Knot and Link Tricolorability

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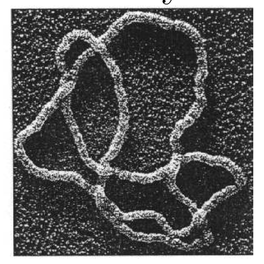
Problem & Importance

Knot Theory, a field of Topology, can be used to model and understand how enzymes (called topoisomerases) work in DNA processes to untangle or repair strands of DNA. In a human cell nucleus, the DNA is linear, so the knots can slip off the end, and it is difficult to recognize what the enzymes do.

However, the DNA in mitochondria is circular, along with prokaryotic cells (bacteria), so the enzyme processes are more noticeable in knots in this type of DNA.



A strand of DNA as the 4_1 knot. Image from paper by De Witt Sumners.

Invariants prove to be a useful tool in studying when two knots are different. Tricolorability is an easily understood invariant we will use to distinguish doubles (replications) of certain prime knots.

Our team studied knots and links which have been observed in DNA. Specifically, we considered what happens to the colorability after performing a doubling operation.

Definitions

Link- A collection of curves in $\mathbb{R}^3$ that are nonintersecting and closed. A knot is a link of one component.

Tricolorable- For a link to be tricolorable, every intersection is colored either in this way ⊕ or this way ⊕, where at least 2 colors are used when coloring the knot.

Full positive twist- A change within the link of a Whitehead Double. We use n to denote the number of full positive twists. Images of $n = 1$ through $n = 6$ are pictured in the Conjectures section.

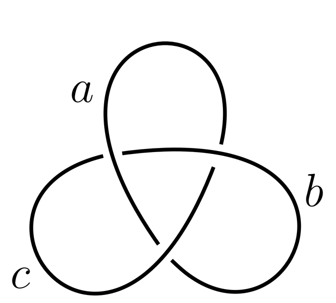
Doubling operator- An operation on a knot which doubles the knot segments. The doubling operators "Whitehead Double" and "Pure Double" are pictured above the corresponding tables to the right.

Skein Relations- A method used to find the Alexander Polynomial of a knot using the equation

$$\triangle_{L_+}(t) - \triangle_{L_-}(t) - (t^{1/2} - t^{-1/2})\,\triangle_{L_0}(t) = 0.$$

A Coloring Matrix of a Knot or Link

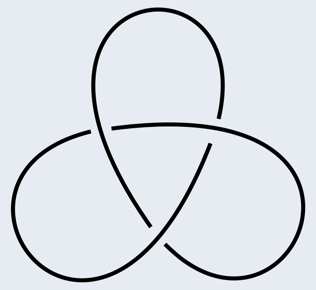
Each intersection of a knot can be represented by an equation. The segments are given names (also known as "colors"), in this case $a, b,$ and c. The "over" segments have coefficient 2, and the "under" segments have coefficient -1. Each equation determines a row in the matrix. Below is a picture of this process, using the knot 3_1.



$$\begin{aligned} 2a - b - c &= 0 \\ 2b - a - c &= 0 \\ 2c - a - b &= 0 \end{aligned} \quad \text{becomes} \quad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

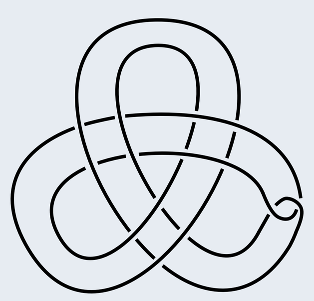
Tables of Characteristics

Original Knot



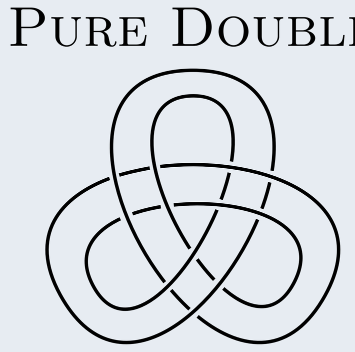
Link L	Colorability	Det(L)	Unknot/Link Number
3_1	3	3	1
4_1	5	5	1
5_1	5	5	2
5_2	7	7	1
6_1	3	9	1
7_1	7	7	3
7_2	11	11	1
2_1^2	2	2	1
5_1^2	2	8	2

Whitehead Double



Link, L	Colorability	Det(L)
WH 3_1	11	11
WH 4_1	1	1
WH 5_1	7, 3	21
WH 5_2	19	19
WH 6_1	7	7
WH 7_1	29	29
WH 7_2	3	27
WH 2_1^2	Any n	0
WH 5_1^2	Any n	0

Pure Double



Link L	Colorability	Det(L)	Unknot/Link Number
PD 3_1	3, 2	6	≤ 4
PD 4_1	0	0	≤ 4
PD 5_1	5, 2	10	≤ 8
PD 5_2	5, 2	10	≤ 4
PD 6_1	3	3	≤ 4
PD 7_1	7, 2	14	≤ 12
PD 7_2	7, 2	14	≤ 4
PD 2_1^2	Any n	0	≤ 4
PD 5_1^2	Any n	0	≤ 8

Colorability

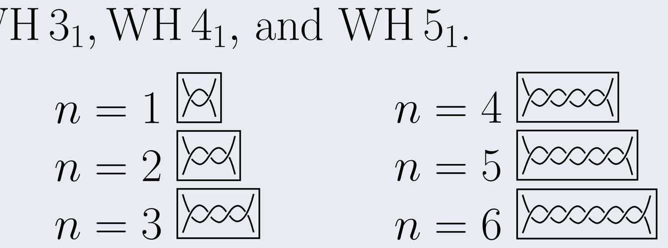
The unknot is not tricolorable, therefore anything that is tricolorable cannot be the unknot. The prime factors of the determinant of the knot or link provides the colorability. For example, if a knot's determinant is 21, it is 3-colorable (tricolorable), and 7-colorable. This is known by the theorem that the determinant of a knot is $0 \mod n$ if and only if the knot is n-colorable.

