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ABSTRACT

Consider the system of linear algebraic equations $L_{h,k}U = f$ which arises from the finite-difference discretization of the singularly perturbed elliptic operator

$$L_K u := -\left\{ \frac{\partial}{\partial x}\, a\, \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\, b\, \frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\, b\, \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\, c\, \frac{\partial u}{\partial y} \right\} + K^{\sigma} \left\{ d \frac{\partial u}{\partial x} + e\, \frac{\partial u}{\partial y} \right\} + Kpu \tag{1.5a}$$

where $1 << K < \infty$, $0 \leq \sigma \leq 1$. In this work we are concerned with preconditioning operators $\tilde{L}_{K,h}$ so that

(i) The condition number $[(\tilde{L}_{K,h})^{-1} L_{K,h}]$ and the condition number $[L_{K,h}(\tilde{L}_{K,h})^{-1}]$ is bounded independently of h <u>and</u>

(ii) there are bounds on the growth of these condition numbers as a function of K .

1. Introduction

The numerical solution of elliptic boundary-value problems leads to the related problem of actually obtaining the solution of a large, sparse system of linear equations

$$A_n U = F \tag{1.1}$$

where A_n is an $n \times n$ matrix and n is the number of grid points. There is a large literature connected with the analysis of iterative methods for the solution of (1.1) - see [1], [2].

Almost all iterative methods, including the multigrid methods [3] can be cast in the framework of a preconditioning followed by iterative improvement. That is, we consider the system

$$B_n^{-1} A_n U = B_n^{-1} F \, , \tag{1.1}$$

or the system

$$(A_n B_n^{-1}) V = A_n (B_n^{-1} V) = F \, , \tag{1.3a}$$

$$U = B_n^{-1} V \, . \tag{1.3b}$$

Then an iterative method is applied to this new problem. Of course, one chooses B_n so that B_n^{-1} is "easy" to compute. Furthermore, it is often advantageous to choose B_n to be positive definite symmetric. With the practical success of multigrid methods for uniformally elliptic problems with positive symmetric part there is a particular interest in preconditioned iterative methods for which the condition number of $B_n^{-1} A_n$ [or $A_n B_n^{-1}$] is bounded independent of the dimension n. It is easy to develop iterative methods that yield estimates of the form

$$\|\varepsilon^j\| \leq K_0 \left[\frac{k-1}{k+1}\right]^j \|\varepsilon^0\| \tag{1.4}$$

where $k = \sqrt{C}$ or C [C = condition number $(B_n^{-1} A_n)$]. Thus, in the case where C is independent of n one has an iterative method which is competitive with multigrid and whose convergence rate is independent of n.

Several authors have studied this problem [4]-[6] in the special case where A_n (and B_n) is positive definite or has a positive definite symmetric part. In that case one can analyze the preconditioned iterative method using the concept of "Spectrally Equivalent Operators" introduced by D'Yakanov [7]. More recently there have been results for the truly indefinite case [8]-[10]. These problems require the concept of "Norm Equivalence" (see [8] for a detailed discussion).

In this work we are concerned with singularly perturbed boundary-value problems of the form

$$L_K u := -\left\{ \frac{\partial}{\partial x}\, a\, \frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\, b\, \frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\, b\, \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\, c\, \frac{\partial u}{\partial y} \right\} \\ + K^\sigma \left\{ d \frac{\partial u}{\partial x} + e\, \frac{\partial u}{\partial y} \right\} + Kpu = f \quad \text{in } \Omega \tag{1.5a}$$

and

$$\underline{\text{specified boundary conditions}} \quad \text{on} \quad \partial\Omega \tag{1.5b}$$

where $1 < K < \infty$, $0 \le \sigma \le 1$. The ellipticity of the system is expressed by the requirement that there are constants $0 < q \le Q$ such that: for all $(x, y) \in \bar{\Omega}$ and all $\xi = (\xi_1, \xi_2)$ we have

$$q\,(\xi_1^2 + \xi_2^2) \le a\,\xi_1^2 + 2b\,\xi_1\xi_2 + c\,\xi_2^2 \le Q\,(\xi_1^2 + \xi_2^2)\,. \tag{1.5c}$$

The domain Ω is taken as the unit square

$$\Omega := (x, y) = 0 < x, y < 1 \tag{1.6a}$$

and the boundary conditions take the form

$$u \ \text{ or } \ \frac{\partial u}{\partial \nu} = 0 \tag{1.6b}$$

along an entire side. The operator L_K is approximated by a finite-difference operator $L_{K,h}$. This is the situation discussed in [8, section 3]. In that work K is kept fixed (indeed it doesn't appear in the discussion) and the authors concentrate on preconditionings B_n for which condition number $(B_n^{-1} A_n)$ is bounded independently of n. In this work we focus our attention on both h and K and discuss preconditioning operators $\tilde{L}_{K,h}$ so that

(i) The condition number $[(\tilde{L}_{K,h})^{-1} L_{K,h}]$ and the condition number $[L_{K,h}(\tilde{L}_{K,h})^{-1}]$ is bounded independently of h <u>and</u>

(ii) there are bounds on the growth of these condition numbers as a function of K.

Throughout this work the symols M, M_0, $\bar{M}$, etc. will denote constants which depend on the coefficients a, b, c, d, e, p and the first derivatives of the coefficients a, b, c, d, e but not on h or K.

Remark 1.1: All of our results are easily extended to the case of the Direchlet problem in a convex polygonal domain Ω whose boundary agrees with the discrete boundary.

