

The Determination of
Bow Sight Settings
Using Arrow Ballistics,
Initial Measurements of Bow and Arrow
Geometry
and
Initial Arrow Velocity

By

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INTRODUCTION

Historically, problems in archery have been solved primarily by the trial and error method. Of these problems, one deals with the increasing popular use of the bow sight as a sighting aid. The problem arises in setting the sight for various distances so that the desired amount of accuracy is achieved. This leads to three areas of difficulty:

- a.) How does the novice archer set the sight setting when their ability does not allow for small enough arrow groups to determine group center?
- b.) As the shooting distance becomes longer, generally the arrow grouping grows in size thus expanding the group to a size too large in order to determine group center.
- c.) The archer does not have access to a shooting area of adequate dimensions or does not possess the time required to effectively use the trial and error method.

The intent of this paper is two fold. First, to use aerodynamics and arrow ballistics with elementary bow and arrow geometry to determine these sight settings; theoretically, thereby eliminating the need for trial and error. This method should only need a minor adjustment at the desired distances to be effective. Second, to learn the Mathcad computer program .

List of Variables

- F_{d_x} = Force of drag along the x-axis, in pounds
- F_{d_y} = Force of drag along the y-axis, in pounds
- F_d = Force of drag at any time, in pounds
- F_w = Force of weight of the arrow, in pounds ($F_w=mg$)
- V = The velocity of the arrow at any time, in ft. / sec.
- V_0 = The initial velocity of the arrow, in ft. / sec.
- V_x = The velocity component of the arrow along the x-axis, in ft. / sec.
- V_y = The velocity component of the arrow along the y-axis, in ft. / sec.
- a = Arrow acceleration, in ft. / sec².
- a_x = Arrow acceleration component along the x-axis, in ft. / sec².
- a_y = Arrow acceleration component along the y-axis, in ft. / sec².
- m = Mass of the arrow, in slugs
- ϕ = The angle the arrow makes with the x-axis at any time
- α = The initial launch angle of the arrow with respect to the x-axis
- g = The acceleration of gravity, 32.16 ft. / sec².
- t = Time of flight, in sec.
- R = Distance to the target, in ft. (target is at eye level)
- x = Displacement along the x-axis as a function of time. Measured from the release point.
- y = Displacement along the line of sight (y-axis) as a function of time. Measured from the release point.

Drag Variables:

C_D = Coefficient of drag, .0045

ρ = Mass density of air, $\rho = .00238$ slug per cubic feet

S_A = Area of surface in contact with the moving air, sq. ft.

V = Relative air velocity, velocity of the arrow in air, ft. per sec.

μ = Coefficient of viscosity of air, lb. sec. per ft.²

l = Characteristic length of surface, (length of arrow) ft.

K = Factor to allow for the area of fletching of the arrow

d = Diameter of the arrow

Bow and Arrow Geometry Variables:

β = Angle between string and arrow at full draw

P = Distance from knock set to peep sight

A = Distance from knock set to arrow rest at full draw

r = Distance from peep to arrow rest at full draw

S = Distance from bow sight to the arrow at right angles to the arrow

c = Distance from arrow rest to a point on the arrow below the sight

α = Initial launch angle of the arrow

Arrow Drag

The drag of an arrow may be separated into two main divisions, one due to skin friction over the surface of the arrow, and the other due to the wake or turbulence formed at the rear. The exterior shape of the arrow immediately classifies it as having primarily skin friction. The general formula for skin friction has been developed to the following general form: (Higgins, 1933, p. 91-92)

$$F_d = \frac{C_D \rho S_A V^2}{2} \left[\frac{\rho V l}{\mu} \right]^{-.15}$$

From this it can be shown that the drag varies primarily as V^2 and secondarily as $1/V^{.15}$. Since the change in velocity of the arrow in flight is not very large, it is assumed the drag varies only as V^2 and a suitable C_D chosen to make allowance for the much smaller factor $1/V^{.15}$. (Higgins, 1933, p. 91-92)

