THE MODULI SPACE OF NON-NILPOTENT COMPLEX 4|1-DIMENSIONAL ASSOCIATIVE ALGEBRAS

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1. $\mathbb{Z}_2$-GRADED ALGEBRAS AS CODIFFERENTIALS

A $\mathbb{Z}_2$-graded algebra structure on a $\mathbb{Z}_2$-graded space V determines an odd coderivation d of the tensor coalgebra $T^c(W)$, where $W = \Pi V$ is the parity reversion of V. The algebra structure is associative precisely when d is a codifferential on $T^c(W)$, that is,

$$[d, d] = 0,$$

where the bracket is the $\mathbb{Z}_2$-graded Lie bracket of coderivations. This complicated way of looking at associative algebras actually leads to a simpler way to study the algebras, because we can use the theory of graded Lie algebras to compute the cohomology of the algebra, and to construct algebras by extensions.

An n-cochain on the vector space V is an element α of $\mathrm{Hom}(V^{\otimes n}, V)$. It determines an element $\varphi = \pi \circ \alpha \circ \pi^{\otimes n} \in \mathrm{Hom}(W^{\otimes n}, W)$ whose parity is $|\varphi| = |\alpha| + n - 1$. Here $\pi : V \to W$ is the identity map, which is odd, since the parity of W is reversed. This element φ extends to a coderivation of $T^c(W)$, and the *coboundary operator* D is given by the formula

$$D(\varphi) = [d, \varphi].$$

. The fact that $D^2 = 0$ follows from

$$D^2(\varphi) = [d, [d, \varphi]] = [[d, d], \varphi] - [d, [d, \varphi]] = -[d, [d, \varphi]] = -D^2(\varphi).$$

Since $\mathbb{C}$ has characteristic zero, this shows that $D^2 = 0$.

2.WEDDERBURN'S THEOREM AND THE FUNDAMENTAL THEOREM

A $\mathbb{Z}_2$-graded algebra is called *simple* if it has no $\mathbb{Z}_2$-graded proper nontrivial ideals. It is called *nilpotent* if there is some n such that the product of n elements is always zero. A direct sum of simple algebras is called *semisimple*. A $\mathbb{Z}_2$-graded division algebra is a unital algebra such that every nonzero *homogeneous* element is invertible.

Theorem 1 (Wedderburn's Theorem) *Every finite dimensional simple algebra is of the form*

$$\mathfrak{gl}(V) \otimes D,$$

where $\mathfrak{gl}(V)$ is the endomorphism ring of a $\mathbb{Z}_2$-graded space V and D is a division algebra.

The definition of a $\mathbb{Z}_2$-graded division algebra above is necessary to extend the classical form of Wedderburn's Theorem to graded algebras.

Theorem 2 (Fundamental Theorem of FD Complex Associative Algebras) *If A is a finite dimensional complex algebra which is neither nilpotent or semisimple, then A is a semidirect product of its maximal nilpotent ideal N and a semisimple subalgebra S; that is $A = N \rtimes S$.*

The reason we state the theorem for complex algebras is to avoid addressing the issue of separability which arises when the algebra is over a field with finite characteristic.

3. SEMISIMPLE ALGEBRAS OF DIMENSION $\leq 4|1$

The dimension of a $\mathbb{Z}_2$-graded space is given in the form $m|n$, where m is the dimension of the even part and n is the dimension of the odd part. The algebra $\mathbb{C}$ has dimension $1|0$, and is a division algebra over $\mathbb{C}$. There is one other $\mathbb{Z}_2$-graded division algebra, $\mathbb{C}_{1|1}$, called the double of $\mathbb{C}$, of dimension $1|1$. The algebra $\mathfrak{gl}(2, \mathbb{C})$ is a simple algebra of dimension $4|0$. All semisimple algebras of dimension less than or equal to $4|1$ are given by direct sums of these three algebras. The semisimple algebras which arise are $\mathbb{C}^2$, $\mathbb{C}^3$, $\mathbb{C}_{1|1} \oplus \mathbb{C}$, $\mathbb{C}^4$, $\mathbb{C}_{1|1} \oplus \mathbb{C}^2$ and $\mathbb{C}_{1|1} \oplus \mathbb{C}^3$, in addition to the 3 simple algebras given above. We construct $4|1$-dimensional algebras by extending these semisimple algebras by nipotent algebras of complimentary dimension. The table below gives the number of nonequivalent algebras arising from each case.