Remark 1.2: The problem (1.5) is frequently rewritten in terms of $K^{-1} := \varepsilon$. These problems arise in models of reaction-diffusion and convection-diffusion. To maintain accuracy, it is often necessary that K and h be constrained by a condition of the form $Kh =$ constant as $K \to \infty$ or $h \to 0$.

Remark 1.3: The operator employed as a preconditioner is typically based on the positive definite symmetric part of the given operator. If $p \leq 0$ in (1.5a) this operator would correspond to the second order derivative terms, which do not depend on K . To see why it is important to consider K as well as h when preconditioning (1.5), consider the following simple stationary wave propagation model $L_K u = -\triangle u - (K + i\delta K^r)u = f$ in Ω , $u = 0$ on $\partial\Omega$, where $\delta > 0$ and $1 \geq r \geq 0$. Suppose that L_K is approximated by the discrete operator, $L_{K,h}$, and we use the discrete Laplacian, $L^0_{K,h}$, as a preconditioner. Since $L_{K,h}$ and $L^0_{K,h}$ have constant coefficients, it can be seen by a simple eigenvalue analysis that

$$C\left((L^0_{K,h})^{-1} L_{K,h}\right) = 0(K^{2-r})$$

as $K \to \infty$ and this estimate is sharp. On the other hand, if we precondition by $L^+_{K,h} \equiv L^0_{K,h} + K$, it can be readily seen that

$$C\left((L^+_{K,h})^{-1} L_{K,h}\right) = 0(K^{1-r})$$

as $K \to \infty$. Hence $L^+_{K,h}$ is a much better preconditioner than $L^0_{K,h}$ even for moderate values of K . Preconditioners for more general variable coefficient wave propagation models in general domains are analyzed in [10] using finite element error analysis.

We briefly outline the remainder of the paper. In Section 2 we define our notation. In Section 3 we establish some basic estimates for the finite difference operators. We state and prove the main results in Section 4 (Theorems 4.1-4.4). The analysis uses the concept of norm equivalence (see (4.1)). The preconditioner can be any operator that is norm equivalent to B_h defined by (4.19), uniformly in K and h . It is shown in [11] how multigrid methods can be employed to give specific preconditioners of this kind. The condition number estimates in Theorems 4.1-4.4 will typically be much worse for large K when positive definite symmetric preconditioning operators other than (4.19) are used. This can be seen in general using the analysis in Section 4 and was demonstrated in Remark 1.3 above. Finally, it is shown in Section 5 how to transform Convection Diffusion type operators into a simpler operator. This new operator can then be preconditioned as in Section 4 with the resulting condition number bounded independently of K and h as $K \to \infty$.

2. Preliminaries I: Notation

Let p and ℓ be integers and set

$$\Delta x = \frac{1}{p+1}\,, \quad \Delta y = \frac{1}{\ell+1}\,, \quad h = \max(\Delta x, \Delta y) \tag{2.1}$$

$$\Omega_h = \{(x_k, y_j) \in \Omega\,, \ \ x_k = k\Delta x\,, \ \ y_j = j\Delta y\} \tag{2.2a}$$

$$\partial\Omega_h = \{(x_k, y_j) \in \partial\Omega\,, \ \ x_k = k\Delta x\,, \ \ y_j = j\Delta y\} \tag{2.2b}$$

$$\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h\,. \tag{2.2c}$$

Note: If $(x_k, y_j) \in \partial\Omega_h$ then either $k = 0$ or $p+1$ or $j = 0$ or $\ell+1$.

Let S_h denote the set of grid vectors $V = \{V_{k,j}\}$ defined on $\bar{\Omega}_h$ that satisfy the appropriate discrete boundary conditions. Thus, if the boundary conditions associated with L_k require

(i) $U(0,y) = 0$, then $V_{0,j} = 0$, $j = 0, 1, ...(\ell+1)$,

(ii) $U_y(x,1) = 0$, then $V_{k,\ell+1} = V_{k,\ell}$, $k = 1, 2, ...p$, and so on.

Remark 2.1: In the case of the boundary condition (ii) above one would probably choose Δy differently so that

$$y_\ell = 1 - \frac{1}{2}\Delta y\,, \quad y_{\ell+1} = 1 + \frac{1}{2}\Delta y\,.$$

However, such a modification has no effect on our analysis. Hence, for the purposes of this discussion we formulate the discrete (finite-dimensional) spaces as above.

Let $G(x,y)$ be a function defined on $\bar{\Omega}$. We write

$$G_{k,j} = G(x_k, y_j)\,, \quad G_{k+\frac{1}{2},j} = G(x_k + \frac{1}{2}\Delta x, y_j)\,, \text{etc.} \tag{2.3}$$

Let $V \in S_h$; we denote the usual forward, backward, and centered difference quotients denoted by subscripts in the following manner

$$[V_x]_{k,j} = \frac{1}{\Delta x}\left[V_{k+1,j} - V_{k,j}\right]\,, \tag{2.4a}$$

$$[V_{\bar{x}}]_{k,j} = \frac{1}{\Delta x}\left[V_{k,j} - V_{k-1,j}\right]\,, \tag{2.4b}$$

$$[V_{\hat{x}}]_{k,j} = \frac{1}{2\Delta x}\left[V_{k+1,j} - V_{k-1,j}\right]. \tag{2.4c}$$

with similar notation for difference quotients in the y directions. Let T_x, T_y denote the shift operators

$$[T_x V]_{k,j} = V_{k+1,j}\,, \quad [T_y V]_{k,j} = V_{k,j+1}\,, \tag{2.5a}$$

$$[T_x^{-1} V]_{k,j} = V_{k-1,j}\,, \quad [T_y^{-1} V]_{k,j} = V_{k,j-1}\,. \tag{2.5b}$$

With this notation, we are able to describe finite difference operators that correspond to the differential operator L_k.