S_A can be written in the form:

$$S_A = K \pi d l$$

Where K is a factor to allow for the additional area of the fletching, (see Table 1.) the expression for the drag force then becomes:

$$F_d = \frac{K C_D \rho \pi d l V^2}{2}$$

**Table 1.
Area Factors for Fletching**

Type of arrow	K
Flight:	
Celluloid vanes	1.00 to 1.02
Feathers	1.03 to 1.05
Target	1.12 to 1.27
Hunting:	
Feathers	1.25 to 1.50
Hunting blades (approx.)	1.20

Equations of Motion

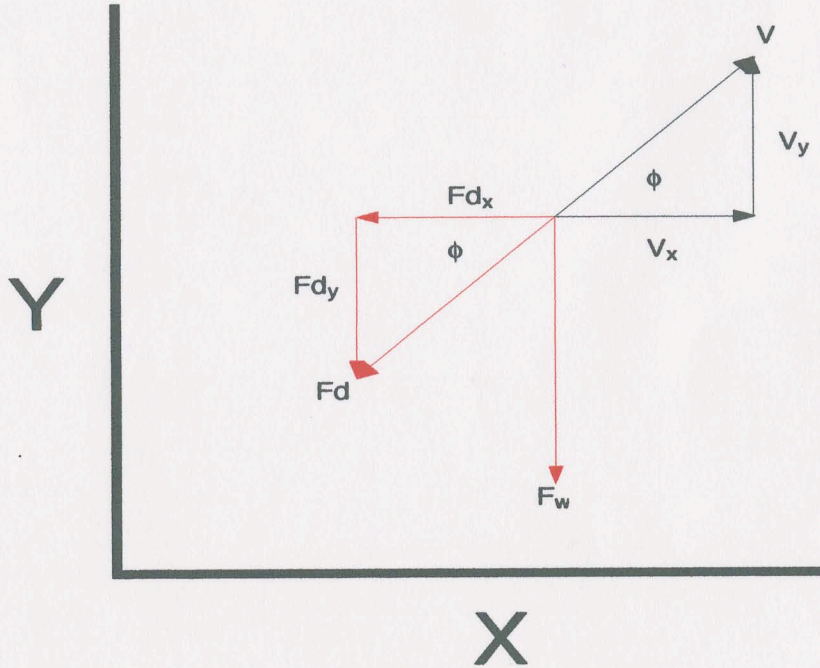


Fig. 1 Force Diagram

The force diagram is shown in Figure 1. The forces can be broken down into their X and Y components. Starting first with the X components:

$$F_x = ma_x$$

Therefore:

$$-F_d = ma_x$$

$$-F_d \cos \phi = ma_x \quad \text{since} \quad F_{dx} = F_d \cos \phi$$

Dividing both sides by the mass, yields:

$$\frac{-F_d \cos \phi}{m} = a_x$$

since $F_w = mg$ and $\frac{g}{F_w} = \frac{1}{m}$ the mass can be substituted out, giving:

$$\frac{-F_d \cos \phi}{F_w} = a_x \quad \text{eq. 1}$$

The force of drag (F_d) was shown to be:

$$F_d = \frac{K C_D \rho \pi d l}{2} V^2 \quad \text{letting } k_1 = \frac{K C_D \rho \pi d l}{2} \quad \text{this expression becomes:}$$

$$F_d = k_1 V^2$$

substituting this into equation 1 gives:

$$\frac{(-k_1) V^2 \cos \phi}{F_w} = a_x \quad \text{eq. 2}$$

expanding equation 2 gives:

$$\frac{(-k_1) g V V \cos \phi}{F_w} = a_x$$

$$\text{let } z = \frac{k_1 g}{F_w} \quad \text{therefore:}$$

$$-z V V \cos \phi = a_x$$

applying right triangle trig. to the velocity vectors gives the expression:

$$V \cos \phi = V_x \quad \text{and} \quad V = \sqrt{(V_x)^2 + (V_y)^2}$$

substituting these into equation 2 gives:

$$\text{then: } (-z V_x) \sqrt{(V_x)^2 + (V_y)^2} = a_x$$

This equation leads to an unsolvable, coupled, second order differential equation. In order to solve the problem the following assumption is made:

