ON THE COMPLEXITY OF REACHABILITY AND MOTION PLANNING QUESTIONS*

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On the Complexity of Reachability and Motion Planning Questions*

(Extended Abstract)

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In this paper we consider from a theoretical viewpoint the complexity of some reachability and motion planning questions. Specifically, we are interested in which generalizations of the basic mover's problem result in computationally intractable problems. It has been shown that for any set of motion-planning problems with bounded degree of freedom, there is a polynomial-time algorithm to solve the motion-planning problem (although the degree of the polynomial may be large), but the two most common generalizations to the problem, multiple movable obstacles and conformable objects, result in much harder problems. It has been shown that the warehouseman's problem is P-space hard; in this paper we show that the reachability problem for one of the simplest types of conformable objects, a two-dimensional linear ("robot arm") linkage is P-space complete. In addition, we demonstrate some motion-planning problems which take exponential time.

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1. Motion Planning Problems

Motion planning problems of various sorts have been studied quite a bit recently. The interest started in 1968 with a paper by W. E. Howden entitled "The Sofa Problem"[1] and today includes various generalizations of the basic problem, such as the "warehouseman's problem," motion planning problems for conformable objects, and optimal path finding problems. The sofa problem is this: suppose you want to move your sofa from the living room to the den. Is there a sequence of motions through which you can move the sofa such that it reaches its goal position in the den without contacting any obstacle, and if so, what is it? The question of the existence of such a sequence of motions is called the reachability problem, and listing the motions is the motion planning problem.

If we generalize the sofa problem to allow for the possibility of moving some of the obstacles out of the way, we have the "warehouseman's problem." For example, we might want to move the dining room table into the corner of the room, move the sofa through the dining room, take the den door off its hinges, and so on. Thus the problem involves an object, a workspace, and a set of obstacles which can be moved, subject to the constraint that the obstacles and object don't collide, or at any rate, don't overlap.

Another generalization is to allow a conformable object, that is, an object which can be changed or rearranged in some way. In that case we may want to arrange the object in some configuration, move it through some tight spot, arrange it in another configuration, and so on. For example, suppose a person carrying a large box wants to get through a doorway. (Here the conformable object is the person *and* the box.) He may have to lift the box above his head and turn sideways to get through the doorway since it may not be wide enough to pass the box with one of his arms on each side.

2. Previous Results

Many polynomial-time algorithms have been presented for the basic sofa problem. Algorithms have been presented by Howden,[2] Lozano-Perez & Wesley,[3] Reif,[4] Brooks,[5] and Schwartz & Sharir,[6] for example. Some of these algorithms are approximation algorithms in that they don't always find a path if there is one, but Schwartz and Sharir give a general algorithm (based on Reif's) which works for arbitrary simple (algebraic) bodies and workspaces in polynomial time.

When one is allowed to move obstacles as in the warehouseman's problem, the situation

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1. Howden, W. E., "The Sofa Problem," *Comput J.* **11** pp. 299-301 (1968).
2. Howden, W. E., "The Sofa Problem," *Comput J.* **11** pp. 299-301 (1968).
3. Lozano-Perez, T. and M. A. Wesley, "An algorithm for planning collision-free paths among polyhedral obstacles," *Comm. ACM* **22** pp. 560-570 (October 1979).
4. Reif, J. H., "Complexity of the Mover's Problem and Generalizations (Extended Abstract)," *Proc. 20th IEEE FOCS*, pp. 421-427 (1979).
5. Brooks, R. A., "Planning collision-free motion for pick-and place operations," *The International Journal of Research* 2(4)(1983).
6. Schwartz, J. T. and M. Sharir, "On the 'Piano Movers' Problem -- II. General Techniques for Computing Topological Properties of Real Algebraic Manifolds," *Advances in Applied Math.* **4** pp. 298-351 (1983).

changes, however. Hopcroft, Schwartz, and Sharir[7] have proven that the warehouseman's problem for an arbitrary number of obstacles is P-space hard even when the workspace, object, and obstacles are all rectangular. (If there is a bound on the number of obstacles, though, then the problem is again solvable in polynomial time, another corollary of Schwartz and Sharir's theorem.)