Let

$$\tilde{a}(x,y) = a(x + \frac{1}{2}\Delta x, y) \tag{2.6a}$$

$$\tilde{c}(x,y) = c(x, y + \frac{1}{2}\Delta y) \tag{2.6b}$$

and for $U \epsilon S_h$ we define

$$[L_{0,h}U] = -\{(\tilde{a}U_x)_{\bar{x}} + (bU_{\hat{x}})_{\hat{y}} + (bU_{\hat{y}})_{\hat{x}} + (\tilde{c}U_y)_{\bar{y}}\} \tag{2.7a}$$

$$[L_{K,h}U] = [L_{0,h}U] + K^\sigma\{dU_{\hat{x}} + eU_{\hat{y}}\} + KpU\,. \tag{2.7b}$$

One can think of (2.7b) as the "natural centered-difference" approximation of L_k. However, there are other reasonable and useful difference approximations. Let

$$[A_{1,h}U] = -\{aU_{x\bar{x}} + 2bU_{\hat{x}\hat{y}} + cU_{y\bar{y}}\}\,, \tag{2.8a}$$

and

$$\begin{aligned}[L^{(1)}_{K,h}U] = [A_{1,h}U] &+ \left(K^\sigma d - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) U_{\hat{x}} \\ &+ \left(K^\sigma e - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x}\right) U_{\hat{y}} + Kpu\,.\end{aligned} \tag{2.8b}$$

Finally, we may also make use of the operator $L^{(2)}_{K,h}$ given by

$$\begin{aligned}[L^{(2)}_{K,h}U] = [L_{0,h}U] &+ \frac{K^\sigma}{2}\{(dU)_{\hat{x}} + dU_{\hat{x}} + (eU)_{\hat{y}} + eU_{\hat{y}}\} \\ &+ \left(Kp - \frac{K^\sigma}{2}\frac{\partial d}{\partial x} - \frac{K^\sigma}{2}\frac{\partial e}{\partial y}\right) U\,.\end{aligned} \tag{2.9}$$

Each of these difference operators has certain advantages. For example the operator $L^{(2)}_{K,h}$ clearly exhibits its symmetric and skew-symmetric parts. These are

$$S_{K,h}U = L_{0,h}U + \left(Kp - \frac{K^\sigma}{2}\frac{\partial d}{\partial x} - \frac{K^\sigma}{2}\frac{\partial e}{\partial y}\right)U \tag{2.10a}$$

$$\tilde{S}_{K,h}U = \frac{K^\sigma}{2}\{(dU)_{\hat{x}} + dU_{\hat{x}} + (eU)_{\hat{y}} + eU_{\hat{y}}\} \tag{2.10b}$$

We now introduce some norms and semi-norms defined on S_h . For every $V \epsilon S_h$ we set

$$\|V\|_g = \left[\Delta x\, \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} |V_{k,j}|^2\right]^{\frac{1}{2}}, \tag{2.11a}$$

$$|V|_{g,1} = \left[\Delta x \Delta y \sum_{k=0}^{p} \sum_{j=0}^{\ell} [(V_x)^2_{kj} + (V_y)^2_{k,j}]\right]^{\frac{1}{2}}, \tag{2.11b}$$

$$|V|_{g,2} = \{\|V_{x\bar{x}}\|^2_g + 2\|V_{\hat{x}\hat{y}}\|^2_g + \|V_{y\bar{y}}\|^2_g\}^{\frac{1}{2}}, \tag{2.11c}$$

$$\|V\|_{g,1} = \{\|V\|^2_g + |V|^2_{g,1}\}^{\frac{1}{2}}, \tag{2.11d}$$

$$\|V\|_{g,2} = \{\|V\|^2_{g,1} + |V|^2_{g,2}\}^{\frac{1}{2}}. \tag{2.11e}$$

Finally, if B_h is a linear operator acting in S_h , we define

$$\|B_h\|_g = \max_{0 \neq V \epsilon S_h} \frac{\|B_h V\|_g}{\|V\|_g}.$$

Remark 2.2. The operators in (2.7)-(2.9) may be defined analogously with the first order centered differences replaced by forward or backward differences (e.g. $dU_{\hat{x}}$ replaced by dU_x , etc.). This is often the case with various singular perturbation models. It can be seen that the results of this paper (in particular, Theorems 4.1-4.4) hold using analogous arguments.

3. Preliminaries II: Estimates

In this section we collect some basic estimates.

Lemma 3.1: Let $V \in S_h$. Let $A_{1,h}$ be given by (2.8a). Let $0 < q \leq Q$ be the constants of (1.5c). Then

$$|V|_{g,2}^2 \leq \frac{2Q^2}{q^4} \|A_{1,h}V\|_g^2 \,. \tag{3.1}$$

Proof: See Lemma A.1 of [8] and [12].

Lemma 3.2: Let $V \in S_h$. Let $L_{0,h}$ be given by (2.7a). Then, there are constants $M > 0$, $h_0 > 0$ such that, for $0 < h \leq h_0$ we have

$$|V|_{g,1}^2 \leq \frac{2}{q}\left[\Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [L_{0,h}V]_{kj}[V]_{kj}\right] . \tag{3.2}$$

Proof: We give a rather detailed proof because this result seems to be a "folk theorem" but we have found no complete proof.

