Enumerative Problems of Doubly Stochastic Matrices and the Relation to Spectra

by

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ABSTRACT

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This work concerns the spectra of doubly stochastic matrices whose entries are rational numbers with a bounded denominator. When the bound is fixed, we consider the enumeration of these matrices and also the enumeration of the orbits under the action of the symmetric group.

In the case where the bound is two, we investigate the symmetric case. Such matrices are in fact doubly stochastic, and have a nice characterization when we are in the special case where the diagonal is zero. As a central tool to this investigation, we utilize Birkhoff's theorem that asserts that the doubly stochastic matrices are exactly the polytope defined by the convex hull of permutation matrices. In particular, we consider the spectra along segments in the Birkhoff polytope.

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1 INTRODUCTION

We will begin by defining some terminology.

Definition 1.1 ([Sza15, p. 363])**.** A probability vector is a numerical vector whose entries are real numbers between 0 and 1 whose sum is 1.

Definition 1.2 ([Sza15, p. 363])**.** A stochastic matrix is a square matrix whose columns are probability vectors. A stochastic matrix is also used to describe the transitions of a Markov chain, also called a Markov matrix.

Definition 1.3 ([Sza15, p. 363])**.** A doubly stochastic matrix is a square matrix of nonnegative real numbers with each row and column adding up to 1.

Definition 1.4. A hollow matrix is a square matrix whose diagonal elements are all equal to 0.

Definition 1.5 ([Sza15, p. 375])**.** A matrix *S* is symmetric if it equals its transpose. Real matrices of the form AA^T and A^TA are symmetric and have real eigenvalues. Note: Symmetric stochastic matrices are always doubly stochastic.

Definition 1.6 ([Bee04, p. 403])**.** Suppose *A* and *B* are two square n×n matrices. Then *A* and *B* are *similar* if there exists a nonsingular n×n matrix, *S*, such that

$$A = S^{-1}BS \tag{1.1}$$

Theorem 1.7 ([Bee04, p. 406])**.** *Suppose* A *and* B *are similar matrices. Then the characteristic polynomials of* A *and* B *are equal, that is,* $p_A(x)=p_B(x)$*. Therefore they have equal eigenvalues.*

Theorem 1.8 ([Dym07, p. 506])**.** *Let* P *in* $\mathbb{R}^{n\times n}$ *be a doubly stochastic matrix. Then P is a convex combination of finitely many permutation matrices.*

Our interest with doubly stochastic matrices comes from the desire to work with Markov chains. We focus on the algebra of doubly stochastic matrices to be able to have a broader discussion. We start by counting doubly stochastic matrices, symmetric stochastic matrices, and hollow symmetric stochastic matrices with the elements $\{0, \frac{1}{2}, 1\}$. We extend this problem by looking at doubly stochastic matrices' eigenvalues to investigate their relationships with similar, and symmetric matrices. The eigenvalues of stochastic matrices relate to Markov mixing times which is important for machine learning and probability topics.[LP17] The enumerations we find allow us to analyze the eigenvalues more thoroughly, especially with the 3×3 case because we know how many matrices we need to examine.

Definition 1.9. [Wei23] A permutation matrix is a matrix obtained by permuting the rows of an $n\times n$ identity matrix according to some permutation of the numbers 1 to n. Every row and column therefore contains precisely a single 1 with 0s everywhere else, and ever permutation corresponds to a unique permutation matrix. Therefore there are n! permutation matrices of size n.

We delve into the case of hollow symmetric stochastic matrices. Of particular interest is the situation where two matrices are similar because then they have the same spectrum. Specifically we consider hollow symmetric stochastic matrices up to conjugacy by permutation matrices. For a given hollow symmetric stochastic matrix we consider the orbit of all matrices obtained by permutation matrix conjugation. By restricting to the case where the entries of the matrices are $\{0, \frac{1}{2}, 1\}$ we are able to describe the orbits combinatorially. Given a matrix with the restricted entries, we clear the denominator by multiplying by 2. We then view the result as an adjacency matrix of a network (i.e. graph). Up to conjugation these matrices correspond to unlabeled networks. The orbits of hollow symmetric stochastic matrices have the network properties that the edges are undirected and have no loops. Futhermore, the degree of each vertex is 2 and therefore are disjoint unions of cyclic networks. The other enumerative problem that occurs is not up to conjugacy where the nodes are labeled. The not up to conjugacy case gives the number of hollow symmetric stochastic matrices for each dimension.

Definition 1.10. [Hed23] A subset *A* of a vector space *V* is said to be convex if λx + (1 - λ)y for all vectors *x, y* which are elements of *A*, and all scalars λ which is an element of [0,1]. Via induction, this can be seen to be equivalent to the requirement that $\lambda_1 \mathrm{x}_1$ + ... + $\lambda_n \mathrm{x}_n$ which is an element of *A* for all vectors x_1, x_2, ..., x_n also elements of *A*, and for all scalars λ_1, λ_2, ..., $\lambda_n \geq 0$ such that $\sum \lambda_i$ = 1. With the above restrictions on the λ_i, an expression of the form $\lambda_1 \mathrm{x}_1$ + ... + $\lambda_n \mathrm{x}_n$ is said to be a convex combination of the vectors x_1, x_2, ..., x_n. Note the set of all convex combinations of vectors in *A* constitute the convex hull of *A* so, for example, if we let *x, y* be elements of *V* and two different vectors in the vector space *V*, then the set of all convex combinations of *x* and *y* constitute the line segment between *x* and *y*.

Another area of interest is the convex combinations of doubly stochastic matrices. We are able to take convex combinations of doubly stochastic matrices because all the rows and columns sum to 1, so it therefore satisfies the definition of convex combinations. It is important to note that when examining convex combinations we are no longer restricting the denominator. The convex combinations are significant because we can plot all possible eigenvalues when pairing two doubly stochastic matrices. Figure 1 is the spectra of two 8×8 doubly stochastic matrices along a line segment through the Birkhoff polytope. The imagine is fascinating, so we want to study the convex combinations, but at lower dimensions to understand the curves fully. We will discuss the 3×3 case exhaustively and then look at some higher dimension graphs that peak our interest.

