CLASSICAL AND QUANTUM LOGICS: TEST SPACES, BOOLEAN ALGEBRAS, AND ORTHOALGEBRAS

by

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ABSTRACT

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In this paper, the connections between classical propositional logic and Boolean algebras are explored. Then, an introduction to quantum logic explains that the distributive property does not hold, which gives rise to the need for a different structure for quantum logic. One of these systems, algebraic test spaces, is described, along with their connection to orthoalgebras, which are also analyzed.

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1 INTRODUCTION

Logic is defined as the study of correct reasoning. Classical propositional logic, which is foundational in many fields, including mathematics, computer science, and philosophy, is concerned with statements, or propositions, that can be either true or false. From those propositions, a rich structure of progressively more complex statements can be formed to assist in reasoning. Many different structures have been developed throughout the years by mathematicians to model this system of building a language of reasoning from simple statements. Here, we analyze the connections between one of those such structures, Boolean algebras, and classical propositional logic.

While classical propositional logic is useful in so many applications, there are cases where this system falls short. In the case of quantum mechanics, a field where much is still unknown, it has become evident the behavior of particles in quantum space does not conform to the expectations that would allow us to reason using classical logic. One of the key differences between classical and quantum logic is the failure of the distributive property. The measurement difficulties described in Heisenberg's Uncertainty Prinicple cause the distributive property to not hold in the case of quantum logic. Here we examine one example, and then we look at one proposed solution to this problem - test space.

First proposed by Foulis and Randall, their manuals are a way of modeling these quantum systems. Here we analyze test spaces and see that these give rise to another mathematical structure, orthoalgebras. Finally, it is shown that the logic of a test space is equivalent to an orthoalgebra, and also that a test space can be constructed from any orthoalgebra, and that a test space constructed in this way is isomorphic to the orthoalgebra that we started with.

2 CLASSICAL LOGIC AND BOOLEAN ALGEBRAS

2.1 Boolean Algebras

Halmos [H] introduces Boolean algebras by giving what he describes as "the shortest definition". The definitions and propositions in this section that follow are also taken from Halmos.

Definition 2.1. A **boolean algebra** is a ring with unit in which every element is an **idempotent;** *i.e.* if p is an element of the ring, then $p^2 = p$.

Recall that in a 'ring with unit', it is assumed that $1 \neq 0$. The simplest example of a Boolean algebra is therefore the field with just two elements, $\{0, 1\}$; we will denote this Boolean algebra by $\mathbf{2}$.

Additional examples of Boolean algebras can be constructed by taking $\mathcal{B} = \mathcal{P}(X)$ for any non-empty set, and taking the sum and product of subsets $A, B \subset X$ to be their symmetric difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$ and intersection $A \cap B$, respectively. The Boolean algebra $\mathbf{2}$ is of this form with X any 1-element set, so that $\mathcal{P}(X) = \{\emptyset, X\}$. (Note, however, that there are Boolean algebras which are not isomorphic to $\mathcal{P}(X)$ for any set$X$.)

Lemma 2.2. *Let $\mathcal{B}$ be a Boolean algebra. Then*

(a) *$\mathcal{B}$ has characteristic 2; i.e. $p + p = 0$ for all $p \in \mathcal{B}$;*

(b) *$\mathcal{B}$ is commutative; i.e. $pq = qp$ for all $p, q \in \mathcal{B}$.*

Proof. (a) Since $p + p$ is an idempotent, we have

$$p + p = (p + p)^2 = p^2 + p + p + p^2 = p + p + p + p, \tag{2.1}$$

from which it follows that $p + p = 0$.

(b) Similarly,

$$p + q = (p + q)^2 = p^2 + pq + qp + q^2 = p + pq + qp + q, \tag{2.2}$$

so that $pq + qp = 0$. Adding qp to both sides and applying (a) with p replaced by qp gives the result. ■

Definition 2.3. A **partial order** on a set X is a binary operator $\leq$ with the following properties for all $a, b, c \in X$:

(a) Reflexive: $a \leq a$

(b) Anti-Symmetric: If $a \leq b$ and $b \leq a$ then $a = b$

(c) Transitive: If $a \leq b$ and $b \leq c$ then $a \leq c$

Definition 2.4. Let X be some set and $\leq$ a partial order. Then, the pair $(X, \leq)$ is a **partially ordered set**, or **poset**.

We can introduce a partial binary relation $\leq$ in any Boolean algebra $\mathcal{B}$ by defining

$$p \leq q \text{ if and only if } pq = p. \tag{2.3}$$

The next lemma shows that $\leq$ is a partial order on $\mathcal{B}$.

Lemma 2.5. *Let $\mathcal{B}$ be a Boolean algebra, $p, q, r \in \mathcal{B}$, and define $\leq$ as in (2.3). Then, $\leq$ is a partial order and $(\mathcal{B}, \leq)$ is a partially ordered set.*

Proof. Reflexivity follows immediately from $p^2 = p$, and antisymmetry from $pq = qp$. For transitivity, suppose that $p \leq q$ and $q \leq r$, so that $pq = p$ and $qr = q$. Then $pr = (pq)r = p(qr) = pq = p$, so that $p \leq r$. ■

Note that the algebra elements 0 and 1 in $\mathcal{B}$ are respectively the least and greatest elements in the poset $(\mathcal{B}, \leq)$.

