviations of the assembly from the process average to the nearest limit. This can be generalized to

$$T_a = f\sqrt{\sum a_i^2 T_i^2}\,. \tag{6a}$$

Gilson (1951, ch. IV) recommended (6) with $f = 1.6$, empirically determined. He approximated this further by

$$T_a = \frac{f}{\sqrt{n}} \sum T_i , \qquad (6b)$$

which is the linear Taylor series approximation to (6) about $\bar{T}$.

Bender (1962) reported that at Delco-Remy, "The component dimensions and their stack-ups were studied over a period of time on many complex assemblies and products. In most cases, the assumptions made in the probability method were invalid. In no case was the over-all variation as great as that calculated by the method of extremes while the probability method stack-up was exceeded in nearly all cases. From these studies and the sense of feel, a modification of the probability method was found to give a realistic method" corresponding to (6) with $f = 1.5$.

Liggett (1993, p. 29-30) recommended a procedure that begins by determining an "effective *n*" equal to the smallest number of components required to include at least 85% of the sum of squares of the component tolerances counting multiple copies of the same part as one (because multiple copies might not be statistically independent). Liggett then uses (6b) with $f = 1$, 1.30, 1.39, or 1.42 if the "effective *n*" is 1, 2, 3, or 4, respectively; if the "effective *n*" exceeds 4, he uses (6) with $f = 1.5$.

For convenience in the following, (5)–(6b) will be called the Ettinger-Bartky-Gilson (EBG) formulae.

MANSOOR-GREENWOOD-CHASE (MGC) FORMULAE

Mansoor (1963) suggested a formula equivalent to

$$T_a = m \sum T_i + (1-m)\sqrt{\sum T_i^2} , \qquad (7)$$

where m is the proportion of the tolerance range devoted to bias. This coefficient $m = \left(1 - C_{pk}/C_p\right) = C_c$ in terms of the process centering coefficient $C_c = 2\left|\bar{X} - \theta\right|/T$ used in process capability studies (Liggett 1993, 60-62). By way of comparison, Motorola's "six sigma" program (Liggett, 1993, p. 60-62; Harry, 1992) assumes that $T_i = 12\sigma_i$ and the process average is at most $1.5\sigma_i$ from the nominal; this makes $m_i = 1.5/6 = 0.25$ and $\gamma_i = 1/9$.

Expression (7) reduces to (6) if

$$f = 1 + m(R-1)$$

where

$$R = \sum T_i \Big/ \sqrt{\sum T_i^2} = \frac{\sqrt{n}}{\sqrt{1+V^2}} \qquad (7a)$$

where V is the coefficient of variation of the T_i's. If the T_i's are all equal, $R = \sqrt{n}$. Thus, (6) and (7) are incompatible because if m is fixed, f is roughly proportional to $\sqrt{n}$ for large n; if f is fixed, m is roughly proportional to $1/\sqrt{n}$. This incompatibility is explained by noting that (6) assumes that all averages are at the nominal while (7) assumes that averages for component dimensions deviate from the nominal in ways that *never cancel* as *n* increases.

Whether (6) or (7) is most appropriate depends on the stack up of the deviations of process averages from component nominals; a procedure is outlined with (13) below for converting process capability studies into a simple formula interpolating between (6) and (7). For a simple shaft-hole system, if the cost of scrap exceeds that of rework then a lower total cost of production and use might be achieved by offsetting the process average away from the nominal to reduce scrap by doing more rework (Bisgaard and Graves, 1994); this could create a stack-up problem, supporting (7) over (6).

Greenwood and Chase (1987; Shawki, 1963) generalized (7) by assuming that *m* varies with *i*, as

$$T_a = \sum m_i T_i + \left(\frac{Z}{3}\right)\sqrt{\sum (1-m_i)^2 T_i^2} , \qquad (8)$$

or more generally (Chase and Greenwood, 1988)

$$T_a = \sum |a_i| m_i T_i + \left(\frac{Z}{3}\right)\sqrt{\sum a_i^2 (1-m_i)^2 T_i^2} , \qquad (8a)$$

where a_i is as in (3a), Z is the number of standard deviations from the assembly process average to the nearest specification limit and $(1-m_i)T_i = 6\sigma_i$. If $m_i = 0$ for all i and $Z = 3$, (8) becomes (2).

