On the Finite Difference Solution for the Recovery of Lame Parameters of a Linear Elastic Membrane

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Abstract

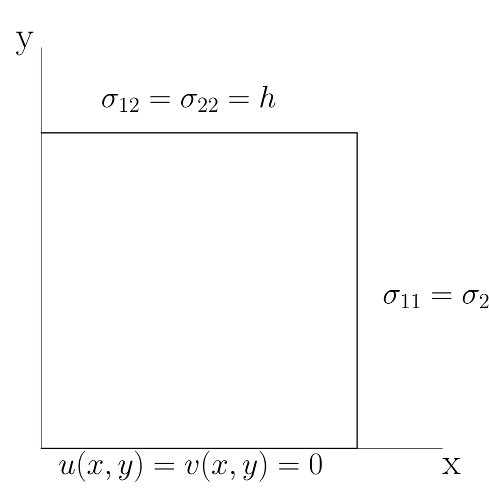
In this project we present the mathematical equation that governs the displacement of an isotropic membrane due to a traction force applied to part of its boundary. This leads to a linear elliptic boundary value problem with two parameters representing the Lamé moduli, which measure the elastic properties of the membrane. The Lamé parameters are constants for a homogeneous material but functions of the position otherwise. Since it is possible to measure interior displacements in human tissue (for example using ultrasound), and since cancerous tumors differ markedly in their elastic properties from healthy tissue, it may be possible to detect and locate tumors by solving the inverse problem for the Lamé parameters.

The objective of this project is to use the Matlab software to develop a numerical code for solving the forward problem for the displacements when the elasticities of the membrane are known. We utilize a finite difference method to solve the system of equilibrium equations.

Future research will utilize these results to estimate the (non-constant) Lamé moduli for a given traction force and a given measurement of the membrane dsplacement. The numerical results obtained in are compared to those previously obtained and, if possible, with experimental data. This is known as the inverse problem.

Boundary Conditions

Thus we have three partial differential equations and three unknown functions. In addition to the equations we also have the boundary conditions. We work on the rectangular region $\Omega = (0,1)\text{x}(0,1)$ and apply a traction force along the top of the region. We also assume that no pockets form on the bottom of the object and no slippage occurs so that $u(x,y) = v(x,y) = 0$ along the bottom. We are left then to define the boundary conditions on the sides of the object. On each of the sides we know that the stress (σ) is zero. The following diagram shows the setup with boundary conditions.



Forward Problem - Basic Equations

For a perfectly elastic solid the relation between stress and strain is described by Hooke's law:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

where σ is the stress tensor, ϵ the strain tensor, and C a tensor of elastic constants independent of stress or strain. For an isotropic medium this simplifies to:

$$\sigma = 2\mu\epsilon(u) + \lambda tr(\epsilon(u))I$$

where the function u is the displacement of the elastic membrane. The parameters μ and λ are the Lamé parameters. These parameters model the physical characteristics of the membrane material. For the forward problem the goal is to solve the equations of equilibrium for the displacements u given certain Lamé parameters. The Lamé parameters are related to Young's Modulus (E), the shear modulus (G) and Poisson's ratio (ν) by

$$\lambda = \frac{2G\nu}{1-2\nu} = \frac{E\nu}{(1+\nu)(1-2\nu)}; G = \frac{E}{2(1+\nu)}$$

In our case, since the medium is assumed to be incompressible $\nu = \frac{1}{2}$ and λ converges to a pressure term.

Finite Difference Solution

In the case of the forward problem the elasticity of the object is known (and with it the Lamé parameters) and the equations must be solved for the unknown displacements. The equations are discretized using finite difference approximations for the derivatives over the region using an MxM grid. The three equations for each grid point are stacked together to produce a composite system. This system can be written in matrix form as $Ax = b$ where A is an $3M^2$x$3M^2$ matrix of coefficients and x is a composite vector of the unknowns at each grid point. At the point (i,j) the discretization of (2) results in

$$\left(1+\frac{\varepsilon_1 h}{2}\right)v_{i+1,j} + (1+\varepsilon_2 h)\,v_{i,j+1} + \left(1-\frac{\varepsilon_1 h}{2}\right)v_{i-1,j} +$$
$$(1-\varepsilon_2 h)\,v_{i,j-1} - 4v_{i,j} + \frac{\varepsilon_2 h}{2}(u_{i,j+1} - u_{i,j-1}) + \lambda h(p_{i,j+1} - p_{i,j-1}) = 0$$

The discretization of (1) is similar. These equations are first calculated over all the unknown v grid points then over all the u grid points and finally the discretization of (3) is calculated over all the p grid points. The resulting equation $A\mathbf{x}= b$ where $\mathbf{x}= [\mathbf{v}\ \mathbf{u}\ \mathbf{p}]^T$ where $\mathbf{v}$ is the vector of all unknown v points in order of increasing index and likewise for $\mathbf{u}$ and $\mathbf{p}$. We use Matlab to solve this system using an iterative method.

Equilibrium Equations

Let $E(x,y) = E_0\varepsilon(x,y)$ represent Young's modulus of elasticity and $u(x,y), v(x,y)$ represent the horizontal and vertical components of displacement respectively.

The results are the 2-D equations of equilibrium:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\varepsilon_1\frac{\partial u}{\partial x} + \varepsilon_2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \lambda\frac{\partial p}{\partial x} = 0 \quad (1)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + 2\varepsilon_2\frac{\partial v}{\partial y} + \varepsilon_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \lambda\frac{\partial p}{\partial y} = 0 \quad (2)$$

where $\varepsilon_1 = \lambda\frac{\partial\varepsilon}{\partial x}, \varepsilon_2 = \lambda\frac{\partial\varepsilon}{\partial y}$ and $\lambda = \frac{1}{\varepsilon}$. Since we are assuming the object to be incompressible, an unknown pressure term $p(x,y)$ must be added to the equations.

The incompressibility condition also results in

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

The Inverse Problem

An area of future research for the summer of 2007 is to utilize the results of this project to find possible solution methods for the inverse problem. The inverse problem in this case is to find the Lamé parameters (the elastic properties of the membrane) for given displacements.

In the forward problem utilizing the finite difference method we obtained a system of equations in the form A$\mathbf{x} = \mathbf{b}$ where the coefficient matrix A depends on the elasticity of the membrane (i.e. A$(\lambda,\mu)\mathbf{x} = \mathbf{b}$).

The values of the Lamé parameters will be estimated, the forward problem solved using the estimated parameters, then the solution will be compared to the given displacements and the resulting difference fed back to refine the estimate of the Lamé parameters.This will be repeated until the difference between the calulated displacements and the given displacements is less than some tolerance value. Additionally, we will add some noise to the solutions to check the stability of the inverse problem solutions.

The project will be to develop and implement an efficient scheme using Matlab software for performing this error correcting iterative method. This method has a number of applications include medical imaging where the displacements of tissue can be measured using ultrasound and the elasticities of normal and diseased tissue differ markedly.

References

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