We can now make the connection to our second characterization of Boolean algebras: they are distributive, orthocomplemented lattices.

Definition 2.6. Let A be a subset of some poset $\mathcal{P}$, with $x \in A$. Then, x is an upper bound of A if $x \geq y$ for all $y \in A$. Likewise, x is a lower bound of A if $x \leq y$ for all $y \in A$. If $a \in A$ is a lower bound and $a \geq b$ for all lower bounds $b \in A$, then a is the greatest lower bound of A. If $a \in A$ is an upper bound and $a \leq b$ for all upper bounds $b \in A$, then a is the least upper bound of A.

For any two elements $x, y \in \mathcal{P}$, the least upper bound is called the **join** and is denoted as $x \vee y$. The greatest lower bound is called the **meet** and is denoted as $x \wedge y$.

Definition 2.7. A **lattice** $\mathcal{L}$ is a poset in which the meet and join exist for each pair of elements in $\mathcal{L}$.

Definition 2.8. Let $\mathcal{L}$ be a lattice. Then $\mathcal{L}$ is **orthocomplemented** if, for all $a, b \in \mathcal{L}$, there exists $a', b' \in \mathcal{L}$ such that:

(a) $(a')' = a$

(b) If $a \leq b$ then $b' \leq a'$

(c) $a \wedge a' = 0$

(d) $a \vee a' = 1$

Then, a' is called the **orthocomplement** of a.

Definition 2.9. Let $\mathcal{L}$ be a lattice. Then $\mathcal{L}$ is **distributive** if, for all $a, b, c \in \mathcal{L}$:

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Theorem 2.10. *Let $\mathcal{B}$ be a Boolean algebra and define $\leq$ as in (2.3). Then $\mathcal{B}$ becomes a distributive, orthocomplemented lattice in which the meet and join of any two elements are given by*

$$p \wedge q = pq, \qquad p \vee q = p + q + pq, \tag{2.4}$$

and the orthocomplement of any element is $p' = 1 + p$.

Proof. We show first that $p \wedge q = pq$. From $p(pq) = p^2q = pq$, it follows that $pq \leq p$; similarly, $pq \leq q$. Now suppose that $r \leq p$ and $r \leq q$. Then $r(pq) = (rp)q = rq = r$, so that $r \leq pq$. Therefore, pq is the greatest lower bound of p and q; *i.e.* $pq = p \wedge q$. The proof that $p + q + pq = p \vee q$ is similar. Since the meet and join of any two elements exist, $(\mathcal{B}, \leq)$ is a lattice.

We now show that $p \mapsto 1 + p$ is an orthocomplementation. First, $p'' = 1 + (1 + p) = p$, since $1 + 1 = 0$. Next, if $p \leq q$, then $pq = p$, so that $p'q' = (1 + p)(1 + q) = 1 + p + q + pq = 1 + p + q + p = 1 + q = q'$, and $q' \leq p'$. Finally, $p \wedge p' = p(1 + p) = p + p^2 = p + p = 0$, and $p \vee p' = p + (1 + p) + p(1 + p) = 1$, completing the proof.

Note that, as expected, the meet, join, and orthocomplementation operations are not independent, In fact, we have

$$(p' \vee q')' = 1 + ((1 + p) + (1 + q) + (1 + p)(1 + q)) = pq = p \wedge q, \tag{2.5}$$

and it easily follows (or can be independently verified) that $p \vee q = (p' \wedge q')'$.

We next show associativity of the meet and join operations. In fact,

$$(p \wedge q) \wedge r = (pq)r = p(qr) = p \wedge (q \wedge r), \tag{2.6}$$

so that the meet operation $\wedge$ is associative. A similar argument shows that the join operation is associative; alternatively, this can be deduced from the meet result and the representation of joins in terms of meets and orthocomplements. Finally,

$$(p \vee q) \wedge r = (p + q + pq)r = pr + qr + (pr)(qr) = (p \wedge r) \vee (q \wedge r), \tag{2.7}$$

so that meet distributes over join; that join distributes over meet now follows on taking orthocomplements. ∎

Conversely, we have the following result.

Theorem 2.11. *Let $(\mathcal{B}, \leq, ')$ be a distributive, orthocomplemented lattice with at least two elements, and define a sum and product in $\mathcal{B}$ by*

$$p + q = (p' \wedge q')' \wedge (p \wedge q)', \quad pq \equiv p \cdot q = p \wedge q. \tag{2.8}$$

Then $(\mathcal{B}, \cdot, +)$ is a ring with identity in which every element is an idempotent.

Proof. The definitions of addition and multiplication make it clear that these operations are commutative, and that multiplication is associative. It is also clear that (with 0 and 1 denoting the least and greatest elements of $\mathcal{B}$), $p + 0 = (p' \wedge 1)' \wedge (p \wedge 0)' = p'' \wedge 0' = p$, and $p \cdot 1 = p \wedge 1 = p$, so that 0 is the additive identity and 1 is the multiplicative identity. Moreover, the definition of multiplication gives $p^2 = p \wedge p = p$, so that every element of $\mathcal{B}$ is an idempotent. Note also that $1 + p = (1 \vee p) \wedge (0 \vee p') = 1 \wedge p' = p'$.