In this article, (7)–(8a) are called the Monsoor-Greenwood-Chase (MGC) formulae. As noted following (7a), they differ from the EBG formulae (5)–(6b) by the consideration of how biases contribute to tolerance stack-up.

SUMMARY OF THE LITERATURE

We can generalize both (5) and (8) by letting $\gamma_i = \sigma_i / \left[(1-m_i)T_i\right] = 1/(6C_{pk})$ to obtain

$$T_a = \sum |a_i| m_i T_i + 2Z\sqrt{\sum a_i^2 \gamma_i^2 (1-m_i)^2 T_i^2} . \qquad (9)$$

If $m_i = 0$, (9) is equivalent to (5) above. If $\gamma_i = 1/6$, (9) becomes (8a).

In summary, the existing tolerance analysis literature includes four different formulae with variations:

1. linear tolerancing; (1)
2. simple statistical tolerancing; (2)
3. the Ettinger-Bartky-Gilson (EBG) modification (5)–(6b), that assumes processes centered at the nominal but different numbers of standard deviations within specifications for component and assembly dimensions;
4. the Monsoor-Greenwood-Chase (MGC) formulae (7)–(8a) that assumes the process averages of component dimensions are consistently biased in ways that never cancel as n increases.

Finally, (9) includes all of these as special cases. We now consider alternative assumptions regarding costs and manufacturing distributions and see what they imply for tolerance analysis.

COSTS, MANUFACTURING DISTRIBUTIONS, AND TOLERANCING

We consider three cases in this section corresponding to cases 1–3 of Table 1; case 4 is a special case of both cases 2 and 3.

CASE 1. LINEAR TOLERANCING: 0–1 LOSS AND MINIMAX OPTIMIZATION

Suppose the cost of having X_a deviate from the nominal is zero if X_a is within specifications and a constant k_1 if outside. If the limits on the assembly X_a are $[L_a, U_a]$, the tolerance width $T_a = U_a - L_a$, and (3a) is satisfied at a cost less than k_1, then the total cost of production and use of any legal assembly (i.e., with $L_i \le X_i \le U_i$ for all i) will only be this production cost. If any tolerance is relaxed further (without a corresponding tightening of other tolerances), some legal assemblies will fall outside $[L_a, U_a]$ at a cost of k_1 plus production cost; thus, the maximum cost over all legal assemblies exceeds k_1. This latter case is avoided via (3a), which is called the *minimax* (Von Neumann and Morgenstern, 1947) solution because it minimizes the maximum cost or loss; see the discussion of minimax following case 3 below.

This strategy uses no knowledge of the production distributions; the cost of this ignorance can be seen by noting that (1) increases with n and (2) increases with $\sqrt{n}$ (approximately).

CASE 2. STATISTICAL TOLERANCING AND VARIATIONS: EXPECTED VALUE OF 0–1 LOSS

We now assume that X_a is approximately normally distributed. For 0 – 1 loss as in case 1, E(Loss) is as follows [at least approximately, as discussed following (4) above]:

$$E(\text{Loss}) = k_1\left\{\Phi\left(\frac{L_a-\mu_a}{\sigma_a}\right)+\left[1-\Phi\left(\frac{U_a-\mu_a}{\sigma_a}\right)\right]\right\}, \quad (10)$$

where μ_a and σ_a are the mean and standard deviation of X_a, and $\Phi(.)$ is the cumulative distribution function for the standard normal (Wadsworth, Stephens and Godfrey, 1986, p. 414).

With μ_l and μ_u as the anticipated lower and upper limits of the assembly process average and Z the number of standard deviations to the closest limit, L_a and U_a can be written as

$$L_a = \mu_1 - Z\sigma_a,$$

and

$$U_a = \mu_u + Z\sigma_a.$$

Hence,

$$T_a = U_a - L_a = (\mu_u - \mu_l) + 2Z\sigma_a. \quad (10a)$$

If the biases of component dimensions stack up in the worst possible way we get

$$\mu_u - \mu_l = \sum |a_i| m_i T_i. \quad (10b)$$

If the distributions of the X_i's are sufficiently well behaved to justify using (approximate) normality for the distribution of X_a [see the discussion following (4)], the obvious choice for σ_a can be substituted into (10a) with (10b) to obtain the EBG / MGC formula (9).

