Introduction to $D_8 \times D_8$ and its Subgroup Lattice

Dan Schilcher

Faculty Mentor: Dandrielle Lewis

University of Wisconsin-Eau Claire

1. Purpose and History

Let A and B be groups and consider the direct product $A \times B$. What can one say about the subgroups of $A \times B$? In 1889, Edouard Goursat proved a theorem that provides the structure of subgroups in a direct product. We have applied the subgroup containment theorem given in [4] to the specific case of $D_8 \times D_8$. Our intention is to improve and make more effective this containment theorem for further use and to identify the extraspecial group of order 32 that lies inside of $D_8 \times D_8$.

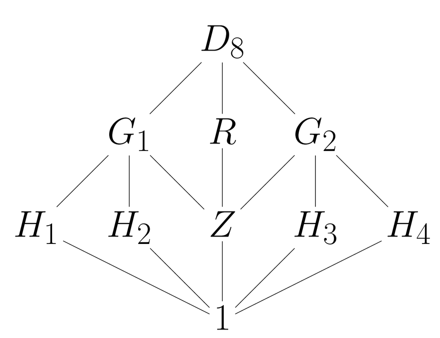
2. Properties of D_8

D_8, the dihedral group of order 8, is an extraspecial group. In particular, the center, frattini subgroup, and commutator subgroup coincide. It has the following presentation:

$$D_8 = < r, s \mid r^4 = s^2 = 1, rsr = s >$$

The subgroup lattice of D_8 and our notation for its subgroups are given below.

$D_8 = \langle r, s \rangle = \{1, r^2, r^3, s, rs, r^2s, r^3s\}$
$G_1 = \langle r^2, rs \rangle = \{1, r^2, rs, r^3s\}$
$G_2 = \langle r^2, s \rangle = \{1, r^2, s, r^2s\}$
$R = \langle r \rangle = \{1, r, r^2, r^3\}$
$Z = \langle r^2 \rangle = \{1, r^2\}$
$H_1 = \langle rs \rangle = \{1, rs\}$
$H_2 = \langle r^3s \rangle = \{1, r^3s\}$
$H_3 = \langle r^2s \rangle = \{1, r^2s\}$
$H_4 = \langle s \rangle = \{1, s\}$
$1 = \langle 1 \rangle = \{1\}$



3. Goursat's Theorem

<u>Theorem 1</u> [3]

Let A and B be groups. Then there exists a bijection between the set of all subgroups of A × B and the set of all triples $\left(\frac{I}{J}, \frac{L}{K}, \sigma\right)$, where $\frac{I}{J}$ is a factor group of A, $\frac{L}{K}$ is a factor group of B, and $\sigma : \frac{I}{J} \to \frac{L}{K}$ is an isomorphism between the factor groups.

Consider the projections $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$,and let $U \leq A \times B$ and $\left(\frac{I}{J}, \frac{L}{K}, \sigma\right)$ the triple associated with it. It follows that:

- $U \cap A \triangleleft \pi_A(U), U \cap B \triangleleft \pi_B(U)$, and $\sigma : \frac{\pi_A(U)}{U \cap A} \to \frac{\pi_B(U)}{U \cap B}$ is an isomorphism.
- Now, let $I = \pi_A(U), J = U \cap A, L = \pi_B(U)$, and $K = U \cap B$.
- The subgroup structure given by Goursat's Theorem is $U = \{(a, b) \mid a \in I, b \in L, \text{ and } \sigma(aJ) = bK\}$.

4. Containment Theorem

<u>Theorem 2</u> [4]

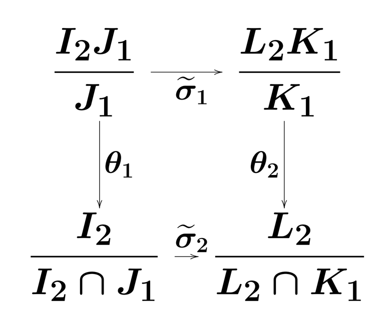
Suppose $U_2, U_1 \leq A \times B$, where U_1 is given by the triple $\left(\frac{I_1}{J_1}, \frac{L_1}{K_1}, \sigma_1\right)$ and U_2 is given by the triple $\left(\frac{I_2}{J_2}, \frac{L_2}{K_2}, \sigma_2\right)$. Then $U_2 \leq U_1$ if and only if:

1. $I_2 \leq I_1, J_2 \leq J_1, L_2 \leq L_1$, and $K_2 \leq K_1$

2. $\sigma_1\left(\frac{I_2 J_1}{J_1}\right) = \frac{L_2 K_1}{K_1}$

3. $\sigma_2\left(\frac{I_2 \cap J_1}{J_2}\right) = \frac{L_2 \cap K_1}{K_2}$

4.