Since V_x is much greater than V_y , most of the drag is in the x direction; therefore, it is reasonable to assume that the drag and the velocity in the y direction is nearly zero.

therefore, substituting V for $\sqrt{(V_x)^2 + (V_y)^2}$ yields:

$$-zV_x V = a_x \quad \text{eq. 3}$$

assume that $V = bV_x$ where b is a constant, then equation 3 becomes:

$$-(V_x)^2 z b = a_x$$

therefore:

$$-(V_x)^2 z b = \frac{dV_x}{dt}$$

separating variables and integrating gives:

$$\int -z b dt = \int (V_x)^{-2} dV_x$$

$$-zbt = \frac{-1}{V_x} + c_1 \quad \text{to determine } c_1, \text{ use the initial conditions;}$$

$$\text{at } t=0 \quad \text{and} \quad V_x = V_0 \cos\alpha$$

then:

$$c_1 = \frac{1}{V_0 \cos\alpha}$$

therefore:

$$-zbt = \frac{-1}{V_x} + \frac{1}{V_0 \cos\alpha}$$

solving for V_x :

$$V_x = \frac{V_0 \cos \alpha}{1 + zbtV_0 \cos \alpha}$$

$$\frac{dx}{dt} = \frac{V_0 \cos \alpha}{1 + zbtV_0 \cos \alpha}$$

separating variables and integrating:

$$\int_0^x dx = \int_0^t \frac{V_0 \cos \alpha}{1 + zbtV_0 \cos \alpha} dt$$

This gives the first of the two parametric equations needed to find the ballistic equation for the path of the arrow.

$$x = \frac{\ln(1 + zbtV_0 \cos \alpha)}{zb}$$

Turning to the y component, since the flight of the arrow is nearly horizontal the F_{dy} can be assumed to be zero.

$$F_y = ma_y$$

$$-F_w = ma_y$$

substituting mg for F_w and dividing both sides by the mass gives:

$$-g = a_y \quad \text{but} \quad a_y = \frac{dV_y}{dt}$$

therefore:

$$-g = \frac{dV_y}{dt}$$

separating variables and integrating:

$$\int -g dt = \int 1 dV_y$$

$$-gt = V_y + c_2$$

to solve for c_2 , use the initial conditions:

$$\text{at } t=0 \quad \text{and} \quad V_y = V_0 \sin\alpha$$

therefore:

$$c_2 = -V_0 \sin\alpha$$

$$-gt = V_y - V_0 \sin\alpha$$

$$-gt = \frac{dy}{dt} - V_0 \sin\alpha$$

separating variables and integrating:

$$(-gt + V_0 \sin\alpha) dt = dy$$

$$\int_0^t (-gt + V_0 \sin\alpha) dt = \int_{-s \cos\alpha}^y 1 dy$$

since α is small $\cos\alpha = 1$ therefore the integration gives

$$\frac{-1}{2} gt^2 + V_0 t \sin\alpha - s = y$$

This gives the second of the parametric equations needed for the ballistic equation:

$$y = \frac{-1}{2} gt^2 + V_0 t \sin\alpha - s$$

Now that both parametric equations have been derived, they must be combined and the resulting equation, the ballistic equation, solved for the variable s . Where s is the distance from the arrow to the sight.

$$x = \frac{\ln(1 + zbtV_0 \cos\alpha)}{zb} \qquad y = \frac{-1}{2} gt^2 + V_0 t \sin\alpha - s$$

Solve the x equation for t and substitute into the y equation giving the ballistic equation:

$$t = \frac{e^{x zb} - 1}{zbV_0 \cos\alpha}$$

therefore:

$$y = -s + \frac{e^{x zb} - 1}{zbV_0 \cos\alpha} V_0 \sin\alpha - \frac{1}{2} g \left[\frac{e^{x zb} - 1}{zbV_0 \cos\alpha} \right]^2$$

$$\text{at } x_i = 0 \qquad y_i = -s$$

This checks the equation at initial conditions.