Let us observe that

$$q \leq \min \{a(x,y)\, , \, c(x,y) : (x,y) \in \bar{\Omega}\} \,. \tag{3.3}$$

A straight-forward summation-by-parts argument gives

$$J := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [L_{0,h}V]_{kj}[V]_{kj} = I_1 + 2I_2 + I_3 \tag{3.4}$$

where

$$I_1 := \Delta x \Delta y \sum_{k=0}^{p} \sum_{j=1}^{\ell} (\tilde{a})_{kj}(V_x)_{kj}^2 \,, \tag{3.5a}$$

$$I_2 := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} [bV_{\hat{x}}V_{\hat{y}}]_{k,j} \,, \tag{3.5b}$$

$$I_3 := \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=0}^{\ell} [\tilde{c}\,(V_y)^2]_{k,j} \,. \tag{3.5c}$$

First let us rewrite I_2 in terms of V_x, $V_{\bar{x}}$, V_y, $V_{\bar{y}}$. We have

$$I_2 = \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} b \left[V_x V_y + V_x V_{\bar{y}} + V_{\bar{x}} V_y + V_{\bar{x}} V_{\bar{y}} \right] . \tag{3.6}$$

Turning to I_1 and I_3 we write

$$\tilde{a} = a + \frac{\Delta x}{2} \left(\frac{\partial a}{\partial x} \right)' , \quad \tilde{a} = (T_x a) - \frac{\Delta x}{2} \left(\frac{\partial a}{\partial x} \right)'' . \tag{3.7}$$

Thus we have

$$\begin{aligned} I_1 = {} & \frac{1}{2} \Delta x \Delta y \sum_{j=1}^{\ell} \left[a_{0,j} (V_x)_{0,j}^2 + a_{p+1,j} (V_x)_{p,j}^2 \right] \\ & + \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} a \left[(V_x^2) + (V_{\bar{x}})^2 \right] + F_1 , \end{aligned} \tag{3.8a}$$

$$\begin{aligned} I_3 = {} & \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \left[c_{k,0} (V_y)_{k,0}^2 + c_{k,\ell+1} (V_y)_{k,\ell}^2 \right] \\ & + \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \sum_{j=1}^{\ell} c \left[(V_y)^2 + (V_{\bar{y}})^2 \right] + F_2 , \end{aligned} \tag{3.8b}$$

where

$$|F_1| + |F_2| \le M h |V|_{g,1}^2 . \tag{3.8c}$$

Collecting these formulae yields

$$\begin{aligned} J = {} & \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} \left[a (V_x)^2 + 2b V_x V_y + c (V_y)^2 \right] \\ & + \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} \left[a (V_x)^2 + 2b V_x V_{\bar{y}} + c (V_{\bar{y}})^2 \right] \\ & + \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} \left[a (V_{\bar{x}})^2 + 2b V_{\bar{x}} V_y + c (V_y)^2 \right] \\ & + \frac{\Delta x \Delta y}{4} \sum_{k=1}^{p} \sum_{j=1}^{\ell} \left[a (V_{\bar{x}})^2 + 2b V_{\bar{x}} V_{\bar{y}} + c (V_{\bar{y}})^2 \right] \\ & + \frac{1}{2} \Delta x \Delta y \sum_{j=1}^{\ell} \left[a_{0,j} (V_x)_{0,j}^2 + a_{p+1,j} (V_x)_{p,j}^2 \right] \\ & + \frac{1}{2} \Delta x \Delta y \sum_{k=1}^{p} \left[c_{k,0} (V_y)_{k,0}^2 + c_{k,\ell+1} (V_y)_{k,\ell}^2 \right] + F_1 + F_2 . \end{aligned} \tag{3.9}$$

Observe that

$$(V_{\bar{x}})_{k,j} = (V_x)_{k-1,j} \tag{3.10a}$$

and

$$a_{0,j} \geq q\,, \quad a_{p+1,j} \geq q\,, \quad c_{k,0} \geq q\,, \quad c_{k,\ell+1} \geq q\,. \tag{3.10b}$$

Thus using (1.5c) we obtain

$$q\,|V|^2_{g_1,1} \leq J + |F_1| + |F_2|\,. \tag{3.11}$$

That is

$$q\,|V|^2_{g,1} \leq J + Mh\,|V|^2_{g,1}\,.$$

Thus, the lemma follows for $h_0 = \dfrac{q}{2M}$.

Lemma 3.3: There are linear operators $\tilde{E}_h, \hat{E}_h$ defined on S_h such that

$$L_{K,h}U = L^1_{K,h}U + \tilde{E}_hU \tag{3.12a}$$

$$L_{K,h}U = L^{(2)}_{K,h}U + \hat{E}_hU \tag{3.12b}$$

and

$$\|\tilde{E}_hU\|_g \leq M\,\|U\|_{g,1}\,, \quad \|\hat{E}_hU\|_g \leq M\,(1+K^\sigma)h\,\|U\|_{g,1}\,. \tag{3.13}$$

Proof: These estimates follow from the estimates and formulae of lemma A.3 of [8]. For estimate (3.13) we require that d and e have bounded second derivatives.