The problem is also made more difficult when the object is conformable, that is, when there are degrees of freedom in the configuration of the object itself. For example, robot arms are conformable; changing the angle of a joint changes the "look" of the object. We know that for any object with a bounded number of degrees of freedom, there is a polynomial time motion planning algorithm, but when the number of degrees of freedom is unbounded, various examples of the problem have been proven P-space hard.

Often the objects (in 3 dimensions) are approximated by polyhedra, and the kind of conformability most commonly considered is that of a set of polyhedra connected by joints. For example, an automobile with doors that open and close might be considered a number of polyhedra connected by joints of a certain type. In trying to find the most generic instance of such a problem, we consider *linkages*, which are physical objects that look like graphs. The links have a particular length, and they are considered to be rigid. The links are connected at their endpoints in universal joints, and one joint may have any number of links incident upon it. In some cases linkages are allowed to have fixed points in the plane, and in other cases simpler linkages are considered, for example, tree-like linkages, and "robot arm" linkages, which have exactly two links incident upon each joint (except at the ends).

Reif proved that the reachability problem is P-space hard for tree-like linkages in a three-dimensional workspace.[8] In addition, Hopcroft, Joseph, and Whitesides[9] proved that the question of whether a planar (graph-like) linkage with fixed points can be moved so that a given joint reaches a certain point in the plane is P-space hard. Hopcroft, Joseph, and Whitesides have also proven NP-hard the reachability problem for "robot arm" (linear) linkages. [10]

The main result of this paper is a proof of the P-space completeness of the reachability problem for two-dimensional linear linkages. This result improves on the latter proof of Hopcroft, Joseph, and Whitesides from NP-hard to P-space complete, and on Reif's proof to two dimensions and simpler linkages. It is not really an extension of the P-space hardness proof for graph-linkages, however, since that proof holds in a fixed, convex workspace.

3. Reachability for Linkages

We now show some results about the complexity of reachability questions for different types of linkages. First, observe that Hopcroft, Joseph, and Whitesides' P-space-hardness proof for graph-

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7. Hopcroft, J. E., J. T. Schwartz, and M. Sharir, "On the complexity of motion planning for multiple independent objects: Pspace hardness of the "Warehouseman's problem"," Preprint (1983).

8. Reif, J. H., "Complexity of the Mover's Problem and Generalizations (Extended Abstract)," *Proc. 20th IEEE FOCS*, pp. 421-427 (1979).

9. Hopcroft, J. E., D. A. Joseph, and S. H. Whitesides, "On the movement of robot arms in 2-dimensional bounded regions," *Proc. 23rd Annual FOCS*, pp. 280-289 (1982).

10. Hopcroft, J. E., D. A. Joseph, and S. H. Whitesides, "On the movement of robot arms in 2-dimensional bounded regions," *Proc. 23rd Annual FOCS*, pp. 280-289 (1982).

like linkages with fixed points [11] can be extended to show that the reachability problem for graph-like linkages without fixed points in a simple convex workspace is P-space hard.

Corollary 1. The reachability problem is P-space hard for (graph) linkages in a simple fixed convex workspace.

We prove this by modifying Hopcroft, Joseph, and Whitesides' construction thus: we add a simple convex workspace around the linkagea square works nicely, although a circle would also work. Then we jam a link in between two corners of the square along a diagonal. We eliminate fixed points and replace them by adding two more links for each fixed pointone from the fixed point to each end of the jammed link (see Figure 1). Thus, those joints which had been fixed are still unable to move.

Next, we show that the reachability problem for 2D linear linkages is P-space complete. This is the main result of the paper.

Theorem 1. The reachability problem for 2D linear linkages is P-space complete.

We will prove this for 2D "robot arm" linkages by a reduction from LBA acceptance, a problem already known to be P-space complete.[12] That is, for any LBA M and word w, we will show how to construct a 2D (non-convex) workspace and put a specified linkage at a certain point such that the linkage can reach the specified goal point if and only if M accepts w.

