Comments

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Comments on "Orthogonality of Exponential Transients"

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It is shown that the problem solved in the above letter[1] can be solved in greater generality and in a more straightforward manner by using classical vector-space methods.

I. Introduction

In the above letter,[1] the authors discuss the representation of a decaying time function $x(t)$, $0 \le t < \infty$, by a weighted sum of exponentials of the form $\Sigma_{k=0}^{N-1} c_k e^{-\alpha_k t}$, where the $\{\alpha_k\}_{k=0}^{N-1}$ are given nonnegative decay rates. The authors assume that

$$\alpha_k = k\Delta, \qquad k = 0, \cdots, N-1, \tag{1}$$

for some given fundamental decay rate, $\Delta > 0$. They then exploit the substitution $z = e^{-\Delta t}$ in order to determine the coefficients, $\{c_k\}_{k=0}^{N-1}$. In this note we point out that the assumption (1) can be removed if one takes a more classical, vector-space approach. In addition, the approach outlined below will yield an expression for the error between the representation of the signal and the signal itself (cf. Remark 2 below).

The problem of signal representation by exponential transients has been studied extensively [1]–[6]. However, the approach presented below is not found there. Note that we assume the decay rates are given. The much more difficult question of how to find simultaneously the best decay rates and the best coefficients was considered in [6].

In Section II we summarize a few results about inner product spaces. Then in Section III we apply these results to a certain function space containing the exponential transients.

II. Preliminaries

We appeal to the following well-known result (for a simple proof, see Luenberger [7, Theorem 1, p. 50]).

Theorem 1: Let X be an inner product space over the real or complex numbers. Denote the inner product by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $\|\cdot\|$. Let M be an arbitrary subspace of X. Let $x \in X$ be given. Then a vector $\hat{x} \in M$ has the property

$$\|x - \hat{x}\| \le \|x - m\| \qquad \text{for all } m \in M, \tag{2}$$

if and only if

$$\langle x - \hat{x}, m \rangle = 0 \qquad \text{for all } m \in M. \tag{3}$$

Further, there is at most one element $\hat{x} \in M$ satisfying (2) and (3).

As an immediate consequence of Theorem 1 we have the formula

$$\|x - \hat{x}\|^2 = \|x\|^2 - \|\hat{x}\|^2. \tag{4}$$

This follows by noting that since $\hat{x}$ itself belongs to M, (3) implies that $(x - \hat{x})$ and $\hat{x}$ are orthogonal. Expanding $\|x\|^2 = \|(x - \hat{x}) + \hat{x}\|^2$ yields (4). It is also worth noting that if x itself belongs to M, then $\hat{x} = x$ is the unique vector in M that satisfies both (2) and (3).

We now suppose that M is a subspace spanned by a finite subset, say $\{v_0, \cdots, v_{N-1}\}$. Then equation (3) holds if and only if

$$\langle x - \hat{x}, v_k \rangle = 0, \qquad k = 0, \cdots, N-1. \tag{5}$$

Note that $\{v_0, \cdots, v_{N-1}\}$ need not be linearly independent. Next, since $\hat{x}$ is an element of M, there exist scalars, $c_0, \cdots, c_{N-1}$, such that

$$\hat{x} = \sum_{i=0}^{N-1} c_i v_i. \tag{6}$$

Hence, (5) reduces to

$$\sum_{i=0}^{N-1} c_i \langle v_i, v_k \rangle = \langle x, v_k \rangle, \qquad k = 0, \cdots, N = 1. \tag{7}$$

In the special case that $\{v_0, \cdots, v_{N-1}\}$ is an orthonormal set, (7) reduces to $c_k = \langle x, v_k \rangle$. Otherwise, set $A_{ki} = \langle v_i, v_k \rangle$ and $b_k = \langle x, v_k \rangle$, and write (7) in matrix-vector notation as $Ac = b$, where $c = [c_0, \cdots, c_{N-1}]'$ and $b = [b_0, \cdots, b_{N-1}]'$. The solution of $Ac = b$ is unique if and only if $\{v_0, \cdots, v_{N-1}\}$ are linearly independent [7, Proposition 1, pp. 56–57]. Now, let c^* denote the complex-conjugate transpose of c. Substituting (6) into (4) yields

$$\begin{aligned} \|x - \hat{x}\|^2 &= \|x\|^2 - c^* A c \\ &= \|x\|^2 - c^* b, \qquad \text{since } Ac = b. \end{aligned} \tag{8}$$

III. Application to Exponential Transients

For fixed $\lambda \ge 0$, we shall take as our real vector space, X, the set of all Borel measurable functions $f: [0, \infty) \to \boldsymbol{R}$ such that

$$\int_0^\infty e^{-\lambda t} |f(t)|^2 \, dt < \infty.$$

The inner product will be

$$\langle f, g \rangle = \int_0^\infty e^{-\lambda t} f(t) g(t) \, dt.$$

If X is to contain the constant functions, we need $\lambda > 0$. Otherwise, $\lambda = 0$ will suffice.

Let $\{\alpha_0, \cdots, \alpha_{N-1}\}$ be given decay rates. If any rate is zero, we require that $\lambda > 0$. Let M denote the subspace consisting of all linear combinations of the signals $\{e^{-\alpha_k t}\}_{k=0}^{N-1}$. Having observed a waveform $x(t)$, $0 \le t < \infty$, our problem is to find scalars $c_0, \cdots, c_{N-1}$ so as to minimize

$$\left(\int_0^\infty e^{-\lambda t} \left| x(t) - \sum_{k=0}^{N-1} c_k e^{-\alpha_k t} \right|^2 dt \right)^{1/2}. \tag{9}$$

Remark 2: The authors' approach[1] assumes *a priori* that $x(t) = \Sigma N - 1_{k=0} \beta_k e^{-\alpha_k t}$ for some constants $\{\beta_k\}$; under this assumption, $x \in M$ and it is clear that the minimum value of (9) is zero.