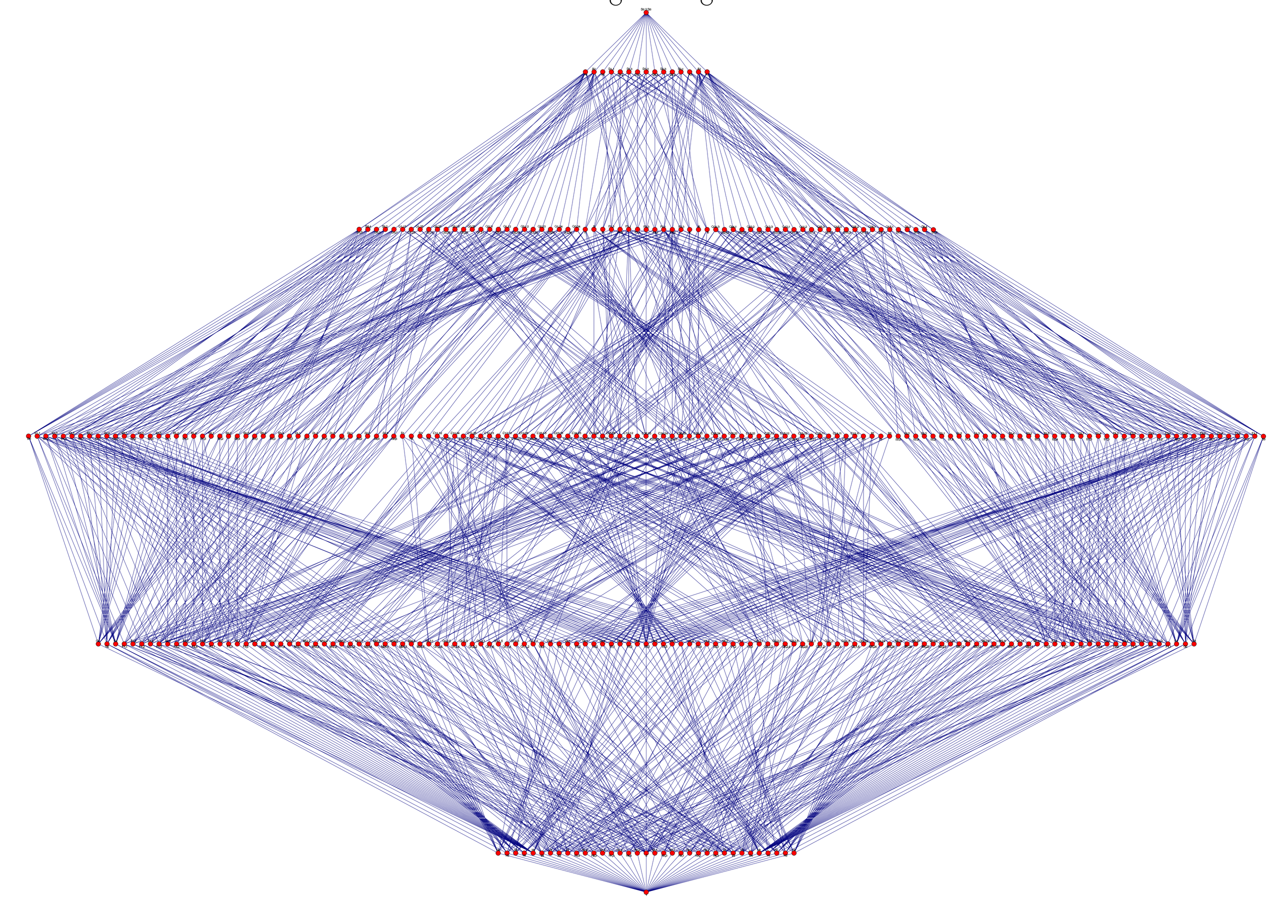
5. Results

- We first identified all factor groups of D_8.
- Secondly, we counted the number of subgroups by identifying all possible isomorphic factor groups and finding the automorphism group of the factor groups. This method is given in [5].
- We calculated the order of each subgroup using $|U| = \frac{|I| \cdot |L|}{|I/J|}$, given in [4], and concluded that $D_8 \times D_8$ has 389 subgroups:1 subgroup of order 64, 15 subgroups of order 32, 67 subgroups of order 16, 143 subgroups of order 8, 127 subgroups of order 4, 35 subgroups of order 2, 1 subgroup of order 1.
- We developed notation for each subgroup based on the projections and intersections which define it. This assisted in organizing the lattice and recognizing group structures. Given that we are working in a p-group, determining the maximal subgroups of each subgroup is sufficient to give the group's structure. A chart with a sampling of the notation, number of subgroups with that notation, group structure, and maximal subgroups of a group of order 4 through 32 follows:

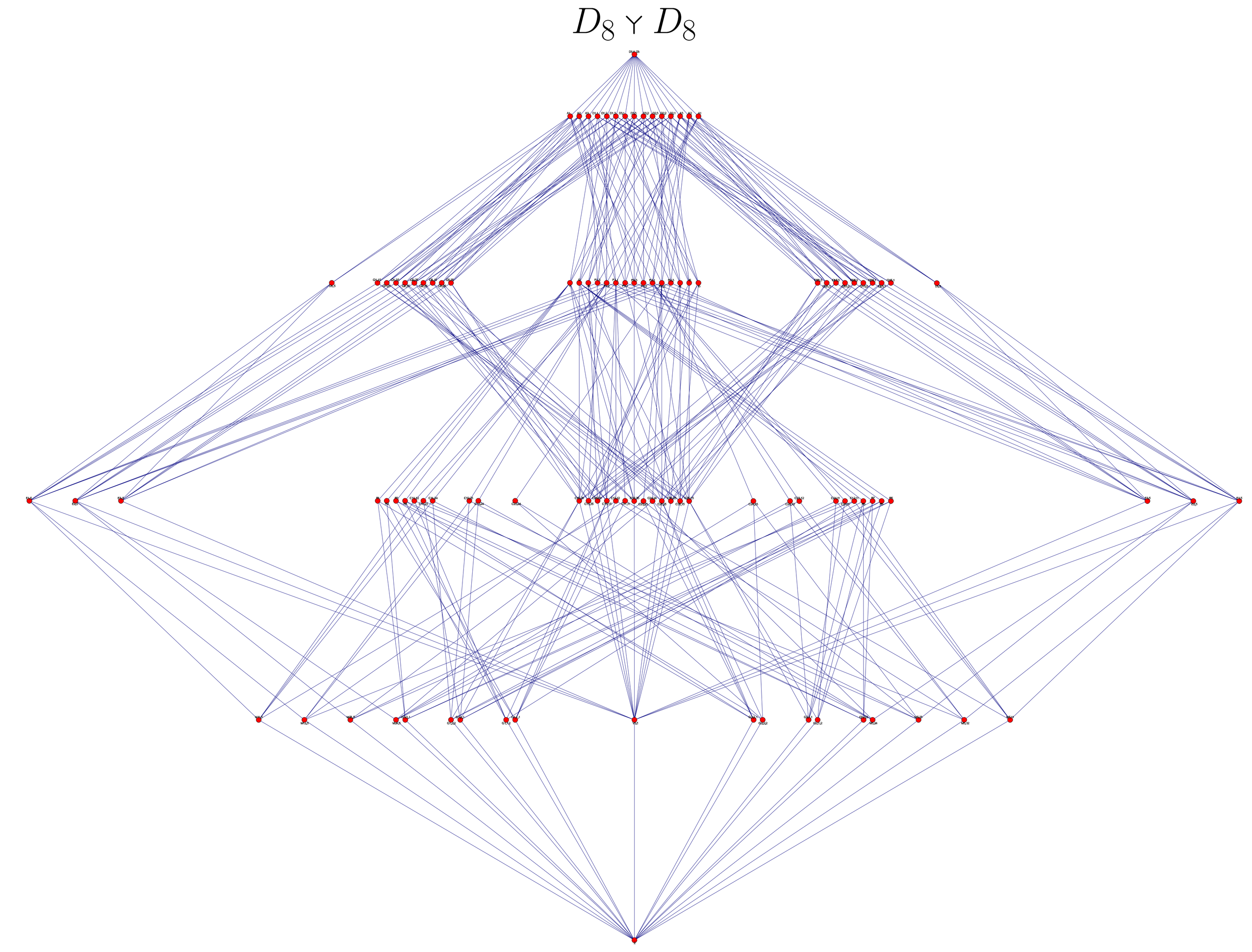
Notation	Order	I,J	L,K	# of Subgps.	Group Structure	Maximal Subgroups
$\overrightarrow{D_8}$	8	1,1	D_8, D_8	1	D_8	$\overrightarrow{G_1}, \overrightarrow{G_2}, \overrightarrow{G_3}$
$Q_{i,35}$ $1 \leq i \leq 3$	16	D_8, G_i	R, Z	3	$C_4 \rtimes C_4$	$\overleftarrow{E_{i,5}}, C_{j5,35}, C_{k5,35}$ $1 \leq j \leq 3, j \neq i$ $1 \leq k \leq 3, k \neq i, j$

6. Lattices

$D_8 \times D_8$



$D_8 \curlyvee D_8$



7. Future Research

Now that we have finished determining subgroup containment for all of the subgroups of $D_8 \times D_8$ and identified the extraspecial group of order 32, we will examine possible applications for $D_8 \times D_8$. We will also try to characterize containment of normality inside of a direct product. This process of determining subgroup containment can be applied to any direct product of groups and is highly methodical. We would like to make it more effective by possibly writing a program to implement it.

References

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[4] Lewis, D. & Brewster, B. (2011). A Characterization Of Subgroup Containment In Direct Products. *Ricerche di Matematica.* **61**(2) p. 347-354

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Acknowledgments

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