At the target $y_f = 0$ and $x = R$

therefore:

$$0 = -s + \left[\frac{e^{R zb} - 1}{zbV_0 \cos\alpha} V_0 \sin\alpha - \frac{1}{2} g \left[\frac{e^{R zb} - 1}{zbV_0 \cos\alpha} \right]^2 \right]$$

simplifying, yields:

$$s = \frac{e^{R zb} - 1}{zbcos\alpha} \sin\alpha - \frac{1}{2} g \left[\frac{e^{R zb} - 1}{zbV_0 \cos\alpha} \right]^2 \qquad \text{eq. 4}$$

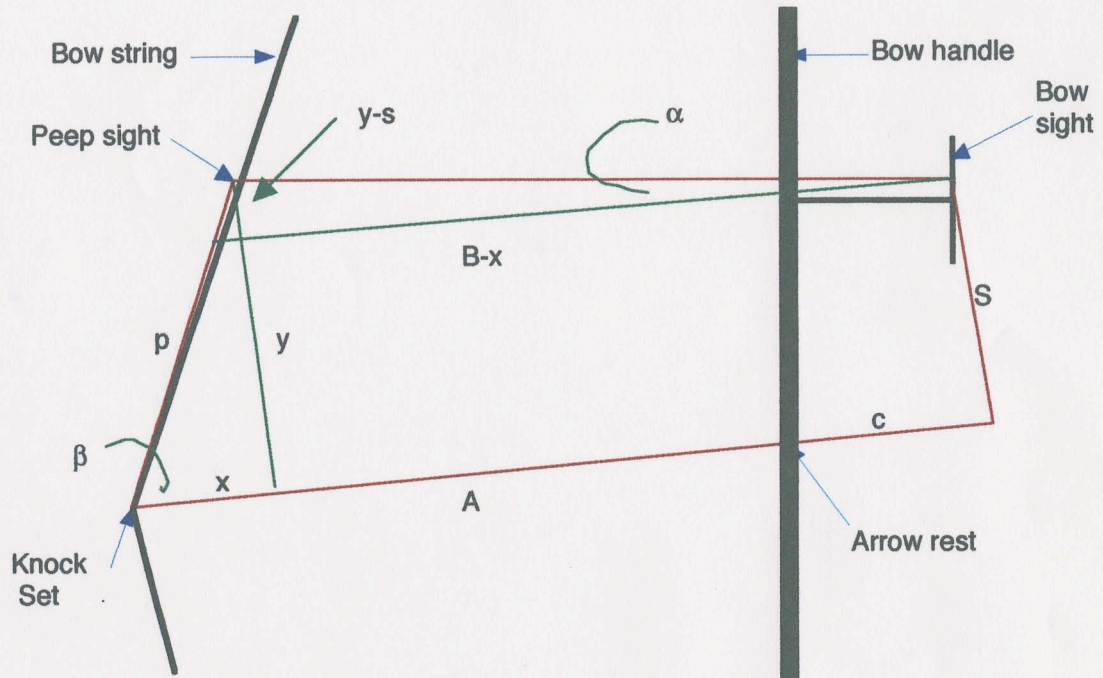


Fig. 2 Bow and Arrow Geometry

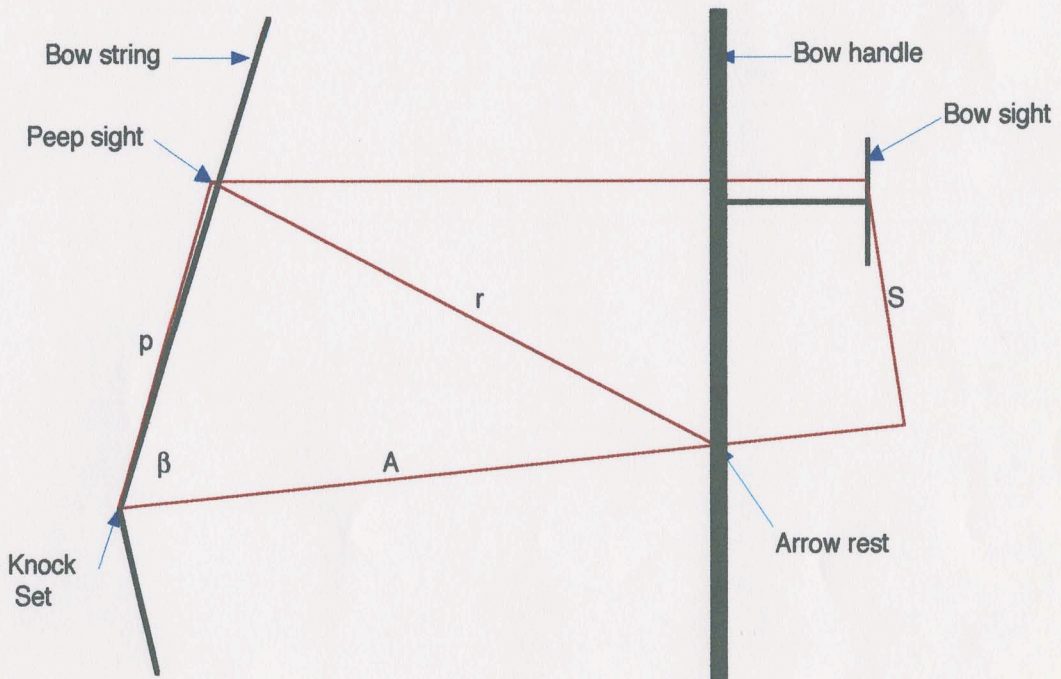


Fig. 3 String Angle

All of the variables found in equation 4 are easily found or measured with the exception of α , the initial launch angle of the arrow. α is very hard to measure directly but can be found indirectly using the relationship between the bow and the arrow at full draw. Referring to figure 2, it can be shown that:

$$\tan\alpha = \frac{y - s}{B - x}$$

since $y = p\sin\beta$ and $x = p\cos\beta$ and $B = A + c$

$$\tan\alpha = \frac{p\sin\beta - s}{B - p\cos\beta} \quad \text{eq. 5}$$

This equation however, presents another problem, The angle β , which is the angle between the arrow and the string at full draw, can't be measured directly. This angle can, however, be found quite easily by referring to Figure 3 using the law of cosines:

$$r^2 = p^2 + A^2 - 2 pA\cos\beta$$

solving for $\cos\beta$ gives:

$$\cos\beta = \frac{p^2 + A^2 - r^2}{2 pA} \quad \text{from this equation } \beta \text{ can be found.}$$

let $bb = p\sin\beta$ and $cc = B - p\cos\beta$ then substitute into equation 5