It only remains to show that addition is associative, and to verify the distributive property. The proofs of each of these results is a computation, which is perhaps best understood intuitively by thinking of $(\mathcal{B}, \leq)$ as a lattice of sets and drawing Venn diagrams.

For associativity of addition, it is convenient to observe first that

$$
\begin{aligned}
p + q &= (p \vee q) \wedge (p' \vee q') \\
&= (p \wedge p') \vee (q \wedge p') \vee (p \wedge q') \vee (q \wedge p') \\
&= (p \wedge q') \vee (p' \wedge q). \qquad (2.9)
\end{aligned}
$$

(Recall the connection with symmetric difference of sets.) Now we have

$$
\begin{aligned}
(p + q) + r &= [((p \wedge q') \vee (p' \wedge q)) \wedge r'] \vee [((p \wedge q') \vee (p' \wedge q))' \wedge r] \\
&= [((p \wedge q') \wedge r') \vee ((p' \wedge q) \wedge r')] \vee [((p' \vee q) \wedge (p \vee q')) \wedge r] \\
&= [(p \wedge (q \vee r)') \vee (q \wedge (p \vee r)')] \vee [(p \wedge q \wedge r) \vee (p' \wedge q' \wedge r)] \\
&= (p \wedge q \wedge r) \vee (p \wedge (q \vee r)') \vee (q \wedge (p \vee r)') \vee (r \wedge (p \vee q)'). \qquad (2.10)
\end{aligned}
$$

This last expression is manifestly invariant under permutations of p, q, and r, and it now follows easily that $p + (q + r) = (p + q) + r$.

For distributivity, we have

$$\begin{aligned} pr + qr &= ((pr) \wedge (qr)') \vee ((pr)' \wedge (qr)) \\ &= ((p \wedge r) \wedge (q' \vee r')) \vee ((p' \vee r') \wedge (q \wedge r)) \\ &= (p \wedge r \wedge q') \vee (p' \wedge q \wedge r) \\ &= ((p \wedge q') \vee (p' \wedge q)) \wedge r \\ &= (p+q)r, \end{aligned} \tag{2.11}$$

and the proof of the theorem is complete. ■

We have now shown the equivalence of the definitions of Boolean algebras as idempotent rings, and as distributive lattices.

2.2 Classical Logic

In this section, we discuss classical propositional logic (CPL) and its relation to Boolean algebras. We will analyze the **syntax** of CPL: the algebraic structure of sentences and propositions in this system, and the notions of theorem and proof. A useful and easily available reference for CPL (and the more general first-order logic) is the draft textbook by Buss [B]. Unless otherwise stated, the definitions and theorems in this section are taken from there.

The **language**, $\mathcal{L}$, of a **propositional calculus** is defined in terms of two disjoint sets: a set of **atomic (primitive) sentences** and a set of **propositional connectives**. The **sentences,** or **well-formed formulas (wffs)**, of $\mathcal{L}$ are constructed from its primitive sentences according to the formal rules appropriate to its propositional connectives.

For CPL, the propositional connectives are

$$\neg,\ \wedge,\ \vee,\ \implies,\ \text{and}\ \iff, \tag{2.12}$$

referred to as **negation, conjunction, disjunction, implication** or **material conditional,** and **biconditional,** respectively. The rules for forming sentences are as follows:

(1.) Any primitive sentence is a sentence;

(2.) If ϕ and ψ are sentences, then $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \implies \psi)$, and $(\phi \iff \psi)$, are sentences;

(3.) The set of sentences is the smallest set consistent with rules (1.) and (2.).

Note that parentheses are used in the recursive step (rule (2.)) to avoid ambiguity; although it is not part of the strict definition of CPL, they may be omitted if unnecessary. The number of primitive logical connectives may also be reduced, as in the axiom systems we will discuss below. There are many ways of doing this, but a popular one is to take $\implies$ and $\neg$ as primitive, and define the others in terms of them by $p \vee q := \neg p \implies q$, $p \wedge q := \neg(p \implies \neg q)$, and $p \iff q := (p \implies q) \wedge (q \implies p)$.

In the syntactic conception of proof, some finite list of sentences are accepted as **axioms.** The **theorems** of the deductive system consist of these axioms, together with any sentences which follow from the axioms and repeated applications of **deduction rules.** The first version of such an axiomatic system is due to Frege [F]. Frege's system was simplified by Łukasiewicz [L, p.136] to the system now known as P_2. In P_2, the axioms are taken to be

(A1) $\phi \implies (\psi \implies \phi)$;

(A2) $(\phi \implies (\psi \implies \chi)) \implies ((\phi \implies \psi) \implies (\phi \implies \chi))$;

(A3) $(\neg\phi \implies \neg\psi) \implies (\psi \implies \phi)$.

The single deduction rule, as in Frege's system, is **modus ponens:**

If ϕ and $(\phi \implies \psi)$ are theorems, then ψ is a theorem.