The linkage is a set of line segments of specified lengths, connected end to end, like a snake (see figure 2). The segments can rotate freely about the joints, and they are allowed to cross over one another. One third of the segments will be of one set length and the other two thirds of various but shorter lengths, with two shorter segments between each long segment. The shorter segments come in lengths varying from 1/2 the length of the long segments to 3/4 the length, arranged in ascending order of length along the linkage.

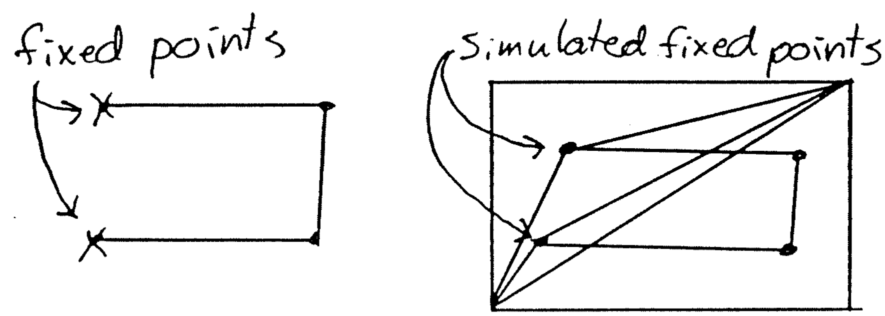

Figure 1: Simulating Fixed Points

11. Hopcroft, J. E., D. A. Joseph, and S. H. Whitesides, "On the mover's problem for 2-dimensional linkages," *SIAM J. Comput.*, (1984). To appear.

12. Garey, M. R. and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, W. H. Freeman and Company, San Fransisco (1979).

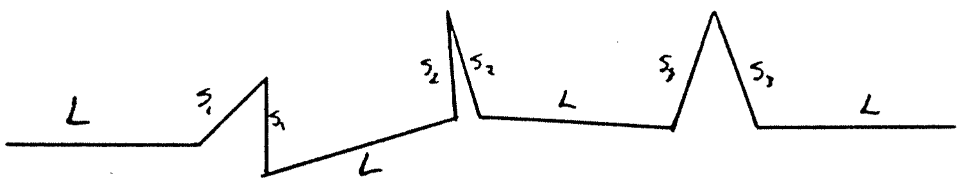

Figure 2: The Linkage

The linkage is used to encode information, each pair of shorter segments acting as a switch (see figure 3). The switches have value 0 or 1 depending on whether they are in the backward or forward position (forward means pointing toward the end of the linkage with the longer "short segments." They are used to encode (in any straightforward way which doesn't waste too much space) the tape contents, tape head position, and state that M would be in while reading input w.

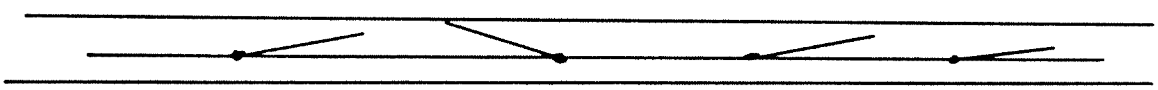

Figure 3: Encoding information in the linkage

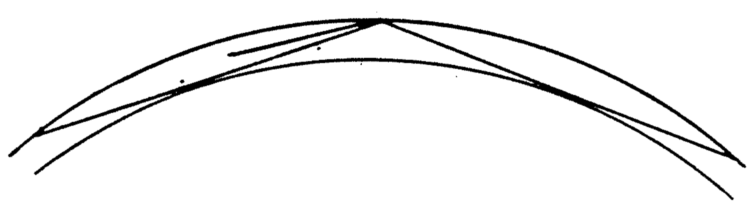

Figure 4: Corners which don't allow switches to switch

The linkages will pass through corridors which are too narrow for the switches to change position when we don't want them to. Note that we can also have "corners" which are sufficiently narrow (figure 4). We must now construct gates through which the linkage can pass if and only if certain switches are in certain positions.

