# Minimal Complexity C-complexes for Colored Links

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## Introduction

A link is a collection of strings tangled together with their ends fused together. Given any link there is a **two-dimensional surface** it bounds. The **genus** of this surface gives a measure of the complexity of the link. In this project, we study the analogous measure of complexity given by a generalization of a surface called a **C-complex**. In order to show that this measure captures some information, we present **an infinite family of links** for which this new measure of complexity is arbitrarily high.

Pictured below are examples of a bounded surface and a C-complex.

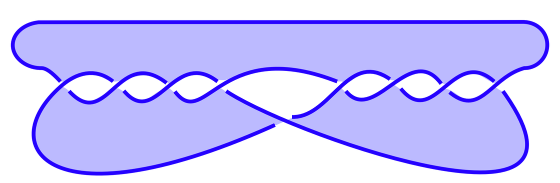

**Figure 1:** Surface

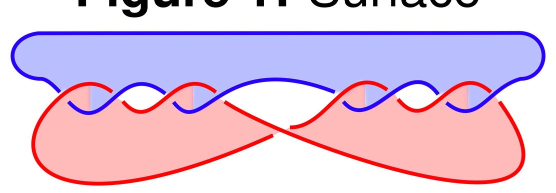

**Figure 2:** C-complex

## Questions and Summary

For a link L the **complexity** of L, $\mathcal{C}(L)$, is defined as the minimum of the complexity of all C-complexes bounded by L. The complexity of a C-complex is defined to be the number of curves needed to "fill up" the complex. For Figure 3, the complexity is at most 1 because it takes one curve to "fill up" the complex. Given a C-complex and the curves on it, one can write a matrix by counting linking number of push offs of curves.

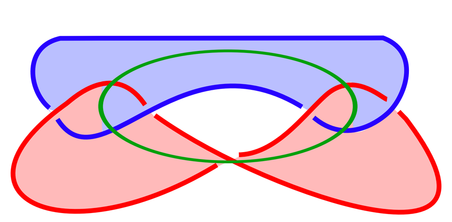

**Figure 3:** A Link and C-complex of complexity 1

$$A^{++} = [0] \qquad A^{--} = [0]$$
$$A^{+-} = [-1] \qquad A^{-+} = [-1]$$

There is a function valued invariant of $\sigma_L(\omega_1, \omega_2)$ such that $|\sigma_L(\omega_1,\omega_2)| \le \mathcal{C}(L)$.

The signature is the summation of the signs of eigenvalues of the link.

**Theorem 1** *The absolute value of the signature must be less than or equal to the complexity.*

## Tools: The linking matrix and signature

Given a C-complex, X, of complexity $k$. There are $k$ curves which fill up X, $l_1, l_2, ..., l_k$. There is a $k \times k$ matrix $A^{\epsilon_1\epsilon_2}$ whose $i^{th}j^{th}$ entry is $\text{lnk}(l_i, l_j^{\epsilon})$, where $l_j^{\epsilon}$ is the $\epsilon$-pushoff. Linking counts crossings between $l_i$ and $l_j^{\epsilon}$.

For example for Figure 3, the Cimasoni-Florens linking matrix is given by:

$$H(\omega_1,\omega_2) = \sum (1-\omega_1^{\epsilon_1})(1-\omega_2^{\epsilon_2})A^{\epsilon_1\epsilon_2}$$
$$= (1-\overline{\omega_1})(1-\overline{\omega_2})(\omega_1+\omega_2)$$

The signature of $L$ is defined by the summation of the signs of the eigenvalues of the link. The link $L$ is Hermitian, therefore it has real eigenvalues.

For the link of Figure 3 when $\theta_1$ and $\theta_2$ are very close to $-1$, the signature is

$$\sigma_L(\omega_1,\omega_2) = 1$$

Since the signature is 1, the complexity of Figure 3 is exactly 1 because $1 \le \mathcal{C}(L) \le 1$.

Facts:

1. Any complex for L produces the same $\sigma$.
2. Since a $k \times k$ matrix has $\le k$ eigenvalues, thus $|\sigma| \le k$. Thus the absolute value of the signature must be less than or equal to the complexity. *In fact this is the proof of Theorem 1.*

## Examples with high complexity

For Figure 5, $\mathcal{C}(L) \le 2k+1$ because there is a C-complex of complexity exactly $2k+1$.