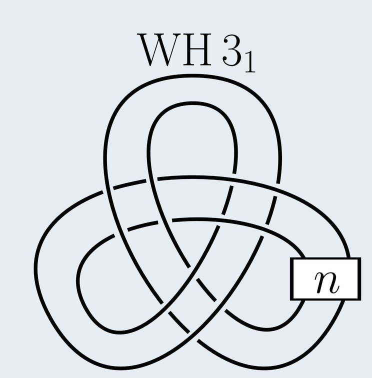
Theorem: If $\det(L) = 0$, then L is n-colorable for all n.

Proof: Suppose $\det(L) = 0$ and A is the minor of the coloring matrix for L. Then $A\vec{x} \equiv \vec{0}$ has a non-trivial solution $0 \equiv 0 \mod n$ for all n. Recall the equivalent condition that because the $\det(L) = 0$ then A is linearly dependent. Since A is linearly dependent it has only the nontrivial solution.

Results

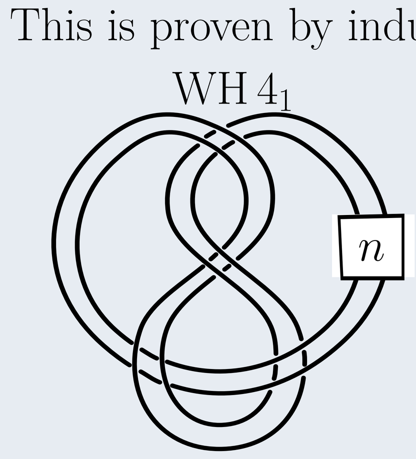
As the number of full positive twists in the link (n) changes, the determinant of the knot changes. We have discovered a patterned format in the determinants for knots WH 3_1, WH 4_1, and WH 5_1.





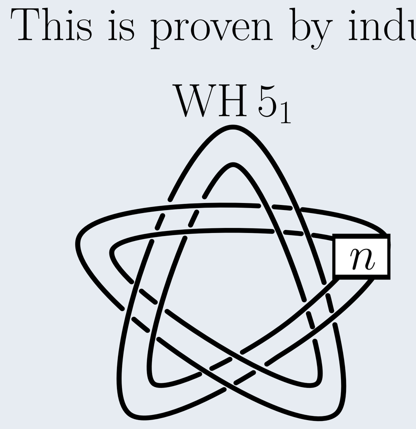
for $n = 1$, the determinant is 11
for $n = 2$, the determinant is 17
for $n = 3$, the determinant is 23
for $n = 4$, the determinant is 29
for $n = 5$, the determinant is 35
for $n = 6$, the determinant is 41

Theorem: The det. of n-twisted WH 3_1 is $\boxed{5 + 6n}$.
This is proven by induction using Skein Relations.



Regardless of the n, the determinant is 1.

Theorem: The det. of n-twisted WH 4_1 is always $\boxed{1}$.
This is proven by induction using Skein Relations.



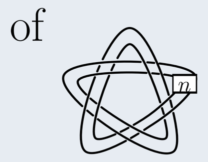
for $n = 1$, the determinant is 21
for $n = 2$, the determinant is 31
for $n = 3$, the determinant is 41
for $n = 4$, the determinant is 51
for $n = 5$, the determinant is 61
for $n = 6$, the determinant is 71

Conjecture: The det. of n-twisted WH 5_1 is $\boxed{11 + 10n}$.

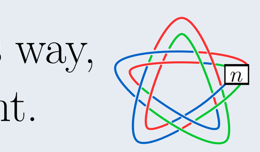
Theorem: For WH 5_1 with n twists, WH 5_1 is tricolorable when $n = 3k + 1$ where $k \in \mathbb{N} \cup \{0\}$.

Proof

Consider WH 5_1 where n is the number of full positive twists.
($n = 1 \ldots n = k$)



WOLOG, let WH 5_1 be colored in this way, excluding coloring the twist component.



If the entrances to the twists are colored in this way: then for every $n = 3k + 1$ where $k = \mathbb{N} \cup \{0\}$, the twist component, and therefore the knot, is tricolorable.

Base Case

Let $k = 0$, $n = 3(0) + 1 = 1$. Then $n = 1$: is tricolorable, and hence the knot is tricolorable.

Inductive Case

For $n = 3(k + 1) + 1$:
$n = 3(k + 1) + 1 = 3k + 3 + 1$.
Adding 1 to k adds 3 full positive twists to n, which looks like .

Placing multiples of this within the twist component does not disrupt the entrance coloring, as the pattern is simply extended. So, adding 3 full twists does not change the twist component's colorability and does not change the colorability of the knot. Therefore, all knots colored in this way for n twists when $n = 3k + 1$ where $k \in \mathbb{N} \cup \{0\}$ are tricolorable.

Further Directions

- Investigate the tricolorable unlinking number which is a value for each knot similar to the unknotting or unlinking number. Instead of how many intersections must be changed to unknot or unlink it, this would be how many crossings we must change for the knot or link to become tricolorable.
- After seeing an initial pattern in the Hopf and Whitehead links and the Borromean rings we want to prove that once links have gone through a doubling operator they will have determinants of 0.
- We are currently working on proving our equation for the determinant the 5_1 knot using Skein Relations.

References

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