Algebra	Dimension of S	Dimension of N	Nilpotents	Algebras
$\mathbb{C}_{1\|1} \oplus \mathbb{C}^3$	$4\|1$			1
$\mathbb{C}_{1\|1} \oplus \mathbb{C}^2$	$3\|1$	$1\|0$	1	5
$\mathfrak{gl}(2, \mathbb{C})$	$4\|0$	$0\|1$	1	1
$\mathbb{C}^4$	$4\|0$	$0\|1$	1	5
$\mathbb{C}_{1\|1} \oplus \mathbb{C}$	$2\|1$	$2\|0$	2	12
$\mathbb{C}^3$	$3\|0$	$1\|1$	2	53
$\mathbb{C}^2$	$2\|0$	$2\|1$	4	235
$\mathbb{C}_{1\|1}$	$1\|1$	$3\|0$	4	4
$\mathbb{C}$	$1\|0$	$3\|1$	9	190

4. ALGEBRAS BY EXTENSIONS

If $W = N \oplus S$ is a semidirect product of an ideal N and a subalgebra S, then the multiplication restricted to N determines an algebra structure on N, which we represent by an odd codifferential μ, and similarly, the algebra structure on S is given by another odd codifferential δ. This means that $[\delta, \delta] = 0$ and $[\mu, \mu] = 0$. If d is an algebra structure on W which is a semidirect product, then the product of something in S with something in N must lie in N. This determines an odd coderivation λ, so that $d = \delta + \mu + \lambda$. The condition that $[d, d] = 0$ reduces to two conditions:

1. $[\mu, \lambda] = 0$, the *compatibility condition.*
2. $[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] = 0$, the *Maurer-Cartan condition.*

If we define D_μ by $D_\mu(\alpha) = [\mu, \alpha]$, then $D_\mu^2 = 0$, in other words, D_μ is a coboundary operator. The compatibility condition says that λ is a cocycle with respect to D_μ; that is, $D_\mu(\lambda) = 0$. If we define the cohomology H_μ determined by D_μ by

$$H_\mu = \ker(D_\mu)/Im(D_\mu),$$

then two equivalent cocycles λ and λ' will determine equivalent algebras (provided that the Maurer-Cartan equation is satisfied.) There is a group of automorphisms of W, which also acts on the cohomology classes, so that one has to consider equivalence classes of cohomology classes under the action of the automorphism group in order to classify the nonequivalent algebra structures on W arising from those on S and N. All of these steps are easy to implement in the computer. This was our method of constructing the 506 different $4|1$-dimensional algebras.

5. CONSTRUCTING THE VERSAL DEFORMATION.

If d is a codifferential determined by an associative algebra structure, then a 1-parameter deformation is a formal power series

$$d_t = d + t\psi_1 + t^2\psi_2 + \cdots ,$$

where $\psi_i : W \otimes W \to W$ are odd coderivations and $[d_t, d_t] = 0$.

All such 1-parameter deformations are obtained from a *miniversal deformation* d^∞ defined as follows. Let $\delta_1, \dots, \delta_m$ be a set of odd 2-cochains whose images in the second cohomology space $H^2(d)$ are a basis for the odd part of H^2. Let $\beta^1, \dots, \beta^n$ be a basis of the even part of $H^3(d)$, and γ^i be odd 2-cochains such that $D(\gamma^i) = \beta^i$. Let $\alpha^1, \dots, \alpha^r$ be a set of even 3-cochains whose images are a basis of the even part of $H^3(d)$. Finally, let $\tau^1, \dots \tau^s$ be a set of even 3-cochains whose image under D give a basis of the even part of H^4. Then there is a unique solution for $x_1, \dots x_n$ such that x_i is a power series in the variables $t_1, \dots, t_m$ of order at least 2 such that

$$d^\infty = d + t_1\delta^1 + \cdots t_m\delta^m + x_1\gamma^1 + \cdots + x_n\gamma^n$$