More formally, we may introduce the symbol $\vdash$ to denote deduceability. Thus, if Γ is some set of sentences, the notation $\Gamma \vdash \phi$ states that there is a valid deduction of ϕ starting from the statements in Γ and the axioms of the system; *i.e.* there is a (finite) sequence of statements $\phi_1, \phi_2, \ldots, \phi_N$ such that ϕ_N is equal to ϕ and, for all $i \leq N$, one of the following three possibilities is true:

- ϕ_i is an axiom;
- $\phi_i \in \Gamma$; or
- ϕ_i is inferred by modus ponens from two earlier statements ϕ_j, ϕ_k, with $j, k < i$.

(In the third case, we would specifically have that ϕ_k is equal to $\phi_j \implies \phi_i$.) In particular, if ϕ is a theorem, there is a valid deduction of ϕ starting from the axioms alone; in this case, we may take $\Gamma = \emptyset$, and we write simply $\vdash \phi$ rather then $\emptyset \vdash \phi$.

It is important to note that if $\Gamma \neq \emptyset$, the notation $\Gamma \vdash \phi$ makes no statement about whether the sentences in Γ or ϕ are in fact theorems. Even less does it make any claim as to the truth or otherwise of any of these statements (which would be a semantic claim). This last observation models the actual situation in mathematics, in which we may have a conditional proof of (say) the Riemann hypothesis, which depends on as-yet-unproved conjectures.

With the $\vdash$ notation, modus ponens may be written as

$$\{\phi, \phi \implies \psi\} \vdash \psi, \tag{2.13}$$

Given that modus ponens is the only deduction rule allowed in the system, it is useful to compile a list of useful theorems that may be used as antecedents in proofs. Here are some—note that part (2) is written using the derived connective $\wedge$. (Parts (3) and (4) are axioms in the system PL used in [B]; parts (5) and (6) were used as axioms by Frege.)

Proposition 2.12. *The following sentences are theorems in P_2.*

(1) $\phi \implies \phi$.

(2) $((\phi \implies \psi) \wedge (\psi \implies \chi)) \implies (\phi \implies \chi)$.

(3) $\neg\phi \implies (\phi \implies \psi)$.

(4) $(\neg\phi \implies \phi) \implies \phi$.

(5) $\phi \implies \neg\neg\phi$.

(6) $\neg\neg\phi \implies \phi$.

As an illustration of what is involved in this proposition, here is a proof of $\phi \implies \phi$.

(L1.) $\phi \implies ((\phi \implies \phi) \implies \phi)$ from (A1).

(L2.) $(\phi \implies ((\phi \implies \phi) \implies \phi)) \implies$
$((\phi \implies (\phi \implies \phi)) \implies (\phi \implies \phi))$ from (A2).

(L3.) $(\phi \implies (\phi \implies \phi)) \implies (\phi \implies \phi)$
from (L1.) and (L2.) by modus ponens.

(L4.) $\phi \implies (\phi \implies \phi)$ from (A1).

(L5.) $\phi \implies \phi$ from (L4.) and (L3.) by modus ponens.

Also, to support the relation with the system PL, here is a proof of $\neg\phi \implies (\phi \implies \psi)$.

(L1.) $(\neg\psi \implies \neg\phi) \implies (\phi \implies \psi)$ from (A3).

(L2.) $((\neg\psi \implies \neg\phi) \implies (\phi \implies \psi)) \implies$
$(\neg\phi \implies ((\neg\psi \implies \neg\phi) \implies (\phi \implies \psi)))$ from (A1).

(L3.) $\neg\phi \implies ((\neg\psi \implies \neg\phi) \implies (\phi \implies \psi))$
from (L1.) and (L2.) by modus ponens.

(L4.) $(\neg\phi \implies ((\neg\psi \implies \neg\phi) \implies (\phi \implies \psi))) \implies$
$((\neg\phi \implies (\neg\psi \implies \neg\phi)) \implies (\neg\phi \implies (\phi \implies \psi)))$ from (A2).

(L5.) $(\neg\phi \implies (\neg\psi \implies \neg\phi)) \implies (\neg\phi \implies (\phi \implies \psi))$
from (L3.) and (L4.) by modus ponens.

(L6.) $\neg\phi \implies (\neg\psi \implies \neg\phi)$ from (A1).

(L7.) $\neg\phi \implies (\phi \implies \psi)$ from (L6.) and (L5.) by modus ponens.

A vital role is also played by the **deduction theorem,** which states that provability of $\phi \implies \psi$ from Γ is equivalent to provability of ψ from $\Gamma \cup \{\phi\}$.

Theorem 2.13. (The Deduction Theorem)

$$\Gamma, \phi \vdash \psi \textit{ if and only if } \Gamma \vdash (\phi \implies \psi). \tag{2.14}$$

The proof of this theorem can be found in [B] (and many other places). As an illustration of the usefulness of this theorem, we can prove the derived deduction rule of **hypothetical deduction:** if $\phi \implies \psi$ and $\psi \implies \chi$ are both theorems, then so is $\phi \implies \chi$.