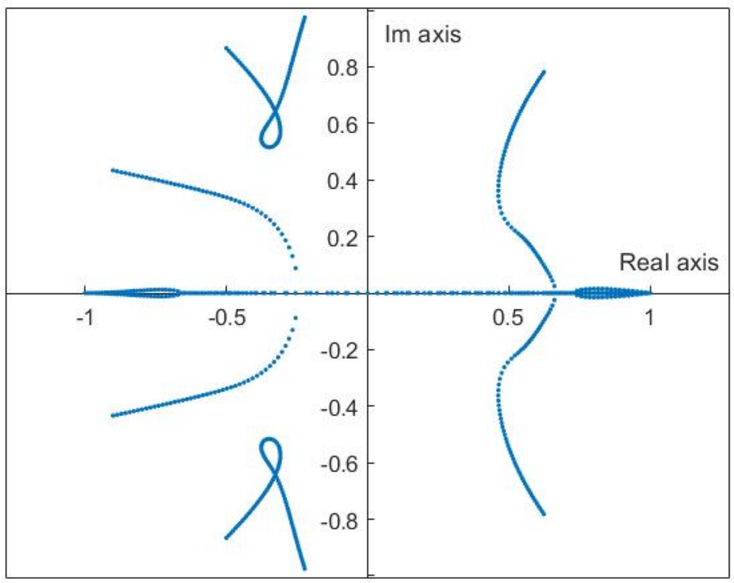

Figure 1: Convex combination of two 8×8 doubly stochastic matrices.

2 ENUMERATING DOUBLY STOCHASTIC MATRICES

Before explaining the process in which we enumerate doubly stochastic matrices with the elements $\{0,\frac{1}{2},1\}$, we will discuss papers that examine the topic. First, the sequence, 1, 1, 3, 21, 282, ..., is on OEIS, the *Online Encyclopedia of Integer Sequences*. With OEIS we are able to execute a thorough literature review. The sequence on OEIS describes "the number of n$\times$n matrices with nonnegative entries and every row and column sum 2."[OFI23] The reason the sequence states that the row and column sums to two and not one is due to the fact that if we multiple the rows and columns of a doubly stochastic matrix with elements $\{0,\frac{1}{2},1\}$ by two, we clear the fraction while changing nothing about the enumeration. The book *Advanced Combinatorics* is a useful reference when studying enumerations. It discusses, "the art of finite and infinite expansions".[Com74] The book *Combinatorial Enumeration* uses generating functions of different types to solve a plethora of enumeration problems. Some of the enumeration questions they answer are integer partitions, permutations by inversion, and coefficient extraction for symmetric functions.[GJ83]. Another book that is essential to our literature review is *A Handbook of Integer Sequences*. The majority of the book includes a table of 2,300 sequences, one of which is our desired sequence.[Slo73] A similar book is *The Encyclopedia of Integer Sequences*. This book contains 5,500 sequences of rational integers, also including our desired sequence.[SP95] An additional book is *Enumerative Combinatorics. Vol 2*, which focuses on exponential generating functions in order to count labeled objects such as permutations and labeled graphs. The book also discusses other generating functions to answer more counting problems.[Sta99] Lastly, the article, "Methods of Successive Restrictions in Computational Problems Involving Discrete Variables", discusses, "two methods that are outlined for dealing with combinatorial problems, especially the construction of orthogonal latin squares and finite projective planes."[Tom63]

In this chapter we also enumerate symmetric stochastic matrices. This sequence is also on

OEIS, `https:https://oeis.org/A000985`. Similar books like, [Slo73] and [SP95] are helpful for the literature review on this enumerative problem. The books include the sequence we find for symmetric stochastic matrices as well. The book [Sta99] includes an enumeration problem of the sequence.

The last two counting problems in this chapter come from hollow symmetric stochastic matrices. We enumerate the number of hollow symmetric stochastic matrices up to conjugacy and not up to conjugacy. OEIS continues to be an essential tool in our literature review. The up to conjugacy OEIS page is `https://oeis.org/A002865` and the not up to conjugacy OEIS page is `https://oeis.org/A002137`. The books *The Encyclopedia of Integer Sequences*, *A Handbook of Integer Sequences*, *Advanced Combinatorics* and *Enumerative Combinatorics. Vol 2* remain useful for this topic as well. The book *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, discusses many different topics, but the chapter on combinatorial analysis and numerical interpolation, differentiation and integration, are the most useful for our problem.

2.1 Enumeration of Doubly Stochastic Matrices

First we enumerate the doubly stochastic matrices with elements $\{0,\frac{1}{2},1\}$ by counting a few dimensions by hand. We count 1, 1×1 matrix, 3, 2×2 matrices, and 21, 3×3 matrices. Once at 3×3, we are able to tell doing the rest by hand is not plausible. We then utilize MATLAB to aid in counting. With the MATLAB code we verify the dimensions done by hand and then find the enumerations for higher dimensions. After dimension 8 the code is too expensive. We obtain 1 for our enumeration.

	Number of Doubly Stochastic Matrices
0	1
1×1	1
2×2	3
3×3	21
4×4	282
5×5	6,210
6×6	202,410
7×7	9,135,630
8×8	545,007,960

Table 1: Number of doubly stochastic matrices with elements $\{0,\frac{1}{2},1\}$.

The code we use in MATLAB to execute the enumeration:

```
clear

dim = 8; % dimension - max is 8 and is still very expensive

% generates all possible rows with ones

one = unique(perms([[1 1] zeros(1,dim-2)]), 'rows', 'stable');

% generates all possible rows with twos

two = unique(perms([2 zeros(1,dim-1)]), 'rows', 'stable');

A = [one;two]; % matrix of all possible rows

% number of possible rows multichoose dimension

S = nmultichoosek(1:length(A),dim);

B = zeros(dim,dim); % preallocate

num = 0; % number of matrices

check = 2*ones(1,dim); % check sums against the 2 vector

m = 0;

for i = 1:length(S)

    B = A(S(i,:),:);

    s = sum(B);

    if (isequal(s,check))

        if (isequal(s,sum(B,2)'))
```

```
            edges = unique(S(i,:));
            counts = histcounts(S(i,:),[edges edges(end)+1]);
            num = num + factorial(dim)/prod(factorial(counts));
        end
    end
end
num
% creates an array with rows n multichoose k
(combinations with repitition)
function combs = nmultichoosek(values, k)
n = numel(values);
combs = bsxfun(@minus, nchoosek(1:n+k-1,k), 0:k-1);
combs = reshape(values(combs),[],k);
end
```

2.2 Enumeration of Symmetric Stochastic Matrices

We start by hand for the enumeration of symmetric stochastic matrices with elements $\{0,\frac{1}{2},1\}$ as well, but once again at dimension four it is too many to do without programming. The program we use in MATLAB is as follows:

```
    clear
dim = 6; %dimension-Max 6
one = unique(perms([[1 1] zeros(1,dim-2)]), 'rows', 'stable');
two = unique(perms([2 zeros(1,dim-1)]), 'rows', 'stable');
A = [one;two];
x = 1:length(A); % possible row indices
C = cell(dim,1); % preallocate
```

```
[C{:}] = ndgrid(x); % creates dim number of x values
y = cellfun(@(x) {x(:)}, C); % converts grids to column vectors
S = [y{:}]; % obtains permutations
B = zeros(dim,dim); % preallocate
num = 0; % number of matrices
check = 2*ones(1,dim); % check sums against the 2 vector
C = []; % container for eigen values
for i = 1:length(S)
    B = A(S(i,:),:);
    s = sum(B);
    if (isequal(s,check))
        if (isequal(B,B'))
            num = num + 1;
            C = [C; eig(B)];
        end
    end
end
num
```

The sequence we obtain is:

	Number of Symmetric Stochastic Matrices
0	1
1×1	1
2×2	3
3×3	11
4×4	56
5×5	348
6×6	2,578
7×7	22,054

Table 2: Number of symmetric stochastic matrices with elements $\{0,\frac{1}{2},1\}$. We obtained the 7×7 from OEIS.