For most manufacturing situations the squared-error loss function, popularized by Taguchi (1986), is more realistic than the 0–1 loss considered in cases 1 and 2.

CASE 3. SQUARED-ERROR LOSS

If the loss from use is $k_0(X_a - \theta_a)^2$ and the component dimensions are distributed as in case 2, then E(Loss) is (at least approximately)

$$E(Loss) = k_0\left\{\left[\sum a_i(\mu_i - \theta_i)\right]^2 + \sum (a_i\sigma_i)^2\right\}, \quad (11)$$

where μ_i and θ_i are the mean and nominal for dimension i. Thus, E(Loss) is proportional to mean square error = [(bias)^2 + variance], as noted in Table 1. Using the notation from (3), (8), and (9), and assuming as with (10b) that none of the biases cancel, (11) can be rewritten as $E(\text{Loss}) = k_0[T_a/(2W)]^2$ where $2W$ is now the number of root mean square errors within assembly tolerance, and

$$T_a = 2W\sqrt{\left[\sum|a_i|m_iT_i\right]^2 + \left[\sum a_i^2\gamma_i^2(1-m_i)^2T_i^2\right]}\,. \quad (12)$$

Note that W here is different from Z in (9), the number of root mean square errors vs. number of standard deviations. The difference is more apparent than real; T_a with either W or Z is chosen to balance the costs of poorly fitting assemblies with the costs of producing finer tolerances.

The choice between (9) and (12) depends on the purpose: use (9) to set go / no-go limits for 100% inspection, use (12) for tolerancing without 100% inspection and to allocate resources for process control and improvement (unless 0–1 loss more closely describes the production situation).

In the example of Table 2 below, using $W = Z = 3$ gives very different answers for (9) and (12). This confirms the observation that W is different conceptually from Z and can not be assigned a value that easily.

EXAMPLE

Table 2 illustrates calculations using some of the above formulae. The example is based on "problem KK" in Chase et al. (1990), who gave tolerances for six components in a linear tolerance chain but not process capability. A suspected relationship between tolerances and process capability was found in data published by Shainin (1949) who gave "natural tolerances" and "drawing specifications" for 48 parts produced at Hamilton Standard Propellers: The ratio of specifications to tolerances is proportional to C_p, and a linear regression of the logarithm of this ratio on log(drawing specification) revealed that C_p was roughly proportional to $\sqrt{T}$. Based partly on this analysis of Shainin's data, we suppose that C_{pk} is proportional to $\sqrt{T}$, C_c is proportional to $\sqrt{C_{pk}}$, and dimension 3 is consistent with Motorola's "six sigma" standard.

For this case, traditional linear tolerancing gives 0.02, which is most likely wider than required. Traditional statistical tolerancing gives 0.0096, which is most likely tighter than required. The EBG / MGC formula (9) gives 0.0085, which is even tighter. The squared-error loss formula (12) gives 0.0272 that is wider than linear tolerancing. Gilson (6) gives 0.0153 that on the surface looks like a plausible compromise between (1) and (2).

The answers for (9) and (12) could be

Table 2.
Illustration of calculations.

						CONTRIBUTIONS:	
	tolerance	C_{pk}	C_c	γ_i	$m_i = C_c$	*linear*	*quadratic*
dimension	T_i	$(\propto \sqrt{T_i})$	$(\propto \sqrt{C_{pk}})$	$=1/(6C_{pk})$	$=(1-C_{pk}/C_p)$	$= m_iT_i$	$= [\gamma_i(1-m_i)T_i]^2$
1	0.006	2.12	0.0884	0.0786	0.177	0.00106	1.51×10^{-7}
2	0.006	2.12	0.0884	0.0786	0.177	0.00106	1.51×10^{-7}
3	0.003	1.50	0.1250	0.1110	0.250	0.00075	6.25×10^{-8}
4	0.003	1.50	0.1250	0.1110	0.250	0.00075	6.25×10^{-8}
5	0.001	0.87	0.2170	0.1920	0.433	0.000433	1.19×10^{-8}
6	0.001	0.87	0.2170	0.1920s	0.433	0.000433	1.19×10^{-8}
					sum=	0.00449	4.50×10^{-7}

traditional linear tolerancing = 0.02
traditional statistical tolerancing = 0.0096