Lemma 3.4: There are linear operators $F_h\,,\ F^{(1)}_h$ defined on S_h such that: for every real α and every $V \in S_h$ we have

$$\begin{aligned} e^{-\alpha x}[L_{K,h}(e^{\alpha x}V)] &= [L_{0,h}V] + K^\sigma[dV_{\hat{x}} + eV_{\hat{y}}] \\ &\quad + [Kp + K^\sigma d\alpha]V + F_hV \end{aligned} \tag{3.14a}$$

$$\begin{aligned} e^{-\alpha x}[L^{(1)}_{K,h}(e^{\alpha x}V)] &= [L_{0,h}V] + K^\sigma[dV_{\hat{x}} + eV_{\hat{y}}] \\ &\quad + [Kp + K^\sigma d\alpha]V + F^{(1)}_hV \end{aligned} \tag{3.14b}$$

and

$$\|F_hV\|_g \leq M(1 + K^\sigma\alpha^4h)\|V\|_{g,1} \tag{3.15a}$$

$$\|F_h^{(1)} V\|_g \leq M(1 + K^\sigma \alpha^4 h)\|V\|_{g,1} \; . \tag{3.15b}$$

Proof: These estimates follow from Lemma 3.3 and several straightforward computations. For example, it is easy to verify that

$$e^{-\alpha x}[a(e^{\alpha x}V)_x]_{\bar{x}} = (aV_x)_{\bar{x}} + 0[(\alpha^2 + \alpha^4 h)\|V\|_{g,1}] \; , \tag{3.16a}$$

$$e^{-\alpha x}(e^{\alpha x}V)_{\hat{x}} = V_{\hat{x}} + \alpha V + 0((\alpha^2 + \alpha^4 h)\|V\|_{g,1}) \; . \quad \blacksquare \tag{3.16b}$$

Corollary: There are linear operators $\tilde{F}_h$, $\tilde{F}_h^{(1)}$ defined on S_h such that: for every real α and every $V \in S_h$ we have

$$\begin{aligned} e^{-\alpha x}[L_{K,h}(e^{\alpha x}V)] = L_{0,h}V &+ \frac{1}{2} K^\sigma [dV_{\hat{x}} + (dV)_{\hat{x}}] \\ &+ \frac{1}{2} K^\sigma [eV_{\hat{y}} + (eV)_{\hat{y}}] + [Kp + K^\sigma d\alpha]V \\ &+ F_h V + K^\sigma \tilde{F}_h V \; , \end{aligned} \tag{3.17a}$$

$$\begin{aligned} e^{-\alpha x}[L_{K,h}^{(1)}(e^{\alpha x}V)] = L_{0,h}V &+ \frac{1}{2} K^\sigma [dV_{\hat{x}} + (dV)_{\hat{x}}] \\ &+ \frac{1}{2} K^\sigma [eV_{\hat{y}} + (eV)_{\hat{y}}] + [Kp + K^\sigma d\alpha]V \\ &+ F_h^{(1)} V + K^\sigma \tilde{F}_h^{(1)} V \; , \end{aligned} \tag{3.17b}$$

and

$$\|\tilde{F}_h V\|_g \leq M(h \mid V \mid_{g,1} + \|V\|_g) \; , \quad \|\tilde{F}_h^{(1)} V\|_g \leq M(h \mid V \mid_{g,1} + \|V\|_g) \; . \tag{3.17c}$$

Proof: Direct computation based on Lemma 3.4. $\blacksquare$

4. Preconditioning Estimates

In this section we are concerned with the condition numbers of families $\{A_h B_h^{-1}\}$ and $\{B_h^{-1} A_h\}$ where A_h denotes one of $L_{k,h}, L_{k,h}^{(1)}$ or $L_{k,h}^{(2)}$ and B_h is a suitable chosen invertible operator defined on S_h. We first discuss the families $\{A_h B_h^{-1}\}$ and then obtain results for the families $\{B_h^{-1} A_h\}$ by the use of the adjoint relationships. One major tool is the following concept.

Definition: Let A_h , B_h be defined on S_h and assume that both are invertible. We say that $\{A_h\}$ is uniformly norm equivalent to $\{B_h\}$ if there exist positive constants $0 < \alpha < \beta$ such that,

$$\alpha \|B_h U\|_g \leq \|A_h U\|_g \leq \beta \|B_h U\|_g \ , \ \ \forall U \in S_h \ , \tag{4.1}$$

α and β are independent of h . This concept was used in [10] and studied in depth in [8]. In the cases of interest in this work, A_h and B_h will depend on the parameter $K >> 0$. In some of these cases it is not possible to find α and β independent of K . Hence we will have occasion to deal with the case where α and β depend on K.

Lemma 4.1: Suppose (4.1) holds. Then

$$\|A_h B_h^{-1}\|_g \leq \beta \ , \tag{4.2a}$$

$$\|B_h A_h^{-1}\|_g \leq \frac{1}{\alpha} \ , \tag{4.2b}$$

so that

$$C(A_h B_h^{-1}) \leq \beta / \alpha \ . \tag{4.3}$$

Proof: To obtain (4.2a) we set $U = B_h^{-1} V$. To obtain (4.2b) we set $U = A_h^{-1} V$. Since $(A_h B_h^{-1})^{-1} = B_h A_h^{-1}$ we obtain (4.3) from (4.2a), (4.2b). ∎

In the work that follows B_h will either be another discrete elliptic operator $\tilde{L}_{K,h}$ or an operator taken from a family $\{B_h\}$ which is uniformly norm equivalent to $\tilde{L}_{K,h}$, with respect to K as well as h .