2.3 Enumeration of Hollow Symmetric Stochastic Matrices

We begin to enumerate the hollow symmetric stochastic matrix case up to conjugacy by drawing out the different networks without labels on the nodes for each dimension. Since these matrices are adjacency matrices, we are able to draw the networks with undirected edges, no loops, nodes with degree 2 and disjoint unions of cycles. Figure 2 is an example of how we draw the networks. For simplicity we multiply the elements $\{0,\frac{1}{2},1\}$ by two. The networks are orbits under the action of the symmetric group. We can draw up to 10 dimensions using the same process and then we can find the sequence on OEIS.

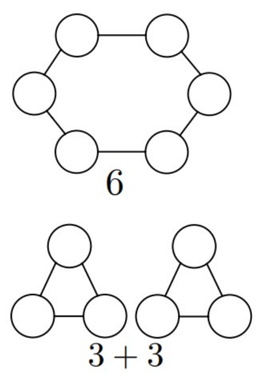

Figure 2: The 6×6 orbits of hollow symmetric stochastic matrices. There are 4 different ways to construct the networks as cycles and disjoint cycles

	Number of Hollow Symmetric Stochastic Matrices up to Conjugacy
0	1
1×1	0
2×2	1
3×3	1
4×4	2
5×5	2
6×6	4
7×7	4
8×8	7
9×9	8
10×10	12

Table 3: Number of hollow symmetric stochastic matrices with elements $\{0,\frac{1}{2},1\}$, up to conjugacy.

Another way of looking at the table 3 is as we go up in dimension we are seeing the partitions of n, where n is the dimension, that does not contain 1 as a part.

We are able to execute the enumeration for the case of all hollow symmetric stochastic matrices with the elements $\{0,\frac{1}{2},1\}$ in a very similar fashion. We enumerate by looking at the networks with labels and computed the sequence up to 5×5 by hand. We are able to count this two separate ways. One of the ways is to physically draw out all the possible permutations of the networks. Another way is to utilize some basic formulas. For example, the 5×5 case can be in the shape of a pentagon with 5 nodes. So we take $\frac{5!}{5*2}$ which gives us 12. We divide by two because if we flip the image, we have the same cycle. The other possible cycle for the 5×5 case is a three cycle and a two cycle. Looking at the three cycle first, we have $\frac{5!}{3!*2!}*\frac{1}{2}$ which is 5. Then we utilize the same method for the two cycle which is also 5. In total the 5×5 case gives us 22 possible orientations. Then we can find the sequence on OEIS. The sequence is shown in 4.

	Number of Hollow Symmetric Stochastic Matrices
0	1
1×1	0
2×2	1
3×3	1
4×4	6
5×5	22
6×6	130
7×7	822
8×8	6,202

Table 4: Number of hollow symmetric stochastic matrices with elements $\{0,\frac{1}{2},1\}$.

Looking at the table 4, we see the 8 dimension case has 6,202. This means that there are 6,202, 8×8 hollow symmetric stochastic matrices with the elements $\{0,\frac{1}{2},1\}$. From the table 3 we have 7, 8×8 hollow symmetric stochastic matrices up to conjugacy. So when we look at the 6,202 matrices under the group action of symmetry we have 7 different partitions.

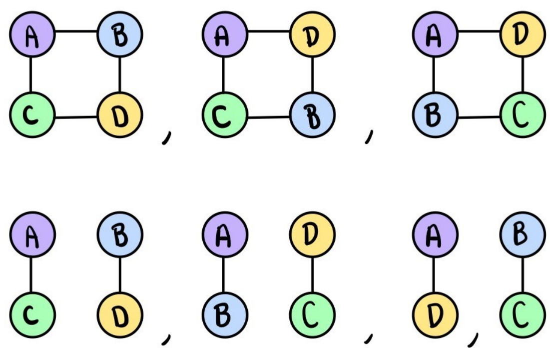

Figure 3: The drawn cycles not up to conjugacy of the 4×4 hollow symmetric stochastic matrices. There are 6 different ways to construct the networks as cycles and disjoint cycles

3 SPECTRA OF DOUBLY STOCHASTIC MATRICES

We will start with a very significant theorem that we use throughout this chapter.

Theorem 3.1. [Spi13] *Let A be a real symmetric stochastic matrix, then*

1. *1 is an eigenvalue of A.*
2. *If λ is an eigenvalue of A then $|\lambda| \leq 1$.*

Proof. *First, if A is a real symmetric stochastic matrix then A1 = 1, since each row (and or column) of A sums to 1. This proves that 1 is an eigenvalue of A.*
Second, suppose there exists $\lambda > 1$ and a nonzero x such that $Ax=\lambda x$.
Let x_i be a largest element of x. Since any scalar multiple of x will also satisfy this equation we can assume, without loss of generality that $x_i > 0$. Since the rows (and or columns) of A are non-negative and sum to 1, each entry in λx is a convex combination of the elements of x. Thus, no entry in λx can be larger than x_i. But $\lambda > 1$, $\lambda x_i > x_i$. Contradiction. Therefore $|\lambda| \leq 1$.

Note: The complex case is proved in a similar fashion, but with the triangle inequality.