The building blocks of a gate will be hairpin turns which the linkage can make if and only if one particular switch is in one specified position (see figure 5). The linkage is too long to simply slide into the dead end and slide out the other side, going the opposite way. And there is no way to make the linkage shorter unless switch S_i is pointed in the direction of motion; but if that is the case, then the switch can be used to temporarily shorten the linkage (as in the diagram) and allow it to pass

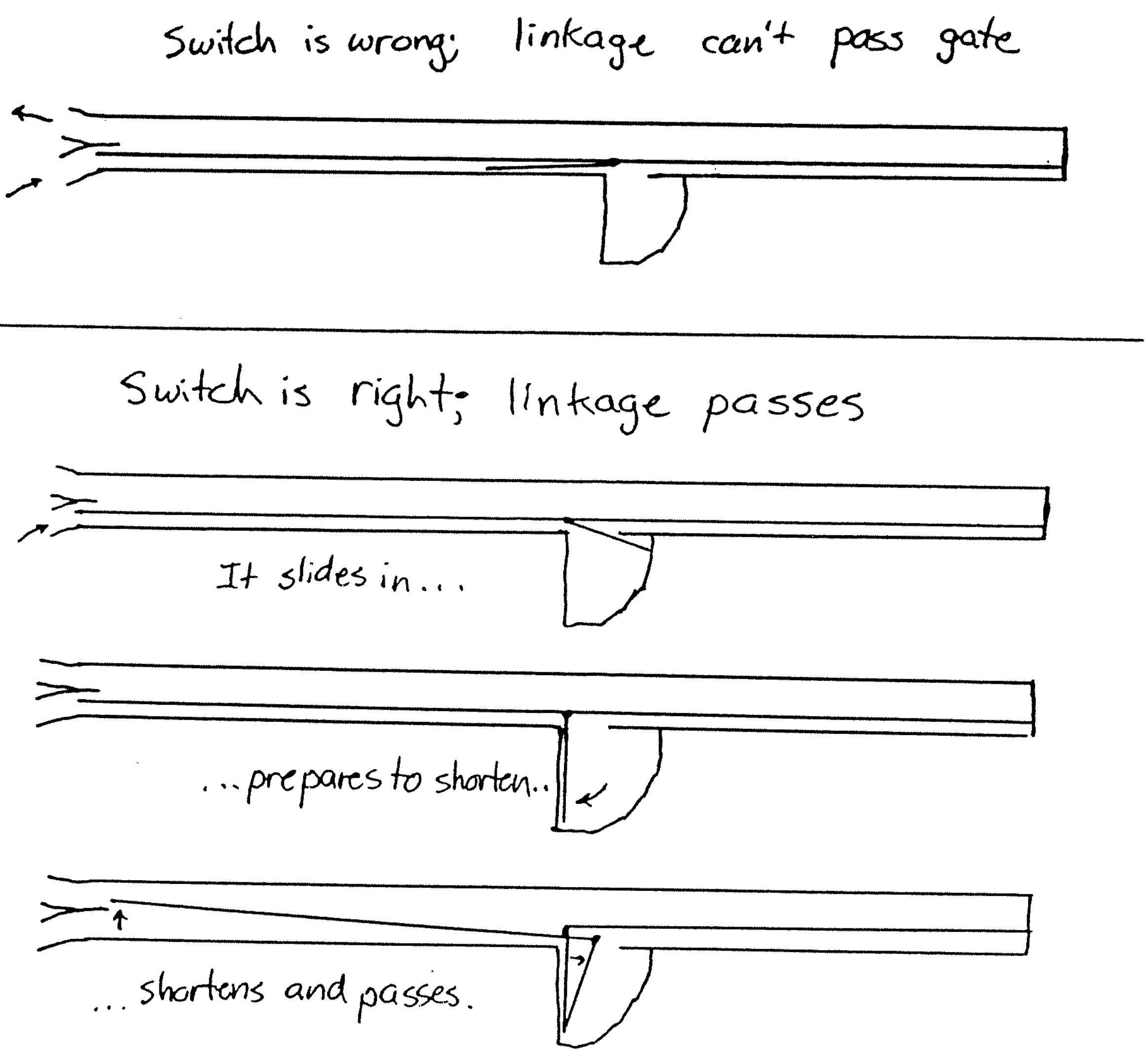

Figure 5: Checking the setting of a switch

out the other path. Note that if we want to check that a switch is set the opposite way, we must check it when it is moving in the opposite direction.