$$\tan\alpha = \frac{bb - s}{cc} \quad (\text{needed in order for Mathcad to solve})$$

substituting for α into equation 4 yields:

$$s = \frac{bb - s}{cc} \frac{e^{Rzb} - 1}{zb} - \frac{1}{2} g \left[\frac{e^{Rzb} - 1}{v_0} \right]^2 \frac{\left[1 + \frac{(-bb + s)^2}{cc^2} \right]}{z^2 b^2} \quad \text{eq. 6}$$

EXAMPLE

Since equation 6 is tedious to solve longhand, I found that using a computer program is more efficient. The program I discovered to be the most useful is Mathcad PLUS 6, by Math Soft. This program allows for the continuous solving of the equation for different values of x_1 , the distance to the target, thus allowing a graph to be drawn of distance versus sight setting. This arrangement makes the finding of sight setting more portable and "user friendly". The graph can be printed once and used to find any sight setting for any distance, quickly and efficiently. The following is an example of the graph using parameters from my personal equipment.

The initial parameters:

To be measured by the archer, refer to diagrams 2 and 3.

$V_0 := 210$	initial velocity of the arrow, in ft. per sec.
$w := 250$	weight of arrow, in grains
$d := .017$	diameter of arrow, in ft.
$l := 2.46$	length of arrow, in ft.
$p := .480$	distance from knock set to string peep, in ft.
$A := 2.29$	distance from knock set to plunger at full draw, in ft.
$r := 2.125$	distance from peep to plunger at full draw, in ft.
$c := .62$	distance from plunger to a point on the arrow below the sight, in ft.