Now, (9) will be minimized if and only if $c = [c_0, \cdots, c_{N-1}]'$ sat-

Manuscript received April 7, 1989; revised July 31, 1989.

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IEEE Log Number 8933162.

[1] G. R. L. Sohie and G. N. Maracas, Proc. IEEE, vol. 76, no. 12, pp. 1616–1618, Dec. 1988.

0018-9219/89/1100-1776$01.00 © 1989 IEEE

isfies $Ac = b$, where, using the notation $v_k(t) = e^{-\alpha_k t}$,

$$A_{ki} = \langle v_i, v_k \rangle = \int_0^\infty e^{-\lambda t} e^{-\alpha_i t} e^{-\alpha_k t}\, dt = (\lambda + \alpha_i + \alpha_k)^{-1}, \tag{10}$$

and

$$b_k = \langle x, v_k \rangle = \int_0^\infty e^{-\lambda t} x(t) e^{-\alpha_k t}\, dt = \int_0^\infty e^{-t(\lambda + \alpha_k)} x(t)\, dt. \tag{11}$$

Observe that b_k is simply the Laplace transform of x, evaluated at $\lambda + \alpha_k$. Note also that having solved $Ac = b$ for c, the error between $x(t)$ and $\hat{x}(t) = \Sigma_{k=0}^{N-1} c_k e^{-\alpha_k t}$ is obtained via (8)

$$\|x - \hat{x}\|^2 = \int_0^\infty e^{-\lambda t} |x(t)|^2\, dt - c'b. \tag{12}$$

In terms of implementation, the integrals (11) and (12) can be approximated by using a numerical integration procedure on $[0, \infty)$ such as Laguerre–Gauss quadrature [8, Section 8.6]. To do this one needs only a finite number of samples of the waveform.

To conclude this section, we set $\lambda = \Delta$ and $\alpha_k = k\Delta$ to obtain the results of the above letter[1] as a special case. First observe that if $\lambda = \Delta$ and $\alpha_k = k\Delta$, (10) reduces to $A_{ki} = (1/\Delta)(1 + i + k)^{-1}$. Next, applying the change of variable $z = e^{-t\Delta}$ in (11) yields

$$b_k = \int_0^\infty e^{-t(\Delta + k\Delta)} x(t)\, dt = \frac{1}{\Delta} \int_0^1 z^k x(-\Delta^{-1} \ln z)\, dz.$$

Now, both A_{ki} and b_k involve the factor $1/\Delta$. Let $G = \Delta \cdot A$. Then $G_{ki} = (1 + i + k)^{-1}$ and we see that G is exactly the matrix found in equation (A6) of the above letter.[1] We can easily write

$$(A^{-1}b)_k = \sum_{l=0}^{N-1} (G^{-1})_{kl} \cdot \int_0^1 z^l x(-\Delta^{-1} \ln z)\, dz. \tag{13}$$

We claim this is exactly equation (8) of the above letter.[1] To see this, substitute (7)[1] into (8)[1] to obtain, using notation defined in the above letter,[1]

$$\int_0^1 \int_0^1 x(y) w(y, z) z^k\, dy\, dz = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} w_{il} \int_0^1 x(y) y^i \left(\int_0^1 z^{l+k}\, dz \right) dy = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} w_{il} G_{kl} \int_0^1 x(y) y^i\, dy = \sum_{i=0}^{N-1} (G^{-1})_{ki} \int_0^1 x(y) y^i\, dy, \tag{14}$$

where the last step follows because the matrix $[w_{il}] \triangleq G^{-2}$ and because G is symmetric. Clearly, (13) and (14) are the same since in (14), $x(y)$ is shorthand for $x(-\Delta^{-1} \ln y)$. We also point out that if $x \in M$, i.e., $x(t) = \Sigma_{k=0}^{N-1} \beta_k e^{-k\Delta t}$, then $x(-\Delta^{-1} \ln z)$ is a polynomial in z and thus (13) can be evaluated exactly using N-point Legendre–Gauss quadrature as pointed out in the above letter.[1]

IV. Conclusions

We have solved the problem considered in the above letter[1] more simply and in greater generality by specializing classical vector-space analysis (e.g., Luenberger [7, Chapter 3]) to a particular function space containing the exponential transients.

References

[1] W. H. Kautz, "Transient synthesis in the time domain," *IRE Trans. Circuit Theory*, vol. CT-1, pp. 29–39, Sept. 1954.

[2] W. H. Huggins, "Signal theory," *IRE Trans. Circuit Theory*, vol. CT-3, pp. 210–216, Dec. 1956.

[3] H. L. Armstrong, "On finding an orthonormal basis for representing transients," *IRE Trans. Circuit Theory*, vol. CT-4, pp. 286–287, Sept. 1957.

[4] T. Y. Young and W. H. Huggins, "'Complementary' signals and orthogonalized exponentials," *IRE Trans. Circuit Theory*, vol. CT-9, pp. 362–370, Dec. 1962.

[5] —, "On the representation of electrocardiograms," *IEEE Trans. Bio-Medical Electronics*, vol. BME-10, pp. 86–95, July 1963.

[6] R. N. McDonough and W. H. Huggins, "Best least-squares representation of signals by exponentials," *Trans. Automat. Contr.*, vol. AC-13, pp. 408–412, Aug. 1968.

[7] D. G. Luenberger, *Optimization by Vector Space Methods*. New York, NY: Wiley, 1969.

[8] F. B. Hildebrand, *Introduction to Numerical Analysis*. New York, NY: McGraw-Hill, 1956.