Theorem 2.14. (Hypothetical Syllogism) *For any sentences ϕ, ψ, and χ,*

$$\vdash (\phi \implies \psi) \implies ((\psi \implies \chi) \implies (\phi \implies \chi)). \tag{2.15}$$

Proof. Each line in the following list is equivalent to the previous one, by an application of the deduction theorem:

$$\begin{aligned} &\vdash (\phi \implies \psi) \implies ((\psi \implies \chi) \implies (\phi \implies \chi)) \\ \phi \implies \psi &\vdash ((\psi \implies \chi) \implies (\phi \implies \chi)) \\ \{\phi \implies \psi,\ \psi \implies \chi\} &\vdash \phi \implies \chi \\ \{\phi \implies \psi,\ \psi \implies \chi,\ \phi\} &\vdash \chi \end{aligned} \tag{2.16}$$

The last line is true, since the claimed proof follows from two applications of modus ponens, and the theorem follows. ■

As an example of the usefulness of this theorem, let us reprove the theorem $\neg\phi \implies (\phi \implies \psi)$. Using Theorem 2.14, the proof reduces to

(L1.) $\neg\phi \implies (\neg\psi \implies \neg\phi)$ from (A1).

(L2.) $(\neg\psi \implies \neg\phi) \implies (\phi \implies \psi)$ from (A3).

(L3.) $\neg\phi \implies (\phi \implies \psi)$ from (L1.) and (L2.) by hypothetical syllogism.

(The previous proof was, in fact, reconstructed from this one, using the ideas in the proof of the deduction theorem.)

The parallels between the connectives of CPL and the operations in a Boolean algebra certainly suggest that CPL might give rise to examples of Boolean algebras. However, the 'algebra' of sentences (wffs) is not yet identical to that of a Boolean algebra. For example, if ϕ is a sentence of CPL, then $\phi \wedge \phi$ is not the same sentence as ϕ (even if the punctuational parentheses are removed), whereas $p \wedge p = p$ in any Boolean algebra. On the other hand, intuition would suggest that the sentence $\phi \wedge \phi$ should have the same logical force as ϕ (at least if $\wedge$ is given some interpretation such as 'and'). Any discussion of meaning or interpretation strictly belongs to semantics rather than syntax, but observations such as these motivate the following definitions.

Definition 2.15. Sentences (wffs) ϕ and ψ are said to be **syntactically,** or **provably, equivalent** if $\phi \vdash \psi$ and also $\psi \vdash \phi$.

We will write $\phi \sim \psi$ if ϕ and ψ are syntactically equivalent.

Syntactic equivalence is indeed an equivalence relation on the set of sentences: $\phi \sim \phi$ because of the theorem $\phi \implies \phi$; symmetry follows directly from the definition; and transitivity follows from hypothetical syllogism (note that two application of the deduction theorem to (2.15) yield $\{\phi \implies \psi,\ \psi \implies \chi\} \vdash \phi \implies \chi$). We will denote the equivalence class of the sentence ϕ by $[\phi]$; these equivalence classes will be referred to as **propositions.** Informally, therefore, a proposition is a collection of sentences, any two of which are mutually provable.

Definition 2.16. The **Lindenbaum-Tarski algebra** of CPL is the set of propositions $[\phi]$, with operations $\wedge$, $\vee$, and $'$ defined by

$$[\phi] \wedge [\psi] = [\phi \wedge \psi], \quad [\phi] \vee [\psi] = [\phi \vee \psi], \quad \text{and} \quad [\phi]' = [\neg\phi]. \tag{2.17}$$

It is now a matter of unpacking definitions to show that the Lindenbaum-Tarski algebra of CPL is a Boolean algebra. In particular, note that $[\phi]'' = [\phi]$ due to Proposition 2.12(4)(5). Also, $[\phi] \wedge [\phi] = [\phi]$, since both $(\phi \implies (\phi \wedge \phi))$ and $((\phi \wedge \phi) \implies \phi)$ are theorems. More generally, note that that $[p] \leq [q]$ in the Lindenbaum-Tarski algebra if and only if $p \vdash q$. This can be seen to follow from the classical definition of partial order on sentences, but can also be deduced through the following chain of equivalent statements (where, of course, 0 denotes the least element of the algebra):

$$\begin{aligned} [p] &\leq [q] \\ [p] \wedge [q]' &= 0 \\ [p \wedge \neg q] &= 0 \\ &\vdash \neg(p \wedge \neg q) \\ &\vdash p \implies q, \end{aligned} \tag{2.18}$$

Definition 2.17. A **filter** F in a Boolean algebra $\mathcal{B}$ is a subset $F \subseteq \mathcal{B}$ such that

(F1) if $p \in F$ and $p \leq q$, then $q \in F$; and

(F2) if $p \in F$ and $q \in F$, then $p \wedge q \in F$.

For purposes of logic, it is more convenient to replace (F1) with

(F1′) if $p \in F$, then $p \vee q \in F$.

That this is equivalent to $(F1)$ in a Boolean algebra follows from the facts that (i) $p \leq p \vee q$, and (ii) $p \leq q$ if and only if $p \vee q = q$. Noting that $[p] \leq [q]$ in the Lindenbaum-Tarski algebra if and only if $p \vdash q$, we see that the set of provable statements in CPL will be a filter in the Lindenbaum-Tarski algebra.

In fact, this filter turns out to be the trivial filter $\{1\}$ which contains only the greatest element of the algebra. This follows from an application of the deduction theorem to axiom (A1), which yields $\phi \vdash \psi \implies \phi$; *i.e.* that if ϕ is a theorem, then so is $\psi \implies \phi$ for any ψ. In particular, if ϕ and ψ are both theorems, then $[\phi] = [\psi]$, so the theorems form a single equivalence class; and since any proposition implies any theorem, this equivalence class must be $\{1\}$.