3.1 3 by 3 Matrices and their Spectra

First, we will discuss the 3×3 case when we have $\{0,\frac{1}{2},1\}$ as our element choices in the stochastic matrix. Using the enumeration found in the previous chapter, we should have 21 unique 3×3. The 21 matrices are as follows:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\1&0&0\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&0&1\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}$$

Observing these 21 matrices we can find their spectra. In order to find their spectra and plot them, we utilize the following MATLAB code:

```
clear
dim = 3;
one=unique(perms([[0.5 0.5] zeros(1,dim-2)]), 'rows', 'stable');
% generates all possible rows with ones
two = unique(perms([1 zeros(1,dim-1)]), 'rows', 'stable');
% generates all possible rows with twos
```

```
A = [one;two]; % matrix of all possible rows
% Generates all permutatations of x with repetition
x = 1:length(A); % possible row indices
C = cell(dim,1); % preallocate
[C{:}] = ndgrid(x); % creates dim number of x values
y = cellfun(@(x) {x(:)}, C); % converts grids to column vectors
S = [y{:}]; % obtains permutations
B = zeros(dim,dim); % preallocate
num = 0; % number of matrices
check = ones(1,dim); % check sums against the 2 vector
C = []; % container for eigenvalues
for i = 1:length(S)
    B = A(S(i,:),:);
    s = sum(B);
    if (isequal(s,check))
        num = num + 1;
        C = [C; eig(B)];
    end
end
plot(real(C),imag(C),'LineStyle','none','marker','*')
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin')
```

Note: This code can be ran for up to 6 dimensions, but after that the code is too expensive. The code generates all possible permutation matrices with the elements $\{0,\frac{1}{2},1\}$ for the dimension given by the variable dim. It then produces all eigenvalues for each permutation matrix and plots them. When the code is ran we obtain the following graph:

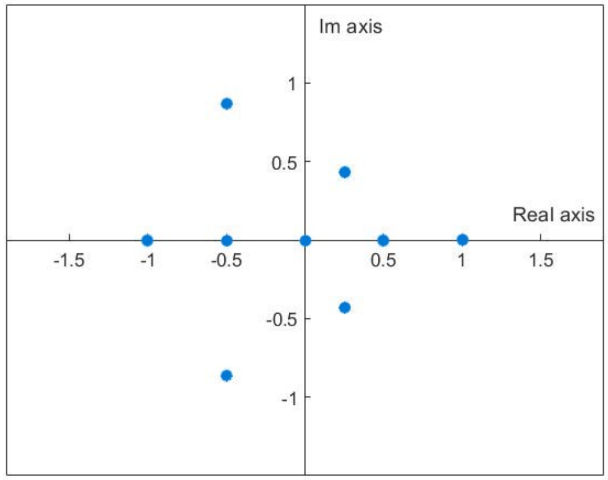

Figure 4: There are a total of nine unique spectra plots.

The graph includes the points (-1,0), (-$\frac{1}{2}$,0), (0,0), ($\frac{1}{2}$,0), (1,0), (-$\frac{1}{2}$,-$\frac{1}{2}$+$\frac{\sqrt{3}}{2}$i), (-$\frac{1}{2}$,-$\frac{1}{2}$-$\frac{\sqrt{3}}{2}$i), ($\frac{1}{4}$,$\frac{1}{4}$+$\frac{\sqrt{3}}{4}$i), ($\frac{1}{4}$,$\frac{1}{4}$+$\frac{\sqrt{3}}{4}$i). We can see that five points lie within the unit circle whereas the other four lie directly on the unit circle. This observation stems from theorem 3.1. To show this more clearly we have 5.

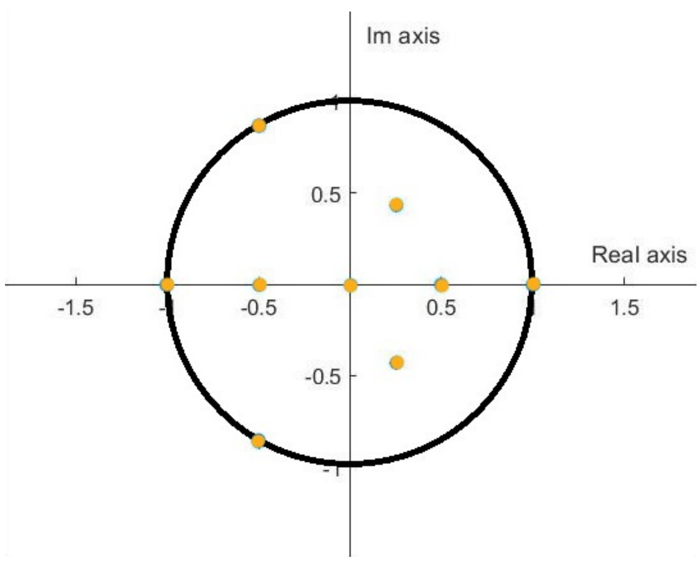

Figure 5: The same as Figure 2, but with the unit circle incorporated

3.1.1 Symmetric 3 by 3 Matrices

We have already looked at all 21 matrices of 3 by 3, the symmetric case is just a subset of those. The symmetric matrices are of interest because they are automatically doubly stochastic, and the transition probabilities are symmetric. The transition probabilities being symmetric means that going from state A to state B is the same as going from state B to A. The symmetric matrices for the 3×3 case are:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

$$\begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&0&\frac{1}{2}\end{pmatrix} \quad \begin{pmatrix}0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&0&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix} \quad \begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix}$$

The spectra of all the 3×3 symmetric stochastic matrices together are 1, -1, 0, -$\frac{1}{2}$ and $\frac{1}{2}$.

3.1.2 Convex Combinations of 3 by 3 Matrices

When working with convex combinations we are looking at specifically permutation matrices. We start by using the following MATLAB code:

```
clear
A = [1,0,0;0,1,0;0,0,1];
B = [1,0,0;0,1,0;0,0,1];
```

```
C=[];
    for i = 0:0.001:1
    M = i*A + (1-i)*B;
    C = [C; eig(M)];
    end
plot(real(C),imag(C),'LineStyle','none','marker','*')
set(gca, 'XAxisLocation', 'origin', 'YAxisLocation', 'origin')
xlabel('Real axis')
ylabel('Im axis')
xlim([-1.5,1.5])
ylim([-1.5,1.5])
```

In the MATLAB code we have the line, M = i*A + (1-i)*B, this is where we are performing the convex combination. As the variable i goes from 0 to 1, we are plotting all the possible eigenvalues as the random permutation matrix A goes to B. The figures in this subsection are made up of many points which then create the curves that we will discuss further. We use the code above to look at each 3×3 doubly stochastic matrix combined with all other possible 3×3 doubly stochastic matrices in order to examine every graphing output. Note that we manually have to input all the different matrices.

The convex combination of two non-hollow symmetric matrices that are not the identity produces the figure 6.

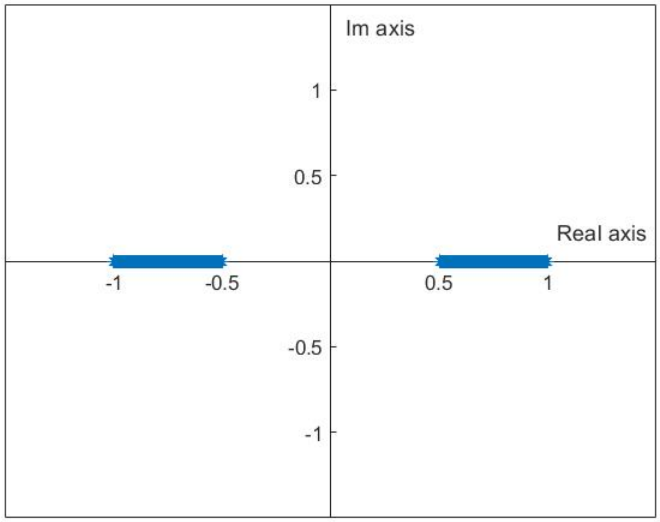

Figure 6: The convex combination of two non-hollow symmetric matrices that are not the identity.