Ettinger-Bartky-Gilson (EBG) / Monsoor-Greenwood-Chase (MGC) (9) (with $Z = 3$) = 0.0085
squared-error loss (12) (with $W = 3$) = 0.272
Gilson (6) = 0.0153

"improved" by "better" choices for m_i, y_i, Z and W, balancing the costs of producing to given tolerances with the losses incurred by having assemblies deviate from target. Future Computer-Integrated Manufacturing systems could include a data base on costs and process capabilities to support such analyses.

In the meantime, Tipnis (1992a, p. 11) has suggested that little use is made of any variant of statistical tolerancing because the required data are rarely available at the design stage. This problem might be solved by giving people a vision of how easily obtained data might be simply converted into a formula and a procedure that engineers will use because it is simple and they have confidence that its use will reduce costs. The discussion with (13) outlines how this might be done.

MINIMAX AND GAMES AGAINST ADVERSARIES

Case 1 and the linear tolerancing portions of (9) and (12) all use the minimax principle. They assume that suppliers will tend to shift their process averages in ways that are most disadvantageous to the organization. In the language of game theory, this is a game against an adversary. Even with all the adversarial relationships between different employees of the same organization and between customer and supplier, it may be overly conservative to assume that m in (7), (9), or (12) does not decline as n increases. On the other hand, in the absence of evidence to the contrary, it may be overly optimistic to assume that f in (6) does not increase with n. These cases are two ends of a continuum; a simple compromise can be developed based on process capability studies as discussed in the next section.

DETERMINING A SIMPLE TOLERANCE ANALYSIS FORMULA

Tipnis (1992a, p. 11) noted that statistical tolerancing is rarely used "because reliable data on distributions have rarely been available during the design stages." The increasing availability of process capability studies makes it possible to use regression analysis to develop design rules tailoring any of the above formulae to a specific organization. This section outlines how this might be done, generalizing a study done at Delco-Remy (Bender, 1962).

A first step would be to collect relevant process capability studies and attempt to derive from them simple rules for C_c and C_{pk} [where $m_i = C_c$ and $\gamma_i = 1/(6C_{pk})$, discussed with (7) and (9)]. For example, C_{pk} may be proportional to $\sqrt{T_i}$ and C_c proportional to $\sqrt{C_{pk}}$, as assumed for Table 2. They may also be functions of other indicators of the complexity of the part.

A second step would be to evaluate the extent to which biases in component dimensions cancel as n increases, interpolating between (6a) with f fixed and (7) with m fixed, as discussed above following (7a). To keep things simple, we could consider letting f in (6a) take the form

$$f = 10^{b_0} n^{b_1}, \qquad (13)$$

so (6a) is equivalent to

$$\log\left(\frac{T_a}{\sqrt{\sum a_i^2 T_i^2}}\right) = b_0 + b_1 \log(n). \qquad (13a)$$

The coefficients b_0 and b_1 can be estimated from (13a) using linear regression, replacing T_a and the T_i's by estimates of σ_a and the σ_i's obtained from process capability studies for tolerance chains in assemblies with the number of critical dimensions ranging from two to many. If tolerances are specified at three standard deviations for components but four standard deviations for the assembly and f in (13) is estimated from standard deviations, it must be multiplied by 4/3 to apply it to tolerances.

A scatterplot of the left-hand side of (13) vs. $\log(n)$ will have roughly half the points lying above the regression line and half below. If there is appreciable deviation from the regression line, it suggests that (13) might not be appropriate and a more elaborate model could be used. There is still power in simplicity, and (13) could still be used by using an upper confidence bound for f as discussed in Draper and Smith (1981, p. 31); this is similar to increasing b_0 in (13) so that nearly all assemblies lie below (6) with (13).