3 TEST SPACES

In “What are Quantum Logics and What Ought They to Be?” [FR], authors D. J. Foulis and C. H. Randall answered the title question by developing the structures that they called manuals. They offered an alternative to the traditional algebraic structures that had up to that point been used to try to model the behavior of physical systems or experiments. Foulis and Randall sought to develop a more general apparatus that could in turn adequately handle the complex nature of the behavior of particles at the quantum level. In more recent literature, the name “test space” has replaced “manual.” More specifically, Foulis and Randall’s manuals are the same as “algebraic test spaces,” which will be defined in this section. The more broadly defined “test spaces” are the structures that Foulis and Randall referred to as “quasimanuals.” Throughout this paper, we will use the “test space” terminology to refer to these sets.

In classical probability, an experiment contains a set E of outcomes, which can also be contained themselves in sets called events. For each trial of the experiment, one outcome can occur and this outcome is assigned a probability. This probability is a mapping $\omega : E \to [0, 1]$. Test spaces serve as generalization of this familiar structure. A test space is simply defined as a non-empty set of non-empty sets. Each of these non-empty sets, called tests, is an analog of the classic probability space. The fact that outcomes and events can be shared between tests is what makes these spaces interesting and worth studying in their own right.

3.1 Example

We begin this chapter with an example taken from [FR] (where it is credited to [WR]). We have an urn containing some mixture of 4 types of balls, identical except that on each type of ball there are letters written in different colors, according to the following table.

We also have 3 pairs of goggles, one pair of each color: red, green, and blue. A *test* (or

Ball Type	Red	Green	Blue
I	a	a	x
II	b	z	b
III	y	c	c
IV	y	z	x

Figure 1: Test Space Example.

"operation" in the language of Foulis and Randall) consists of putting on a pair of goggles, removing one ball from the urn, and recording the letters that can be read, where it is assumed that we can only read the letters of the same color as the goggles we are wearing. There are three tests, defined by goggle color, which we can refer to as the Red, the Green, and the Blue test, and for each of these tests we can write down a sample space of all possible outcomes. Denoting these sample spaces by R, G, and B respectively, we have

$$R = \{a, b, y\}, \; G = \{a, z, c\}, \text{ and } B = \{x, b, c\}. \tag{3.1}$$

As in classical probability theory, the subsets of these sample spaces represent the possible events corresponding to each test. There are $2^3 = 8$ possible events for each test, but since there is some overlap there are only 17 distinct events in total:

$$\begin{aligned}
&\emptyset, \; \{a\}, \; \{b\}, \; \{c\}, \; \{x\}, \; \{y\}, \; \{z\}, \\
&\{a,b\}, \; \{a,y\}, \; \{b,y\}, \; \{a,z\}, \; \{a,c\}, \; \{z,c\}, \; \{x,b\}, \; \{x,c\}, \; \{b,c\}, \\
&R, \; G, \; \text{and } B.
\end{aligned} \tag{3.2}$$

Note that the events $\{a, y\}$ and $\{b\}$ are *complementary,* in the sense that there is some test (the Red test) in which one and only one of them can occur. Similarly, the events $\{x, c\}$ and $\{b\}$ are complementary via the Blue test. There is therefore some sense in which we might want to think of the events $\{a, y\}$ and $\{x, c\}$ are being logically equivalent; after all, under the appropriate experimental circumstances, each of them occurs if and only if $\{b\}$ does not, even though there is no single test which tests for them both simultaneously. Note also that $\{b\}$ occurs in each of the two tests under consideration if and only if a type II ball is picked; we may therefore interpret the equivalence class $\{\{a, y\}, \{x, c\}\}$ as the proposition 'a type II ball was not picked'. Note that although we may be able to *infer* the truth of this proposition if we observe either of the events $\{a, y\}$ or $\{x, c\}$, we do not *observe* it directly: what is observed is only the letters on the chosen ball.

3.2 Test Spaces

While Foulis and Randall developed the idea of test spaces, Wilce offered an updated survey of the theory of test spaces and orthoalgebras, in his "Test Spaces and Orthoalgebras." [W] The definitions and theorems in this section are adapted from his work.

Definition 3.1. A **test space** is a non-empty set $\mathfrak{A}$ of non-empty sets that is irredundant;

$\forall\, E, F \in \mathfrak{A}, E \subseteq F \implies E = F$

Elements of $\mathfrak{A}$ are called tests, and elements of tests are outcomes. The set $X = \bigcup_{E\in\mathfrak{A}} E$ is the outcome space of $\mathfrak{A}$. A probability weight, or state, is a mapping

$\omega : X \to [0, 1]$

where $\Sigma_{x\in E}\omega(x) = 1$ for every $E \in \mathfrak{A}$. The **state space**, $\Omega(\mathfrak{A})$, is the set of all such states.

Note that since tests are not necessarily disjoint, the same events can be contained in multiple tests. The probability of that event, however, must be independent of the test it is contained in. This also helps explain why the set of tests in any test space must be irredundant.

Definition 3.2. A test space consisting only of disjoint tests is **semi-classical**. A semi-classical test space with a single test is **classical**.