We have multiple different combinations that display 6 so we will pick one of the options to show the convex combination by hand.

First, we let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then we have the equation

$$M = aA + (1-a)B \tag{3.1}$$

where a is an element of [0,1]. Utilizing 3.1 we obtain M = $\begin{pmatrix} 1-a & a & 0 \\ a & 0 & 1-a \\ 0 & 1-a & a \end{pmatrix}$

Then we can use the equation

$$det(\lambda I - M) \tag{3.2}$$

With 3.2 we acquire $(\lambda\text{-}1\text{+}a)[\lambda(\lambda\text{-}a)\text{-}(\text{-}1\text{+}a)(\text{-}1\text{+}a)]\text{+}a(\text{-}a(\lambda\text{-}a)\text{-}0)\text{+}0$ which simplifies to $3a^2\lambda\text{-}3a^2\text{-}3a\lambda\text{+}3a\text{-}\lambda^3\text{+}\lambda^2\text{+}\lambda\text{-}1\text{=}0$. We can factor, which then gives us $(\lambda\text{-}1)(3a^2\text{-}3a\text{-}\lambda^2\text{+}1)\text{=}0$.

Finally, we can solve for λ. We have λ_1=1, λ_2=$\sqrt{3a^2-3a+1}$, and λ_3= -$\sqrt{3a^2-3a+1}$. Now, we can take the derivative of one λ. Looking inside the square root we get, 6a-3 as our derivative. Setting that equal to zero we get a = $\frac{1}{2}$ which means that we have a minimum at a = $\frac{1}{2}$. Plugging that into λ_2 we get λ_2=$\frac{1}{2}$. This explains why our graph goes from $\frac{1}{2}$ to 1. The same process can be done with λ_3, which explains the -$\frac{1}{2}$ to -1.

The convex combination of the identity matrix with a hollow matrix produces the following graph:

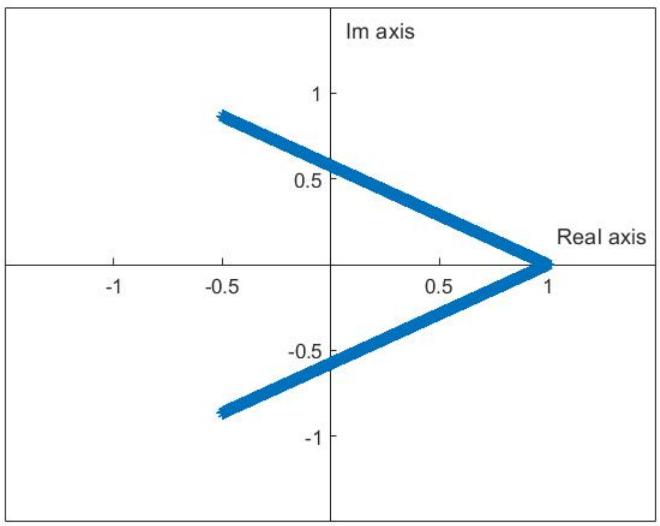

Figure 7: The convex combination of a hollow 3×3 matrix with the identity matrix.

The hollow matrices are:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Observing 7 we can see that the end points of the graph are the roots of unity for x^3-1. The roots of unity for x^3-1 are -$\frac{1}{2}\pm\frac{\sqrt{3}}{2}$i and 1. We can do the same process as before to prove this.

The convex combination of the identity matrix with a non-hollow symmetric matrix displays 8

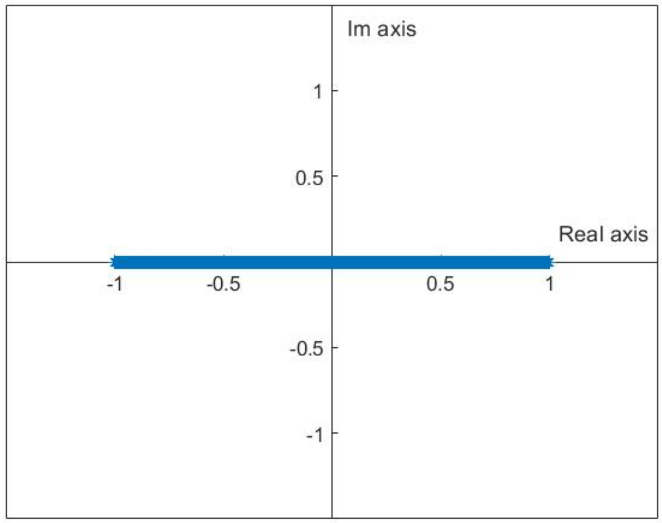

Figure 8: The convex combinations of the identity matrix with a non-hollow symmetric matrix.

The non-hollow symmetric matrices are:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We see in 8 that there is a horizontal line on the real axis going from -1 to 1. Since the convex combination of 8 is with two real symmetric matrices, we have only real numbers for their spectra. We can observe this same property with 6 for the same reason.

The next convex combination is a very appealing one to examine because we see a curve for the first time. The convex combination is of a symmetric and non-symmetric hollow matrix which outputs 9.

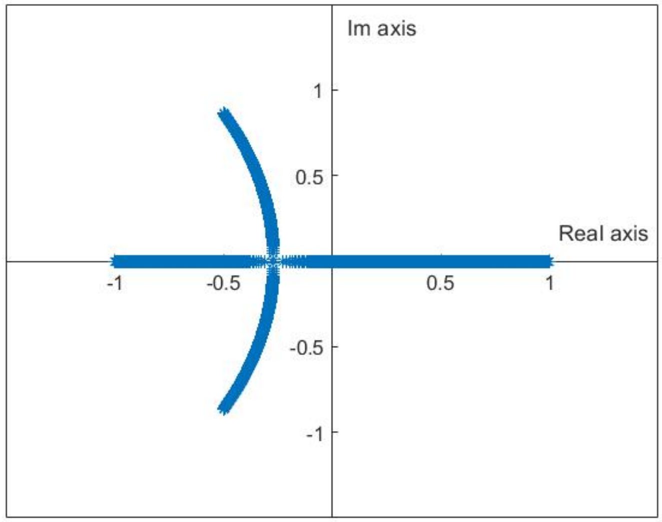

Figure 9: The convex combination of a 3×3 symmetric matrix and a non-symmetric hollow matrix.