Lemma 4.2: Let B_h be a discrete elliptic operator of the form $L_{K,h}$, $L^{(1)}_{K,h}$ or $L^{(2)}_{K,h}$. Let $\sigma = 1$. Assume that $d(x,y) \geq d_0 > 0$ and $Kh \leq M_0$. Then there is a $K_0 > 0$ and an $h_0 > 0$ such that, for all $K \geq K_0$ and all h, $0 < h \leq h_0$ we have the following estimates

$$\|B_h^{-1}\|_g \leq M/K \tag{4.4a}$$

$$|B_h^{-1} f|_{g,1} \leq M K^{-\frac{1}{2}} \|f\|_g \tag{4.4b}$$

$$|B_h^{-1} f|_{g,2} \leq M(1 + K^{\frac{1}{2}}) \|f\|_g \tag{4.4c}$$

Proof: Let

$$B_h U = f \,. \tag{4.5}$$

Let $U = e^{\alpha x} V$, where the positive number α will be determined later. Invoking the Corollary to Lemma 3.4 we see that there is a constant M_0, depending only on the coefficients of $L_{K,h}$ but not on h or K so that

$$\begin{aligned} L_{0,h} V + \frac{1}{2} K \, [dV_{\hat{x}} + (dV)_{\hat{x}} + eV_{\hat{y}} + (eV)_{\hat{y}}] \\ + K \, [p + \alpha d] V = e^{-\alpha x} f + Q_h V \end{aligned} \tag{4.6a}$$

where

$$\|Q_h V\|_g \leq M_0 (1 + \alpha^4) \|V\|_{g,1} + K \, M_0 \|V\|_g \,. \tag{4.6b}$$

Multiplying by V and summing over Ω_h we use Lemma 3.2 to obtain

$$\begin{aligned} \frac{q}{2} \, |V|^2_{g,1} + K \, [\alpha d_0 - \|p\|_\infty] \|V\|^2_g \leq \|f\|_g \cdot \|V\|_g \\ + M_0 (1 + \alpha^4) \|V\|_{g,1} \|V\|_g + K M_0 \|V\|^2_g \,. \end{aligned} \tag{4.7}$$

Let

$$\alpha = \alpha_0 = \frac{\|p\|_\infty + M_0 + 2}{d_0} \,. \tag{4.8a}$$

Then (4.7) and elementary inequalities yield

$$\frac{q}{2} \, |V|^2_{g,1} + 2K \|V\|^2_g \leq \|f\|_g \cdot \|V\|_g + M_1 \|V\|^2_g + \frac{q}{4} \, |V|^2_{g,1} \tag{4.9}$$

where

$$M_1 = \frac{M_0^2 (1 + \alpha_0^4)^2}{q} + M_0 (1 + \alpha_0^4) \,.$$

Thus, for $K \geq M_1$ we have

$$\frac{q}{4}|V|_{g,1}^2 + K\|V\|_g^2 \leq \|f\|_g \cdot \|V\|_g \tag{4.10}$$

which immediately implies

$$\|U\|_g \leq e^{\alpha_0}\|V\|_g \leq e^{\alpha_0}K^{-1}\|f\|_g \ ,$$

and

$$|U|_{g,1}^2 \leq e^{2\alpha_0}(1+2\alpha_0)^2|V|_{g,1}^2 \leq e^{2\alpha_0}(1+2\alpha_0)^2 \,\frac{4}{q}\,\frac{1}{K}\,\|f\|_g^2 \ .$$

Thus, we have proven (4.4a) and (4.4b).

Using Lemma 3.3 and (2.8a), (2.8b) we see that

$$A_{1,h}U + K(dU_{\hat{x}} + eU_{\hat{y}}) + KpU = f + R_hU \tag{4.11a}$$

where

$$\|R_hU\|_g \leq M\|U\|_{g,1} \ . \tag{4.11b}$$

Using (4.4a) and (4.4b) we see that

$$A_{1,h}U = F$$

where

$$\|F\|_g \leq (1 + K^{\frac{1}{2}})\, M\|f\|_g \ .$$

Using Lemma 3.1 we see that 4.4c holds. ∎

Remark : The estimates (4.4) remain valid if the assumption $d(x,y) \geq d_0 > 0$ is replaced by $-d(x,y) \geq d_0 > 0$, or $e(x,y) \geq e_0 > 0$, or $-e(x,y) \geq e_0 > 0$. In the first case α_0 is replaced by the negative of (4.8a). In the latter two cases $e^{\alpha x}$ is replaced by $e^{\alpha y}$.

Lemma 4.3: Let B_h be a discrete elliptic operator of the form $L_{K,h}$, $L_{K,h}^{(1)}$ or $L_{K,h}^{(2)}$. Assume that $Kh \leq M_0$. Let $\sigma < 1$ and $p(x,y)$ satisfy

$$p(x,y) \geq p_0 > 0 \ . \tag{4.12}$$

Then there is a $K_0 > 0$ and an $h_0 > 0$ such that, for all $K \geq K_0$ and all h, $0 < h \leq h_0$ we have the estimates (4.4a), (4.4b) and

$$|B_h^{-1} f|_{g,2} \leq M\,(1 + K^{\sigma - \frac{1}{2}}) \|f\|_g \,. \tag{4.4c'}$$

Proof: As before, let

$$B_h U = f \,. \tag{4.5}$$

Using Lemma 3.3, we see that

$$L_{K,h}^{(2)} U = f + Q_h U \tag{4.13a}$$

where

$$\|Q_h U\|_g \leq M(1 + K^\sigma h) \|U\|_{g,1} \,. \tag{4.13b}$$

Multiplying by U and summing over Ω_h we use Lemma 3.2 to obtain

$$\begin{aligned} \frac{q}{2} |U|_{g,1}^2 + K p_0 \|U\|_g^2 &\leq \|f\|_g \cdot \|U\|_g \\ &\quad + M K^\sigma \|U\|_g^2 + M(1 + K^\sigma h) \|U\|_{g,1} \cdot \|U\|_g \,, \end{aligned}$$

Thus, for h small enough we have

$$\frac{q}{4} |U|_{g,1}^2 + (\frac{1}{2} K p_0 - M(1 + K^\sigma)) \|U\|_g^2 \leq \|f\|_g \; \|U\|_g \,,$$

and the estimates follow as in the previous lemma.