Definition 3.3. If $\mathfrak{A}$ is a test space, then outcomes a and b are **orthogonal** if and only if they are distinct but belong to some common test, and we write $a \perp b$.

Definition 3.4. Let $\mathfrak{A}$ be a test space. Then, set A is called an **event** if and only if there is some test $E \in \mathfrak{A}$ such that $A \subseteq E$. The set of all events in $\mathfrak{A}$ is denoted $\mathcal{E}(\mathfrak{A})$.

Definition 3.5. Let $\mathfrak{A}$ be a test space with $A, B \in \mathcal{E}(\mathfrak{A})$.

(1) A family $\{A_i\}$, $i \in I$ of events is **compatible** if and only if its members are all contained in a common test; i.e., if $\bigcup_{i \in I} A_i$ is an event.

(2) Disjoint compatible events are **orthogonal**, and we write $A \perp B$. If $\{A_i\}$ is a compatible family of pairwise disjoint events, then it is **jointly orthogonal**.

(3) If $A \perp B$ and $A \cup B$ is a test, then A and B are **complementary** and we write A c B.

(4) If A and B share a common complement, then they are **perspective** and we write $A \sim B$.

Proposition 3.6. *Let $\mathfrak{A}$ be a test space. Then: (a) If A c B, then $A \perp B$; (b) If $A \perp B$, then there exists $C \perp B$ with A c $(B \cup C)$; (c) For any tests $E, F \in \mathfrak{A}$, $E \sim F$.*

Proof (a) By the definition, if events A, B are complementary, then they are orthogonal. (b) If $A \perp B$, then $A \cap B = \emptyset$ and $A, B \subseteq E$ for some $E \in \mathfrak{A}$. Then, there exists some $C \subseteq E$ (which may or may not be empty) such that $(A \cup B)$ c C. So, $E = (A \cup B) \cup C = A \cup (B \cup C)$. Therefore, A c $(B \cup C)$. (c) For any tests $E, F \in \mathfrak{A}$, $E \cup \emptyset = E$ and $F \cup \emptyset = F$. So, E c $\emptyset$ and F c $\emptyset$. Therefore, $E \sim F$.

Definition 3.7. A test space is **algebraic** if and only if perspective events share the same complements. Symbolically:

$$A \sim B \text{ c } C \implies A \text{ c } C$$

Proposition 3.8. *Let $\mathfrak{A}$ be an algebraic test space. Then, perspectivity is an equivalence relation on the set of events in $\mathfrak{A}$. Symbolically:*

(a) $A \sim A$;

(b) If $A \sim B$, then $B \sim A$;

(c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof Let $\mathfrak{A}$ be an Algebraic test space with events $A, B, C \subseteq E$ for some $E \in \mathfrak{A}$.

(a) For any $A \subseteq E \in \mathfrak{A}$, there exists A' such that A c A', so $A \sim A$.

(b) Let B c C. Then, if $A \sim B$, A c C. So $B \sim A$.

(c) If $A \sim B$ then A c B'. If $B \sim C$ then $C \sim B$ and C c B'. Therefore, $A \sim C$.

Lemma 3.9. *Let $\mathfrak{A}$ be an algebraic test space, and let $A \sim A'$ and $B \sim B'$, for $A, B \in \mathcal{E}(\mathfrak{A})$. Then,*

$$A \perp B \implies A' \perp B' \text{ and } A \cup B \sim A' \cup B'$$

Proof. Since $A \perp B, \exists E \in \mathfrak{A}$ with $A \cup B \in E \in \mathfrak{A}$. Let $C = E \backslash A \cup B$. Then, $B \cup C$ c A. Since $\mathfrak{A}$ is algebraic and $A \sim A'$, $B \cup C$ c A'. So, B c $(C \cup A')$, and since $B \sim B'$, B' c $(C \cup A')$. Therefore, $B' \perp A'$ and $(A' \cup B')$ c C c $(A \cup B)$, so $A \cup B \sim A' \cup B'$. □

Corollary 3.10. *Let $\mathfrak{A}$ be an algebraic test space. Then, for all $A, B \in \mathcal{E}(\mathfrak{A})$,*

(a) If $A \perp B$ and $B \perp C$, then $A \perp C$;

(b) If $A \perp B$ and $A \subseteq B$, then $A = B$;

(c) $\mathfrak{A}$ is irredundant. (Even if there is no initial assumption that $\mathfrak{A}$ is irredundant.)

Proof. (a) Since $B \perp C$, let $D = (B \cup C)'$. Then, B c $(D \cup C)$. Since $A \sim B$, A c $(D \cup C)$, and $A \perp C$. (b) Since $A \subseteq B$, $B \backslash A \perp A$. By (a), $B \perp B \backslash A$. So, $B \cap B \backslash A = \emptyset$ and $B \subseteq A$. (c) For any tests $E, F \in \mathfrak{A}$, E c $\emptyset$ c F. So, $E \sim F$. Then, by (b), $\mathfrak{A}$ is irredundant. □

Proposition 3.11. *Let $\mathfrak{A}$ be a classical test space. Then, the set of events in $\mathfrak{A}$ is a Boolean algebra.*

Proof As referenced above, the power set of any set X, $\mathcal{P}(X)$, is a Boolean algebra. Let $\mathfrak{A}$ be a test space such that $\mathfrak{A} = \{E\}$. Then, $\mathfrak{A}$ is a classical test space, containing only one test. So, for any event $A \in \mathfrak{A}$, $A \subseteq \mathcal{P}(E)$, which is a Boolean algebra.