We can see that we have a parabola shape in the figure 9. We use the same process to produce the equation of the curve as we did in the first example. The equation of the curve above the real axis is $\lambda{=}\frac{\sqrt{a^2+6a-3}+a-1}{2}$ and the curve below the real axis is $\lambda{=}\ \frac{-\sqrt{a^2+6a-3}+a-1}{2}$. Note that their is a spectral gap around the real axis at -0.268. To show this more clearly we have 10.

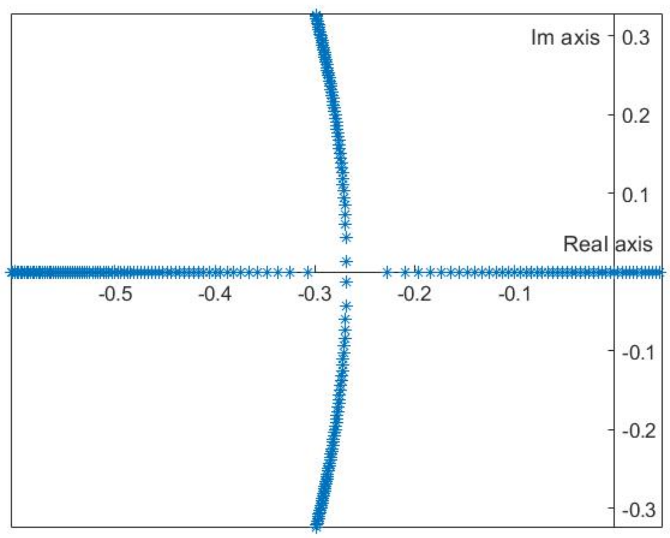

Figure 10: Same figure as 9, but zoomed in.

An example the matrices that produce the plots above are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The first matrix is non-hollow symmetric and the second is the hollow non-symmetric matrix.

Lastly, we have the convex combination of two 3×3 non-symmetric hollow matrices below:

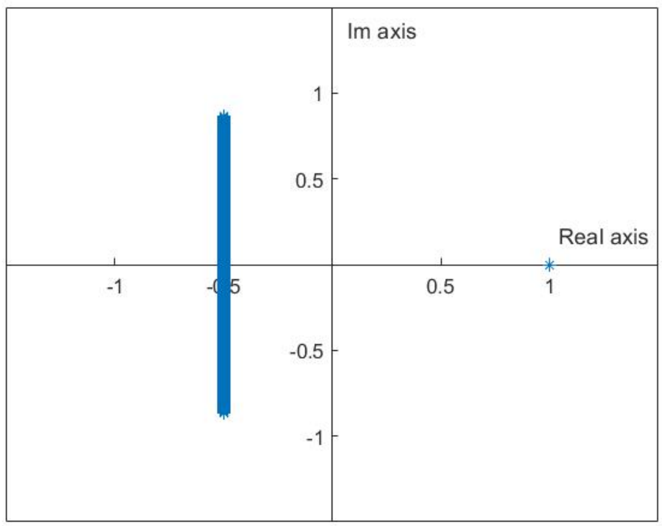

Figure 11: Convex combination of two 3×3 non-symmetric hollow matrices.

The two matrices are:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

With this convex combination we see a vertical line at -$\frac{1}{2}$ from one root of unity to another. We also have a point at 1 which shows our Theorem 3.1. We see the connections of roots of unities throughout this subsection and that is because they are complex conjugates.

3.2 Higher Dimension Convex Combinations and their Spectra

In this section, we will focus on looking at the interesting figures we find when taking convex combinations in higher dimensions. For the 3×3 convex combinations section, we had all the possible eigenvalues being plotted at once, that will continue on in this section. We will also continue to see the complex conjugates.

Looking at the 4×4 dimension of convex combinations, we have two with intriguing spectra. The first figure we have slight curves from 1 to 0, so we know that there are complex conjugates in effect. We also see a small line -1 on the real axis which is from a symmetric matrix.

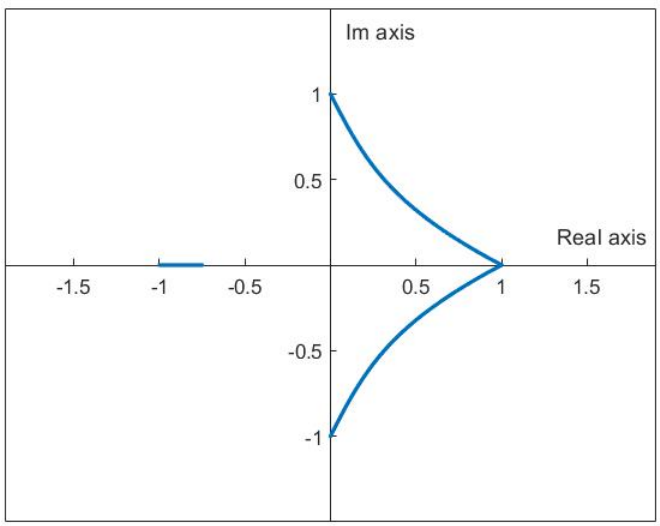

Figure 12: The convex combination of two 4×4 doubly stochastic matrices.

An example of the convex combination of two matrices that produce the above plot are:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

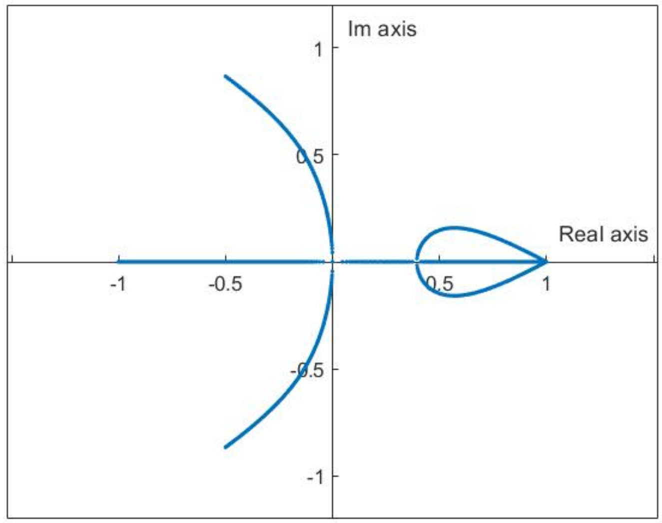

Figure 13: The convex combination of two 4×4 doubly stochastic matrices.

In figure 13 we observe many more curves. Zooming in on 13 we were able to see there is a point at (0,0), but a significant gap around the point. We also notice a spectral gap close to 0.4 on the real axis, but no point on the axis. To emphasize the spectral gap we have the figure 14.