In Lemma 4.2 and Lemma 4.3 we were concerned with cases in which the symmetric part of B_h is positive definite (for K large enough). We now turn to the indefinite case.

Lemma 4.4: Let B_h be a discrete elliptic operator of the form $L_{K,h}$, $L_{K,h}^{(1)}$ or $L_{K,h}^{(2)}$ with

$$K h \leq M_0 \,.$$

Suppose

$$\|B_h^{-1}\|_g \leq M_1 \,. \tag{4.14}$$

Then there is an h_0, such that for all h, $0 < h \leq h_0$, we have

$$|B_h^{-1} f|_{g,1} \leq M(1 + K^{\frac{1}{2}}) \, \|f\|_g \; \|B_h^{-1}\|_g \tag{4.15a}$$

$$|B_h^{-1}f|_{g,2} \le M[1 + (K^{\sigma+\frac{1}{2}} + K)\|B_h^{-1}\|_g] \cdot \|f\|_g \ . \tag{4.15b}$$

Proof: Let $B_h U = f$. Then Using Lemma 3.3

$$L_{k,h}^{(2)} U = f + Q_h U \tag{4.16a}$$

where

$$\|Q_h U\|_g \le M(1 + K^\sigma h)\|U\|_{g,1} \ . \tag{4.16b}$$

Multiplying by U and using Lemma 3.2, we have

$$\frac{q}{2}|U|_{g,1}^2 \le K\|U\|_g^2 + \|f\|_g \cdot \|U\|_g + M\|U\|_{g,1} \cdot \|U\|_g \ .$$

So that

$$\frac{q}{4}|U|_{g,1}^2 \le (K + \frac{M}{q})\|U\|_g^2 + \|f\|_g \, \|U\|_g \ .$$

Hence (4.15a) holds.

Using Lemma 3.3 and (2.8) we see that

$$A_{1,h} U = f + R_h U \tag{4.17a}$$

where

$$\|R_h U\|_g \le M[K^\sigma \|U\|_{g,1} + K\|U\|_g] \ . \tag{4.17b}$$

Hence, Lemma 3.1 and (4.15a) yields (4.15b).

Definition: We say A_h is of class I if the estimates (4.4) hold. That is

$$\|A_h^{-1}\|_g \le M/K \ , \quad |A_h^{-1}f|_{g,1} \le MK^{-\frac{1}{2}}\|f\|_g \tag{4.18a}$$

$$|A_h^{-1}f|_{g,2} \le M(1 + K^{\sigma-\frac{1}{2}})\|f\|_g \ . \tag{4.18b}$$

Theorem 4.1: Let A_h be of class I. Let

$$B_h = -\Delta_h + K \ . \tag{4.19}$$

That is

$$B_h V = -[V_{x\bar{x}} + V_{y\bar{y}}] + KV \ ,$$

and B_h satisfies the boundary conditions of A_h. Then

$$C(A_h B_h^{-1}) = \|A_h B_h^{-1}\|_g \cdot \|B_h A_h^{-1}\|_g \leq M(1 + K^{\sigma - \frac{1}{2}})^2 \,. \tag{4.20}$$

Proof: Use (4.18a), (4.18b) and apply Lemma 4.3 to B_h. ∎

Theorem 4.2: Let A_h be an operator which satisfies the estimates (4.14) and (4.15). That is

$$\|A_h^{-1}\|_g \leq M_1 \tag{4.21a}$$

$$|A_h^{-1} f|_{g,1} \leq M\, M_1 K^{\frac{1}{2}} \|f\|_g \tag{4.21b}$$

$$|A_h^{-1} f|_{g,2} \leq M[1 + (K^{\sigma + \frac{1}{2}} + K) M_1] \|f\|_g \,. \tag{4.21c}$$

Let B_h be the operator given by (4.19). Then

$$C(A_h B_h^{-1}) \leq M(1 + K^{\sigma - \frac{1}{2}})[1 + (K^{\sigma + \frac{1}{2}} + K) M_1] \,.$$

Proof: Apply lemma 4.3 to B_h.

We now turn to estimates on $C(B_h^{-1} A_h)$. We observe that

$$\|B_h^{-1} A_h\|_g = \|(B_h^{-1} A_h)^*\|_g = \|A_h^*(B_h^{-1})\|_g \,, \tag{4.22a}$$

$$\|A_h^{-1} B_h\|_g = \|(A_h^{-1} B_h)^*\|_g = \|B_h (A_h^{-1})^*\|_g \,. \tag{4.22b}$$

However, the representation $L_{k,h}^{(2)}$ shows that A_h^* satisfies exactly the same estimates as A_h. Hence we immediately obtain the following results which mirror theorems 4.1 and 4.2.