Constants, changeable if so desired.

$K := 1.00$	arrow fletching factor, table 1
$CD := .0040$	coefficient of drag
$\rho := .00238$	mass density of air, slug per cubic foot
$g := 32.16$	acceleration of gravity, in ft. per sec. per sec.
$b := 1.00$	factor to test assumption
$x := 30.. 300$	distance to target, in ft. computed every foot starting at 30 and ending at 300

Defining various functions needed by Mathcad:

$$\beta := \arccos\left(\frac{p^2 + A^2 - r^2}{2 p \cdot A}\right)$$

$$B := A + c$$

$$bb := p \cdot \sin(\beta)$$

$$cc := B - p \cdot \cos(\beta)$$

$$Fw := w \cdot 1.428571 \cdot 10^{-4}$$

$$k_1 := \frac{K \cdot CD \cdot \rho \cdot \pi \cdot d \cdot l}{2}$$

$$z := \frac{k_1 \cdot g}{Fw}$$

Now that the parameters and functions have been defined, proceed with solving. Mathcad uses a built in function called "root" to solve this type of equation. To begin, set equation 6 equal to 0, then initialize a starting point for the function, s=1, then use the root function to solve.

$$s = \frac{bb - s}{cc} \frac{e^{R z b} - 1}{z b} - \frac{1}{2} g \left[\frac{e^{R z b} - 1}{v_0} \right]^2 \frac{\left[1 + \frac{(-bb + s)^2}{cc^2} \right]}{z^2 b^2}$$

setting equal to zero:

$$0 = \left[\frac{bb - s}{cc} \frac{e^{R z b} - 1}{z b} - \frac{1}{2} g \left[\frac{e^{R z b} - 1}{v_0} \right]^2 \frac{\left[1 + \frac{(-bb + s)^2}{cc^2} \right]}{z^2 b^2} \right] - s$$

Choosing 1 as a starting value for the root function:

$$s_0 := 1$$

Using the secant method the root function searches for solutions. In this case, many solutions, since the equation is set up to give solutions for all values of R between 30 and 300 feet.

$$f(R, s) := \text{root} \left[\left[\frac{bb - s}{cc} \cdot \frac{e^{R \cdot z \cdot b} - 1}{(z \cdot b)} - \frac{1}{2} \cdot g \cdot \frac{(e^{R \cdot z \cdot b} - 1)^2}{(V_0)^2} \cdot \frac{\left[1 + \frac{(bb - s)^2}{cc^2} \right]}{(z^2 \cdot b^2)} \right] - s, s \right]$$

$$s_R := f(R, s_{R-1})$$

The results of this function are stored by the program under variable names R and s_R . They are stored in table format but in this case, since a graph is the desired final product, there is no need to view the tables.

Since s_R is the sight setting distance, the q_R function has been defined to translate the s_R function from feet to a more "user friendly" unit, inches.