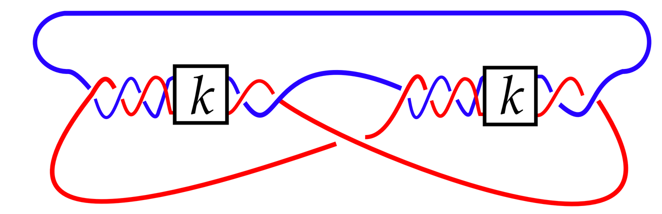



**Figure 4:** A Link L

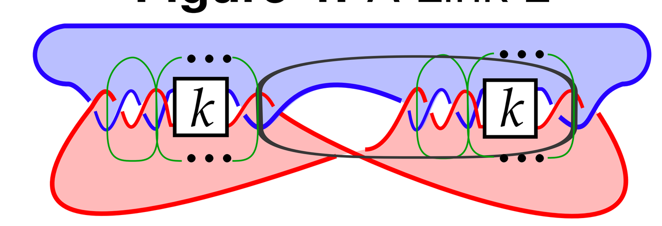



**Figure 5:** A C-complex for L

By showing $|\sigma_L(\omega_1,\omega_2)| = 2k-1$ for some $(\omega_1,\omega_2)$ we see $2k-1 \le \mathcal{C}(L) \le 2k+1$. In particular we have examples of arbitrarily high complexity.

The linking matrix of $L$, call it $H$, is given by:

$$(1-\overline{\omega_1})(1-\overline{\omega_2})\left|\begin{array}{cc|c|c} G_1 & & 0 & \begin{matrix}0\\0\\\omega_1\end{matrix} \\ \hline 0 & & G_2 & \begin{matrix}0\\0\\-1\end{matrix} \\ \hline 0\dots\ \omega_2 & & 0\dots\ -\omega_1\omega_2 & 0 \end{array}\right|$$

Where

$$G_1 = \begin{vmatrix} \omega_1+\omega_2 & -\omega_1 & 0 & 0 \\ -\omega_2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -\omega_1 \\ 0 & 0 & -\omega_2 & (\omega_1+\omega_2) \end{vmatrix}$$

and

$$G_2 = \begin{vmatrix} -1-\omega_1\omega_2 & \omega_1\omega_2 & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \omega_1\omega_2 \\ 0 & 0 & 1 & -1-\omega_1\omega_2 \end{vmatrix}$$

are tridiagonal $k-1 \times k-1$ matrices.

**Proposition 2** *The tridiagonal matrix*

$$\begin{vmatrix} a & b & 0 & 0 \\ \overline{b} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & b \\ 0 & 0 & \overline{b} & a \end{vmatrix}$$

*diagnolizes to*

$$\begin{vmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & p_{k-1} \end{vmatrix}$$

*where $p_1 = a$ and $p_{j+1} = a + \frac{b\overline{b}}{p_j}$.*

By solving this recurrence relation, we that $G_1$ has eigenvalues (up to sign) $p_j = \frac{-\sin\left((k+1)(\pi - \frac{\phi_1+\phi_2}{2})\right)}{\sin\left(k(\pi-\frac{\phi_1+\phi_2}{2})\right)}$ where $\omega_1 = e^{i\theta_1}$ and $\omega_2 = e^{i\theta_2}$, and $G_2$ has eigenvalues (again, up to sign) $q_j = \frac{-\sin\left((k+1)(\pi - \frac{\phi_1-\phi_2}{2})\right)}{\sin\left(k(\pi-\frac{\phi_1-\phi_2}{2})\right)}$.

For $\theta_1+\theta_2$ and $\theta_1-\theta_2$ positive and very close to 0, $p_1,\dots,p_k$ and $q_1,\dots,q_k$ are all negative. Unfortunately the last eigenvalue is positive.

Thus, for this choice of $\theta_1$ and $\theta_2$, $\sigma_K(\omega_1,\omega_2) = -2k+1$

## Future Research

- If we add n-additional positive twists, we hope to able to compute the complexity more precisely.

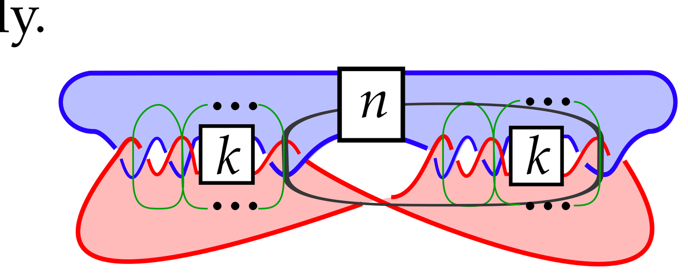



**Figure 6:** C-complex for L with additional positive twists

- We are interested in studying how C-complex complexity compares with genus. For example, is it possible to find a C-complex of low complexity and high genus (vice versa)?
- Since the signature function and genus are related, we're curious as to whether the discontinuity of the signature functions affects the genus.
- In the future, we plan to attempt to generate links which cannot be distinguished by their linking forms.

## References

[1] David Cimasoni and Vincent Florens. Generalized Seifert surfaces and signatures of colored links. *Trans. Amer. Math. Soc.*, 360(3):1223–1264 (electronic), 2008.

[2] Colin C. Adams. *The Knot Book*. American Mathematical Society, 2nd edition, 2004.

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