Theorem 4.3: Let A_h be of class I. Let B_h be given by (4.19). Then

$$C(B_h^{-1} A_h) \leq M(1 + K^{\sigma - \frac{1}{2}})^2 \,. \quad ∎ \tag{4.23}$$

Theorem 4.4: Let A_h be an operator which satisfies the estimates (4.21). Let B_h be the operator given by (4.19). Then

$$C(B_h^{-1} A_h) \leq M\,(1 + K^{\sigma - \frac{1}{2}})[1 + (K^{\sigma + \frac{1}{2}} + K) M_1] \,. \quad ∎ \tag{4.24}$$

5. An Alternative Method for Convection - Diffusion Type Models

Suppose that A_h is of class I (see estimates (4.18)) and let B_h be given by (4.19). If $\sigma \leq \frac{1}{2}$, it follows from Theorems 4.1 and 4.3 that

$$C(A_h B_h^{-1}) = O(1) \text{ and } C(B_h^{-1} A_h) = O(1) \text{ as } K \to \infty \ . \tag{5.1}$$

On the other hand, these condition numbers can increase with K if $\sigma > \frac{1}{2}$. This includes the important case of convection-diffusion models for which $\sigma = 1$.

In this section, we assume that $\sigma > \frac{1}{2}$ and show how to replace problem (1.5) by one of the same form for which (5.1) holds. Suppose that either

$$\text{(i) } d(x,y) \geq d_0 > 0 \text{ or (ii) } -d(x,y) \geq d_0 > 0 \text{ in } \Omega \tag{5.2a}$$

or

$$\text{(ii) } e(x,y) \geq e_0 > 0 \text{ or (ii) } -e(x,y) \geq e_0 > 0 \text{ in } \Omega \ . \tag{5.2b}$$

If (5.2a) holds, set

$$U = e^{-\alpha K^\sigma x} u \tag{5.3a}$$

and

$$F = e^{-\alpha K^\sigma x} f \ . \tag{5.4a}$$

If (5.2b) holds, set

$$U = e^{-\alpha K^\sigma y} u \tag{5.3b}$$

and

$$F = e^{-\alpha K^\sigma y} f \ . \tag{5.4b}$$

It is easily seen that (1.5a) is transformed into the following equation assuming either (5.2a)-(5.4a) or (5.2b)-(5.4b) holds:

$$\begin{aligned} \hat{L}_K U = &-\left\{ \frac{\partial}{\partial x}(a\frac{\partial U}{\partial x}) + \frac{\partial}{\partial y}(b\frac{\partial U}{\partial x}) + \frac{\partial}{\partial x}(b\frac{\partial U}{\partial y}) + \frac{\partial}{\partial y}(c\frac{\partial U}{\partial y}) \right\} \\ &+ K^\sigma \left(\hat{d}\frac{\partial U}{\partial x} + \hat{e}\frac{\partial U}{\partial y} \right) + K^{2\sigma}\hat{p}U = F \end{aligned} \tag{5.5}$$

with suitable functions $\hat{d}$, $\hat{e}$, and $\hat{p}$. Furthermore, we assume that the boundary conditions for U take the form

$$U \text{ or } \frac{\partial U}{\partial n} = 0 \tag{5.6}$$

along an entire side.

Now suppose we show that α can be chosen such that

$$\hat{p} \geq \hat{p}_0 > 0 \text{ for } K \text{ sufficiently large .} \tag{5.7}$$

Then if we discretize $\hat{L}_K$ using one of the difference schemes, (2.7), (2.8), or (2.9) and replace K by $K' = K^{2\sigma}$, it follows from Lemma 4.3 that the resulting discrete operator, A_h , is of class I for K sufficiently large. Hence we may apply Theorems 4.1 and 4.3 to see that (5.1) holds. We may thus iteratively solve for the approximate solution, U^h , of (5.5) with the number of iterations bounded as $K \to \infty$. The approximate solution, u^h , of (1.1) is now given by either $u^h = e^{\alpha K^\sigma x} U^h$ or $u^h = e^{\alpha K^\sigma y} U^h$. Hence our goal is to prove that (5.7) holds.

Theorem 5.1: Suppose that either (5.2a)-(5.4a) or (5.2b)-(5.4b) holds. Then α may be chosen such that (5.7) holds.

Proof: It suffices to assume (5.2a) since analogous arguments hold assuming (5.2b). In view of (5.3a) and (5.3b), we see using a straightforward calculation that (5.5) holds with

$$\hat{p} = \alpha d - \alpha^2 a + p K^{1-2\sigma} - \alpha \frac{\partial a}{\partial x} K^{-\sigma} . \tag{5.8}$$

Now set

$$\alpha = \beta = \frac{d_0}{2\|a\|_{L^\infty(\Omega)}} \text{ if condition (i) in (5.2a) holds} \tag{5.9a}$$

and

$$-\alpha = \beta = \frac{d_0}{2\|a\|_{L^\infty(\Omega)}} \text{ if condition (ii) in (5.2a) holds} \tag{5.9b}$$

It is readily seen using (5.8) and (5.9) that

$$\begin{aligned} \hat{p} &\geq \beta d_0 - \beta^2 \|a\|_{L^\infty} - \left(K^{1-2\sigma}\|p\|_{L^\infty(\Omega)} + \beta K^{-\sigma}\|\frac{\partial a}{\partial x}\|_{L^\infty(\Omega)}\right) \\ &= \frac{d_0^2}{4\|a\|_{L^\infty(\Omega)}} - \left(K^{1-2\sigma}\|p\|_{L^\infty} + \frac{K^{-\sigma} d_0 \|\frac{\partial a}{\partial x}\|_{L^\infty(\Omega)}}{2\|a\|_{L^\infty(\Omega)}}\right) . \end{aligned}$$

Since $2\sigma > 1$, we see that (5.7) holds. ∎

Note: Computational difficulties can arise using transformation (5.3) for K large since it may be necessary to deal with very large or very small numbers.

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