By concatenating these we can construct a gate which checks that a certain set of switches are all ones, a certain set zeros and the rest of them "don't cares." Thus, for example, we could check that **M** is in state **q**, the tape head is scanning tape position *n*, and the symbol at that tape position is **a** (See Figure 6.). It is important that we can check certain switches without having to check every switch, for in that case we would have an exponential number of checks to make over the course of the simulation.

We must also be able to allow certain switches to change position without any other switches being able to do so (see figure 7). Here a linkage can slide into the dead-end, and when it reaches the end, the switch in question can change directions. None of the switches which preceded it into the corridor were able to toggle because they are all longer than the switch in question. These chambers can be concatenated to let any combination of switches toggle position without allowing any of the others to change.

Now we have the tools to build a corridor which corresponds to a transition T of M for some particular tape head position. That is, we will build a corridor which the linkage can get through only if the state, head position, and symbol being scanned satisfy the preconditions of T, and if the linkage does go through the corridor, then when it exits, the tape head, tape contents, and state reflect the changes that the transition T should cause. Thus, if the linkage gets through the first "checker" gate, we know it meets the preconditions for that transition, and we allow the transition to take place in a "setter" gate. By way of another "checker" gate, we allow the linkage to exit the transition tunnel (in the forward direction) only if the transition has taken place.

Now we can easily describe the workspace (see figure 8). The linkage starts at the starting point in some initial configuration, passing through the starting gate only with w, head position 1, and the initial state encoded into the switches. It can move around the oval as much as it would like, and it can take transitions in any order the Turing machine could, recording the effects in the

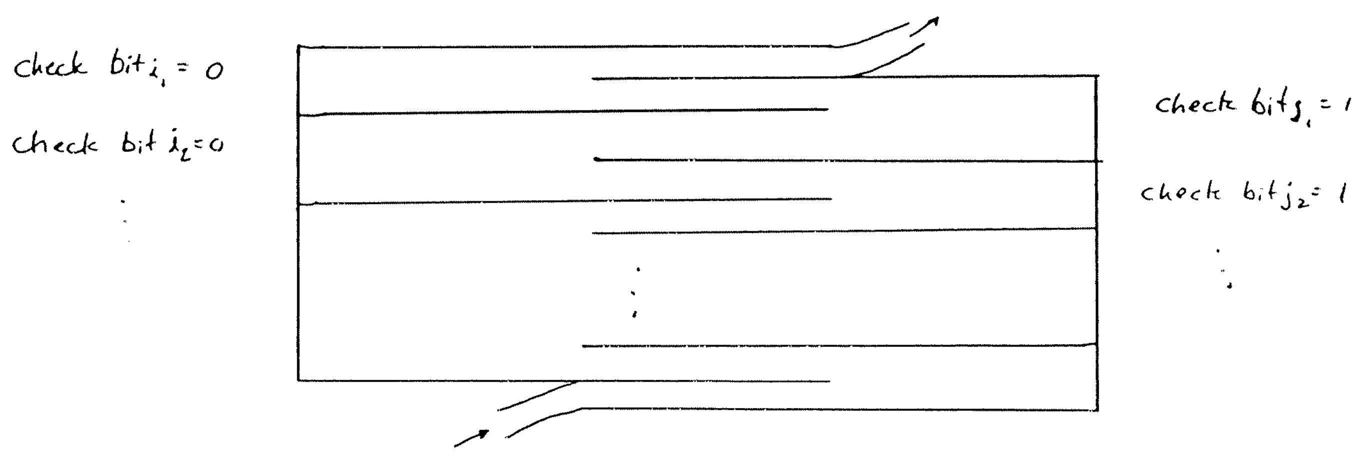

Figure 6: A gate which checks state, head position, and tape symbol being scanned