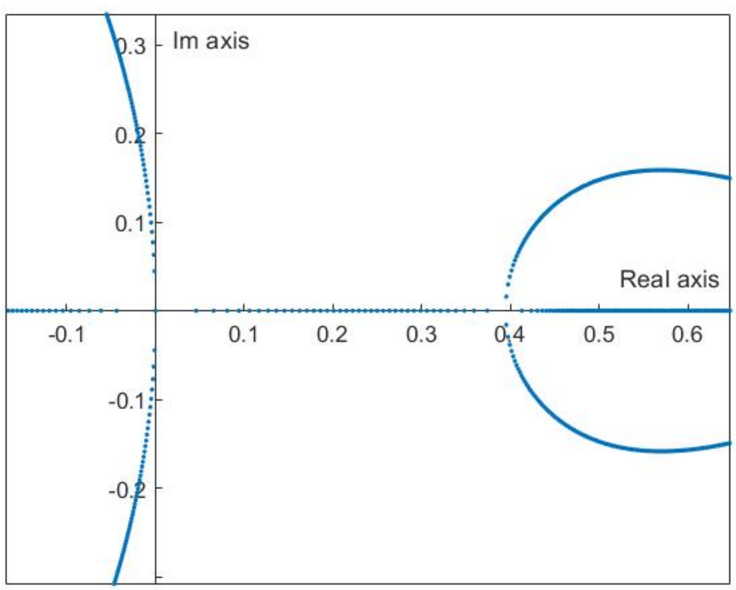

Figure 14: The figure 13 zoomed in.

An example of two matrices that create the plot 13 are:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We also have two 5 dimension convex combinations to investigate.

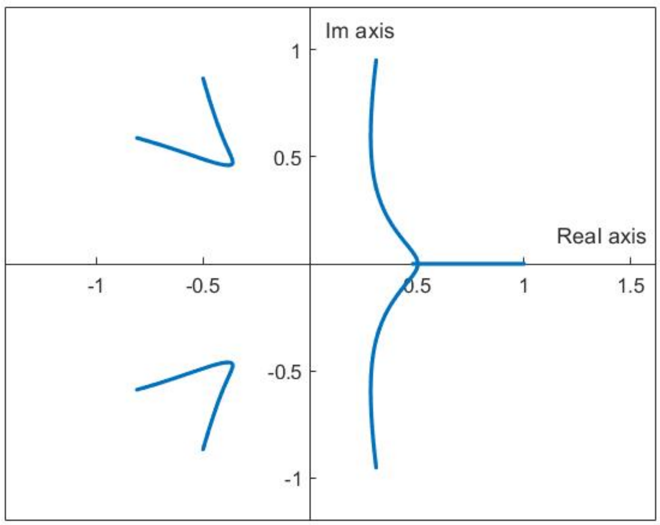

Figure 15: The convex combination of two 5×5 doubly stochastic matrices.

The convex combination of matrices that outputs 15 are:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

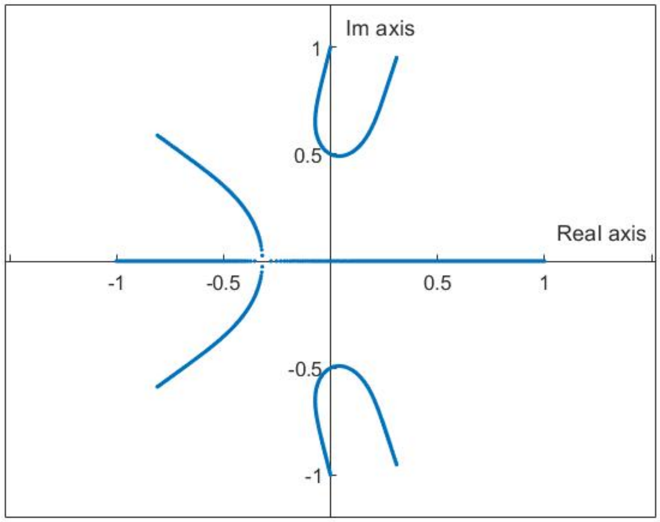

Figure 16: The convex combination of two 5×5 doubly stochastic matrices.

The convex combination of the matrices that create 16 are:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

With the two 5 dimension figures we start to see more defined parabolas and that they only happen when there are complex conjugates as solutions to the matrices. There is also a spectral gap in figure 16.

One of the graphs for 6 dimensions is 17.

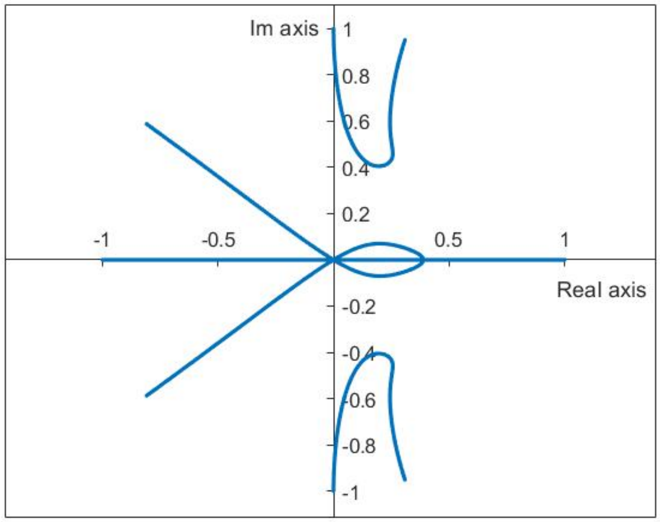

Figure 17: The convex combination of two 6×6 doubly stochastic matrices.

The convex combination of the matrices that gives us 17 are:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We start to see these loops at 6 dimensions. The loops are very fascinating and are ultimately what lead us to examine convex combinations originally.

The other 6×6 convex combination figure we note is 18. The figure is important to include because of its interesting spectral gap.

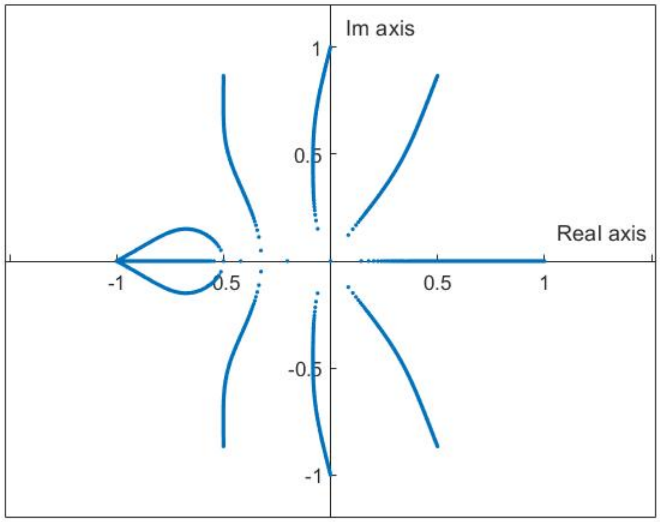

Figure 18: The convex combination of two 6×6 doubly stochastic matrices.