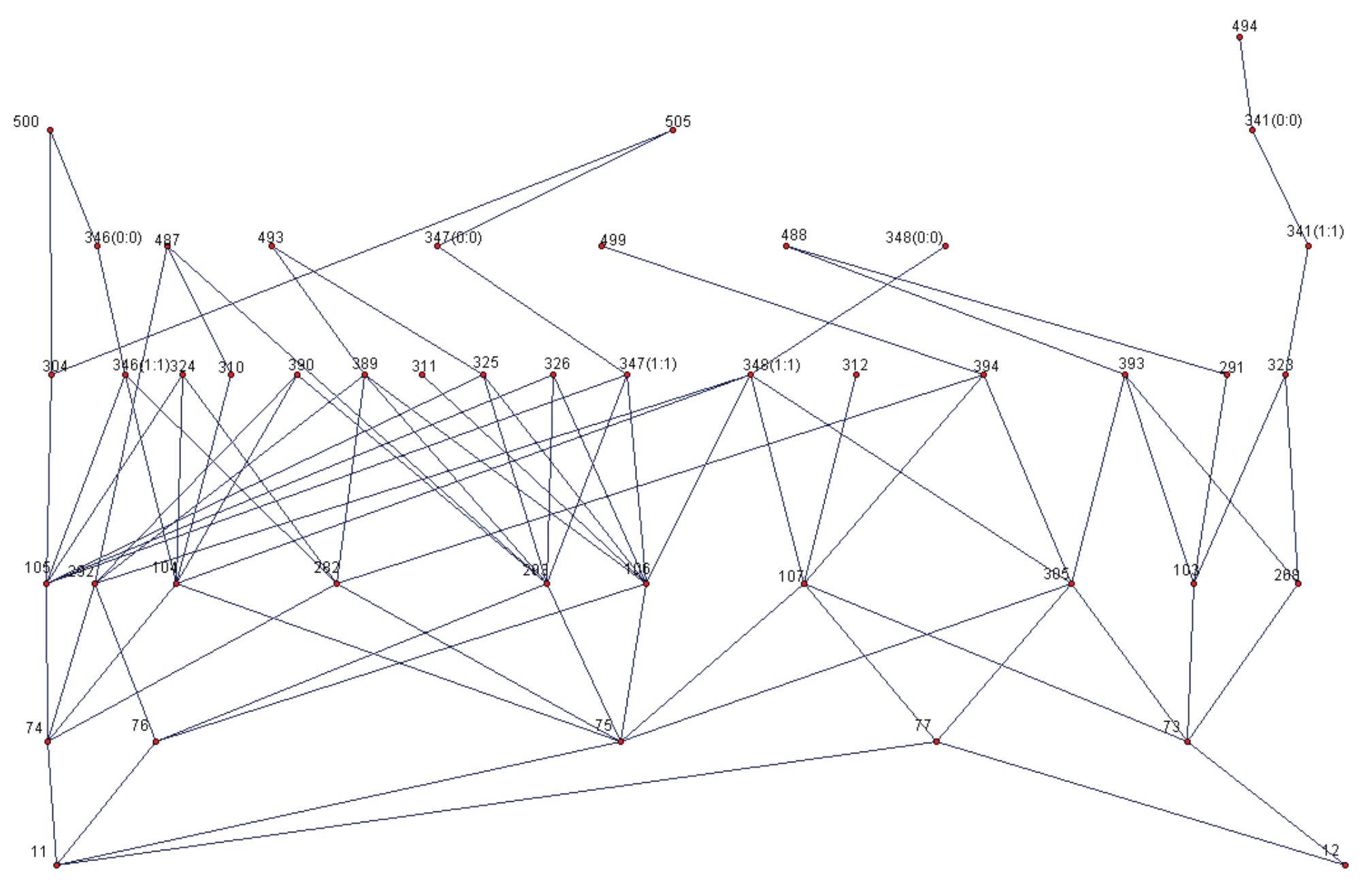
satisfies

$$[d^\infty, d^\infty] = r_1\alpha^1 + \cdots + r_r\alpha^r + u_1\tau^1 + \cdots u_s\tau^s,$$

for some power series r_i, u_i in the t variables. It turns out that the u_i lie in the ideal generated by the r_i, so if we solve $r_i = 0$ for the t_i we get actual deformations.

6. EXAMPLES OF DEFORMATIONS

To give a complete picture of all of the deformations of the 506 algebras would be very difficult. However, it is well known that if an algebra deforms to a graded commutative algebra, then it must be graded commutative as well. Since there are only 35 graded commutative algebras, it is feasible to illustrate their deformations to each other, which we do in the picture below.



References

[1] A. Boyd, M. Loper, A. Meinel, M. Penkava, and M. Sivaperumal. The moduli space of 4|1-dimensional complex associative algebras. In preparation, 2013.

[2] L. Brunshidle, A. Fialowski, J. Frinak, M. Penkava, and D. Wackwitz. Fundamental theorem of finite dimensional graded associative algebras. *Communications in Algebra*, 40(12), 2012.

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