Other factors could be included in a regression model such as (13a) to develop tolerancing rules that utilize any information about the assembly that might be readily available to product designers. This might include an index of manufacturing complexity. Liggett's (1993, p. 29-30) "effective n", mentioned following (6b) above, could be used in place of or in addition to the actual length of the tolerance chain in (13).

If the m_i's and γ_i's in (9) and (12) do not vary with i and the variations in the T_i's are small, expressions (6), (9) and (12) are essentially equivalent. If the first step of this process capability analysis indicated that

C_{pk} and C_c were not constant, or if (13a) does not fit the available data very well, one might attempt to fit other models based on (9) or (12).

Tolerancing rules derived in this way might vary considerably from one organization to another or over time within the same organization. An organization with very poor process control might have appreciably larger f values than an organization with good process control. For a given organization, changes or improvements in technology might induce shifts over time in the parameter estimates for b_0 and b_1 in (13a).

OTHER CONSIDERATIONS

The main point of the comparison of the three cases in the previous section of this article is that different assumptions regarding the loss function and the shape of the production distribution have different implications for tolerancing, and statistical decision theory provides a set of tools for understanding those differences.

For the specific question of tolerance analysis, case 3 is generally preferred over case 2 because it is based on a loss function that is more descriptive of the situation generally encountered in production. Case 3 also has a secondary advantage over case 2 in that it does not depend on features of the distribution of X_a beyond the first two moments. Information on skewness and kurtosis, for example, could be used to modify the EBG / MGC procedure (9) (Evans 1975; Tukey 1957); skewness and kurtosis have no impact on (11) and (12).

This analysis did not explicitly consider the costs of machining to finer tolerances, an issue discussed by many authors (e.g., Zhang, Wang and Li, 1992; He, 1991; Chase, et al., 1990; Wu, ElMaraghy and ElMaraghy, 1988; Juran, 1962, p. 3-20, 21; Bjørke, 1989, p. 88-90; Peat, 1968). Squared-error loss was promoted by Taguchi (1986) as a Taylor series approximation to the actual loss function. If information to support an alternative loss function were available, it could be used (see e.g., Bisgaard, Hunter and Pallesen, 1984; Schmidt and Pfeifer, 1991; Melloy, 1991; Easterling, et al., 1991) and might result in economically important adjustments to tolerances in certain cases.

None of the formulae of this article consider the possibility of correlations between different dimensions (Tukey 1957) that can arise if multiple dimensions for a given part contribute to a given assembly dimension; t may not be reasonable to assume, for example, that the deviation from the nominal of the width of a given tooth on a gear is independent of the deviations of other teeth on that gear. Similarly, if, say, p copies of the same part go into each assembly and all p were produced one after the other, their dimensions could be correlated. If the correlations are appreciable, it might be desirable to modify the formulae discussed above accordingly.

Also, Bisgaard and Graves (1994) found that measurement variability could have a considerable impact on the optimal specification limits for tolerancing a single dimension. Measurement variability was assumed to be negligible in this article. In many manufacturing situations on the cutting edge of technology (e.g. manufacturing small, hand-held transponders for ultrasonic imaging), measurement variability contributes to manufacturing difficulties and might profitably be considered in establishing manufacturing tolerances.

With these limitations, we believe we have shown (in conjunction with Bisgaard and Graves, 1994) that statistical decision theory can respond to many of the concerns raised by Tipnis (1990, 1992a; Srinivasan and Voelcker, 1993; Voelcker, 1993) in calling for improved mathematical foundations for tolerancing.

CONCLUSIONS

We have reviewed existing tolerance analysis formulae considering the costs of deviating from the nominal and the distributions of dimensions of components. If production does not include 100% inspection of parts, then the formula most compatible with the cost structure in typical manufacturing is (12). However, in many applications, more comprehensive procedures are less likely to be used; therefore, a procedure was described with (13) for developing a simple tolerance analysis formula based on process capability studies in a given organization.

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