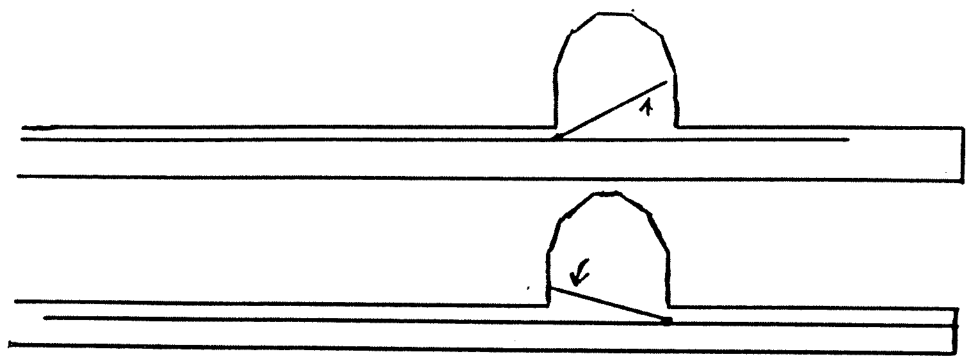

Figure 7: Changing a switch

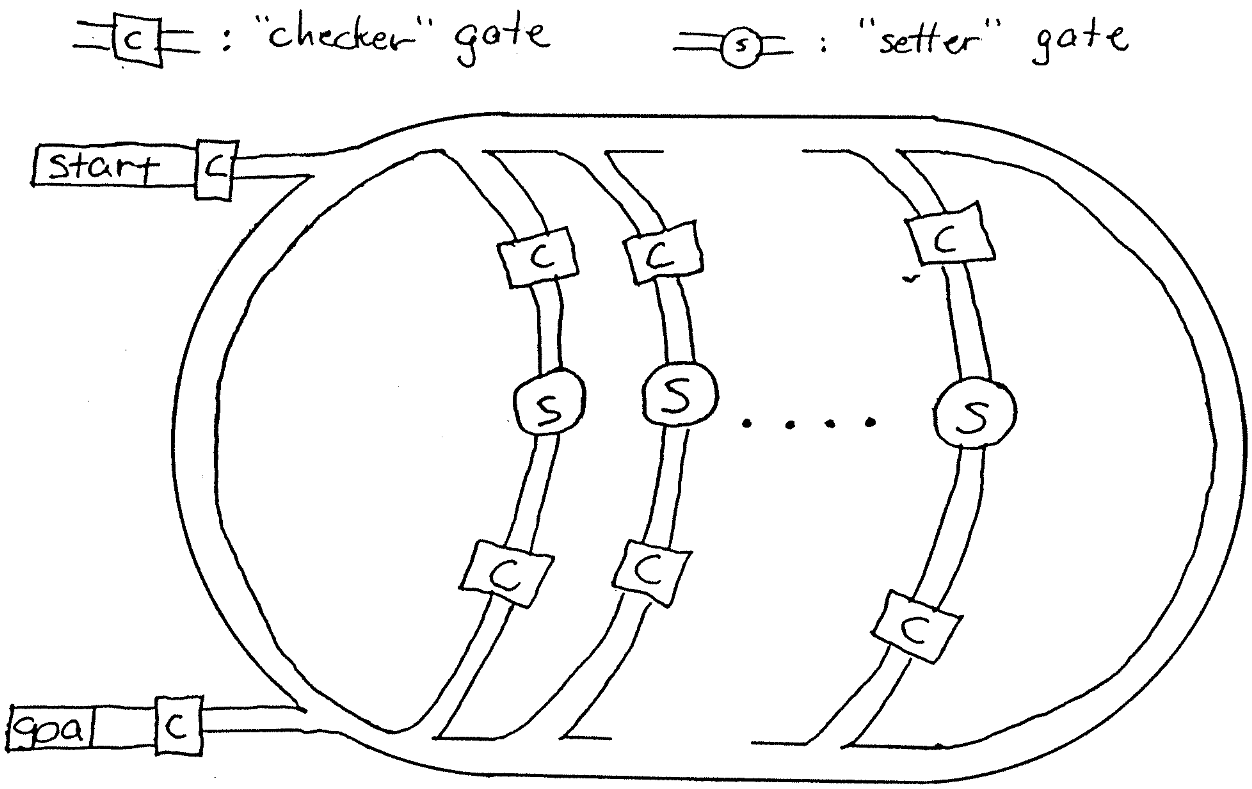

Figure 8: The workspace

configuration of the linkage. It can thus reach the goal position if and only if it can reach a final state, i.e. if and only if M accepts w.