The convex combination of the below matrices plots 18.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The 19 had the convex combination of these matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

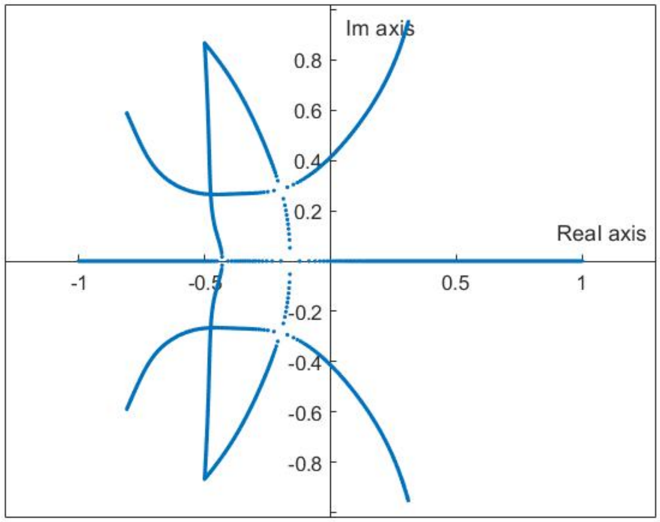

Figure 19: The convex combination of two 7×7 doubly stochastic matrices.

The convex combination of the matrices below display the plot 20.

$$\begin{pmatrix} 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix} \text{ and } \begin{pmatrix} 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix}$$

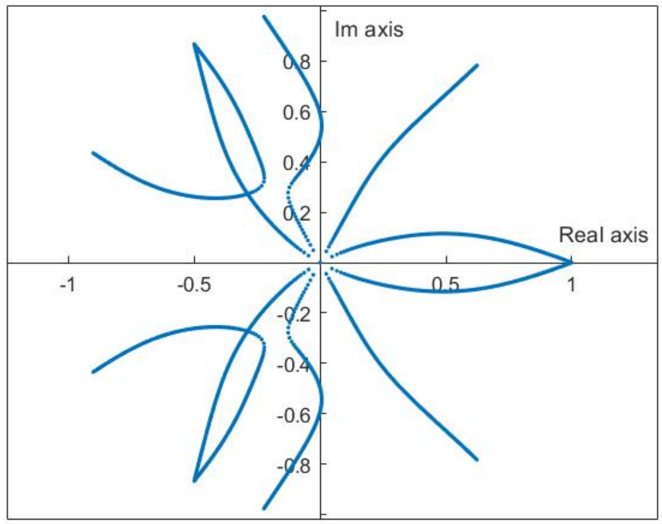

Figure 20: The convex combination of two 7×7 doubly stochastic matrices.

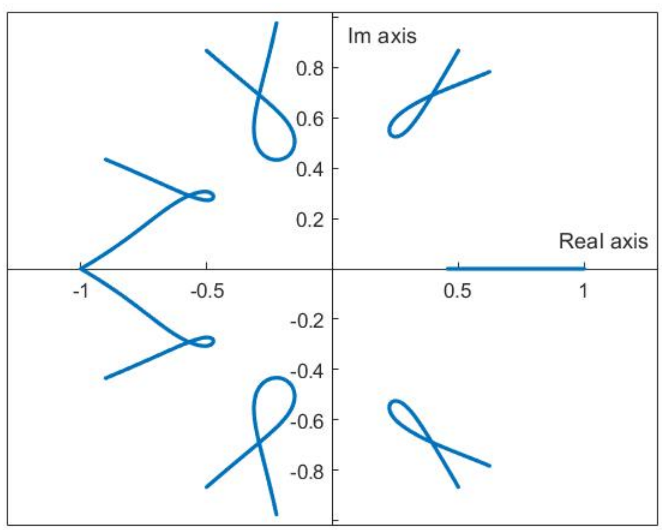

Figure 21: The convex combination of two 8×8 doubly stochastic matrices.

The convex combination of the 8×8 matrices below create the plot 21.

$$\begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \text{ and } \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0 \end{pmatrix}$$

3.3 Spectra of Hollow Symmetric Stochastic Matrices

Let's examine matrix A and B respectively:

$$\begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \text{ and } \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix}$$

Note: These matrices have been multiplied by two in order to clear the denominator.

Starting with matrix A, we have an 8×8 permutation matrix that commutes and is a complete cycle. Since the matrix has these properties, its characteristic polynomial can be written in the form x^8-1. When the polynomial is solved we obtain the following solutions: 1, i, $\frac{1+i}{\sqrt{2}}$, -$\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, -$\frac{1-i}{\sqrt{2}}$, -i, -1.

Matrix B is the transpose (inverse) of A, so the solutions are inverses of A. The solutions are 1, -i, $\frac{1-i}{\sqrt{2}}$, -$\frac{1-i}{\sqrt{2}}$, $\frac{1+i}{\sqrt{2}}$, -$\frac{1+i}{\sqrt{2}}$, i, -1.

When we add A and B we obtain:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Notice that we now have a hollow symmetric stochastic matrix.

Since these matrices commute, they are simultaneously diagonalizable. These properties enable us to add their spectra. First, we have to line up their spectra so that the inverses are paired:

1, i, $\frac{1+i}{\sqrt{2}}$, -$\frac{1+i}{\sqrt{2}}$, $\frac{1-i}{\sqrt{2}}$, -$\frac{1-i}{\sqrt{2}}$, -i, -1

1, -i, $\frac{1-i}{\sqrt{2}}$, -$\frac{1-i}{\sqrt{2}}$, $\frac{1+i}{\sqrt{2}}$, -$\frac{1+i}{\sqrt{2}}$, i, -1

Once we have added up the spectra we have only real numbers because the imaginary parts add to 0. So the spectrum of the 8×8 hollow symmetric stochastic matrix is:

1, 0, $\frac{1}{\sqrt{2}}$, -$\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, -$\frac{1}{\sqrt{2}}$, 0, 1

Note: We have divided the added spectra by 2 in order to show the spectrum with the elements $\{0, \frac{1}{2}, 1\}$.

In summary, we know spectra of hollow symmetric stochastic matrices with $\{0, \frac{1}{2}, 1\}$ because an eigenvalue of any of our matrices is $\lambda + \overline{\lambda}$, where λ is a root of unity.

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