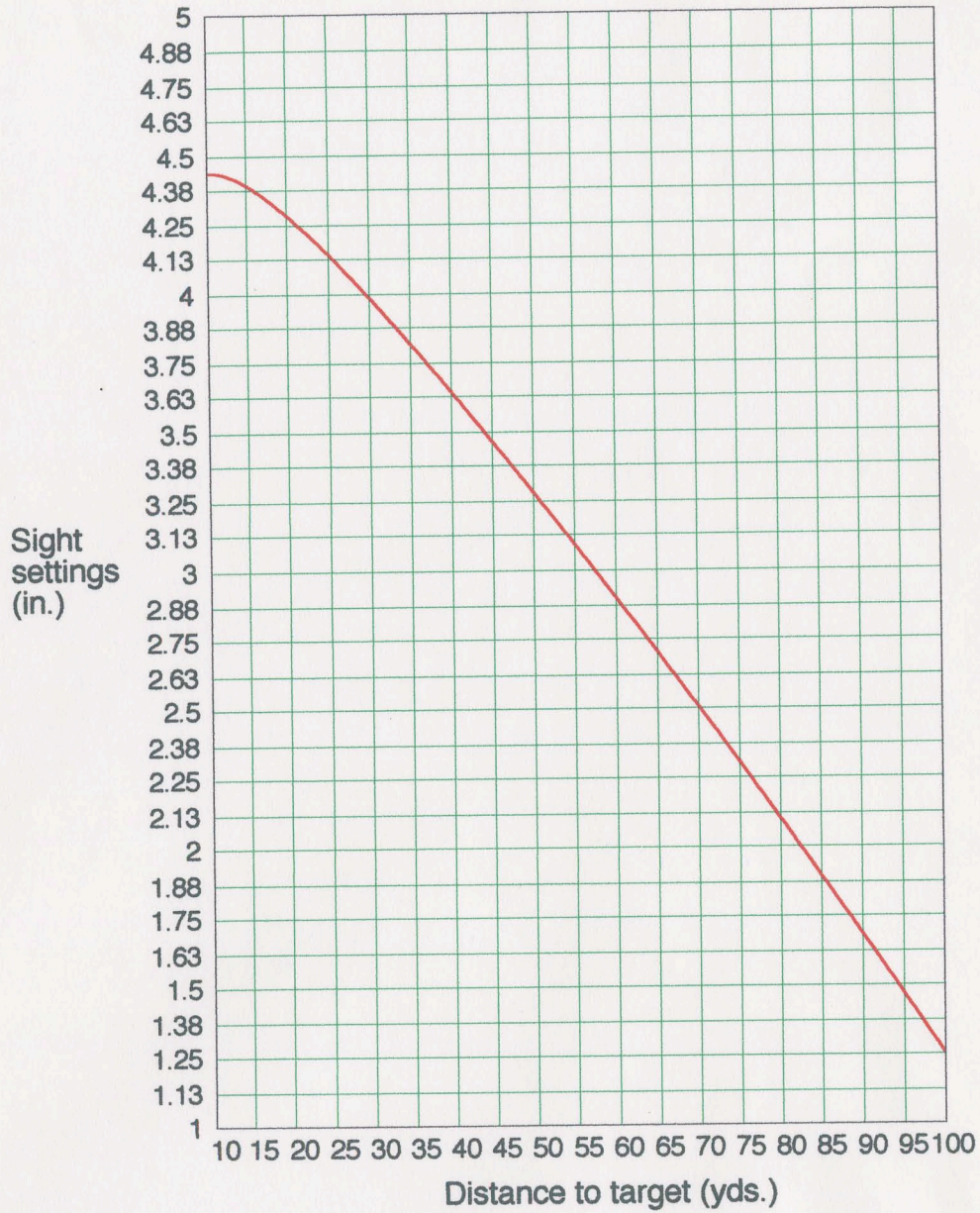
$$q_R := s_R \cdot 12$$

Since R is the distance to the target, the f_R function has been defined to translate the R function from feet to a more "user friendly" unit, yards.

$$f_R := \frac{R}{3}$$

To produce a graph of these results, graph q_R versus f_R . Choosing the appropriate axis dimensions will yield the graph on the following page, labeled "sight setting graph". This graph makes it quite easy to determine any desired sight setting for the required distance. (Mathcad, 1996)

SIGHT SETTINGS GRAPH



CONCLUSIONS

When comparing the theoretical sight settings derived from the graph with the actual sight settings for those distances, there is favorable agreement between the two. Looking at three distances scattered throughout the graph, 32.7, 54.7, and 76.7 yards, the impact point difference ranges from a maximum of 6 inches at 32.7 yards to a minimum of less than 1 inch at 76.7 yards. However, at 98.3 yards the difference balloons to over 3 feet. It appears that the equation and resulting graph are useful only for distances of 80 yards or less.

The equation does, however, give acceptable results up to 80 yards with only a minor adjustment needed to hit center. Therefore the equation and resulting graph are effective tools in determining bow sight settings.

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