We must be a little careful here, thoughwe can't allow M to be nondeterministic, because then it could "back up" through a transition into a configuration that it might not otherwise have been able to reach. We must also not allow M to have transitions from an accepting state, for the same reason. However, these restrictions can easily be met without affecting the construction.

But what is the size of this construction? Let m be the size of the description of the LBA M and the word w. The linkage certainly has size $O(m)$, so each gate or changer can be described with at most $O(m)$ line segments. Thus with $2m$ gates and m changers we get at most $O(m^2)$ line segments or arcs to describe a transition. The number of transitions in the Turing machine clearly cannot exceed m, so the whole construction takes only $O(n^3)$ line segments and arcs to describe.

Thus the reachability problem is P-space hard. The problem is also in P-space. This we can prove by showing that if there is a path to the goal, then there is one in which the moves of the linkage are restricted to ones that can be written down in polynomial space. Thus, a path to the goal can be found (if one exists) by repeatedly guessing a move, writing it down, and taking it. (This proof doesn't suffice to show that the problem is in NP, since an exponential number of moves may be required.)

We must therefore define elementary moves and show that they suffice in any reachability problem. We proceed as follows: Suppose the coordinates of the line segments in the original workspace are written out as decimal fractions. Then we can multiply them all by some number (which depends polynomially on the length of the description of the number with the most decimal digits) to get large integer coordinates. We can also assume that the links in the linkage are of integral length.

Now consider a unit grid just large enough to contain the workspace; say it is m units on a side. The segments of the workspace can only take on a polynomial number of different angles; in order to number these angles, we consider line segments joining every pair of grid points. These segments take on all of the possible angles with some axis. Thus, there are fewer than m^2 angles, and we can number them with a number which has a polynomial bound.

We then define elementary motions to be motions in which the endpoint of the linkage moves from some gridpoint to some other gridpoint, and each joint takes on some new angle. (We must have a fairly powerful notion of "elementary motion" in order to allow us to get out of certain jamming situations in a single motion.)

Now suppose, contrary to what we want to prove, that the linkage can get through to the goal using general motions, but not using these restricted motions. We will show in the full paper that this leads to a contradiction, and thus that these restricted motions and positions suffice. It then follows that the problem is in P-space.

4. The Complexity of Some Motion-Planning Problems

The complexity of reachability problems certainly represents a lower bound on the complexity of the associated motion planning problem, but can we get a better lower bound in the motion planning case? The answer is *yes*, we can show that some motion-planning problems take exponential time, because the number of elementary motions required (for any reasonable definition of "elementary motions") is exponential.

Corollary 1. The motion-planning problem for 2D linear linkages takes exponential time.

This fact is a result of the P-space completeness of the reachability problem: we find a linear bounded automaton which runs for exponential time and then accepts; since the LBA goes through an exponential number of transitions, the corresponding motion-planning problem takes exponential time. For example, consider the LBA which on input x of length n counts to 2^n and accepts. Clearly, under any reasonable definition of elementary moves, the associated linkage must go through through an exponential number of them before it reaches the goal.

Corollary 2. The motion-planning problem related to the warehouseman's problem takes exponential time.

The proof is similar.

5. Directions for Further Research

Thus, we know that the generalization of the basic mover's problem to an unbounded number of movable obstacles is P-space hard, and we know that the generalization of the problem to linear linkages in a non-convex workspace or graph-like linkages in a convex workspace is P-space complete. Are there other generalizations to the problem which result in hard reachability problems? Are there some reachability problems which are in NP? (For example, we know that the robot arm reachability problem is NP-complete in a particular workspace;[13] is that true for *every* fixed workspace, or is there some fixed workspace for which the problem is P-space complete?) Is there some general theorem which entails all of these motion-planning-complexity results?

13. Hopcroft, J. E., D. A. Joseph, and S. H. Whitesides, "On the mover's problem for 2-dimensional linkages," *SIAM J. Comput.*, (1984). To appear.

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