Thus, it follows that in the case of some physical phenomenon that can be modeled with a single set of outcomes, we have the structures needed to model such situations. These can be represented by the familiar Boolean algebra, and so we can also use the tools of classical propositional logic.

It is in the cases where such predictability fails us that the need for newer models reveals itself. In the case of quantum mechanics, it has become evident that the tools of classical logic will prove to be insufficient. The primary difference between classical propositional logic and quantum logic has proven to be the failure of the distributive property in the quantum case. This can be attributed to the well-known Heisenberg Uncertainty Principle. It states that the more precise we intend to be with our measurement of a particle's position, the less precisely we can measure it's momentum, and vice versa. Consider the following example taken from the Wikipedia article on quantum logic:

Let p be the momentum of a particle, q its position. Then, in any quantum state, the Uncertainty Principle tells us that:

$\Delta p \cdot \Delta q \geq \frac{\hbar}{2}$, where $\hbar = \frac{h}{2\pi}$ is the reduced Plank's constant.

Choose units where $\hbar = 1$. Then, we have $\Delta p \cdot \Delta q \geq \frac{1}{2}$. Now consider the following propositions:

p: "The particle has momentum in $[0, \frac{1}{6}]$."

q: "The particle has position in $[-1, 1]$."

r: "The particle has position in $[1, 3]$."

Consider the proposition $p \wedge q$: The particle has momentum in $[0, \frac{1}{6}]$ and its position is in $[-1, 1]$. Then, $\Delta p \cdot \Delta q = \frac{1}{6} \cdot 2 = \frac{1}{3} < \frac{1}{2}$. So, $p \wedge q$ is false. (Similarly, $p \wedge r$ is false.) Now, we examine the proposition $p \wedge (q \vee r)$: The particle has momentum in $[0, \frac{1}{6}]$ and its position is in $[-1, 3]$. Then, $\Delta p \cdot \Delta q = \frac{1}{6} \cdot 4 = \frac{2}{3} \geq \frac{1}{2}$. So, $p \wedge (q \vee r)$ can be true.

Therefore, if we apply the distributive law to our (sometimes) true statement $p \wedge (q \vee r)$, the result is: $(p \wedge q) \vee (p \wedge r)$. This second statement is always false. This counterexamples illustrates one example in which the distributive property fails in the quantum case and gives rise to the need to develop different structures for modeling quantum phenomenon.

Using the perspectivity relation, we can now make the connection between test spaces and orthoalgebras, which will be described in the next section.

The following definition is taken from Wilce[WA]:

Definition 3.12. First, let $\mathfrak{A}$ be any algebraic test space. Then, for any event $A \in \mathcal{E}(\mathfrak{A})$, let $p(A)$ be the equivalence class of A under perspectivity, and let $L(\mathfrak{A})$ be the set of all equivalence classes. Then, for all $p(A), p(B) \in L$, the relation

$p(A) \perp p(B) \Leftrightarrow A \perp B$

is well defined, as is

$p(A) \perp p(B) \Rightarrow p(A) \oplus p(B) := p(A \cup B)$

and the orthocomplement is given by

$p(A)' = p(B) \Leftrightarrow p(A) \oplus p(B) = p(E)$ for some $E \in \mathfrak{A}$.

The set L with the partial operator $\oplus$ (called the orthogonal sum or orthosum) and the orthocomplement, $(L, \oplus, ')$, is called the logic of $\mathfrak{A}$. [W]

4 ORTHOALGEBRAS

4.1 Properties of Orthoalgebras

We will now describe a class of propositional logics which represents "a structure which is sufficiently general to subsume all previously mentioned quantum propositional logics, but which is at the same time sufficiently small to be interesting." [LH1] Unless stated otherwise, the following definitions and propositions are taken from Lock and Hardegree.

Definition 4.1. An **orthoalgebra** is a set, $\mathcal{L}$, a partially defined binary operator (called the orthogonal sum or orthosum) $\oplus : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that $a \oplus b$ is defined if and only if $a \perp b$, a map $' : \mathcal{L} \to \mathcal{L}$ for all $a \in \mathcal{L}$, and two elements $0, 1 \in \mathcal{L}$, such that for all $a, b \in \mathcal{L}$, the following are satisfied:

(i) If $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$;

(ii) $a \perp 0$ and $a \oplus 0 = a$;

(iii) $a \perp a'$ and $a \oplus a' = 1$;

(iv) if $a \perp (a' \oplus b)$, then $b = 0$;

(v) if $a \perp (a \oplus b)$, then $a = 0$;

(vi) if $a \perp b$, then $a \perp (a \oplus b)'$ and $b' = a \oplus (a \oplus b)'$.

Proposition 4.2. *Let $\mathcal{L}$ be an orthoalgebra. Then, for all $a, b \in \mathcal{L}$;*

(a) $0' = 1$ *and* $1' = 0$

(b) $(a')' = a$

(c) if $a \oplus b = a \oplus c$ *then* $b = c$

(d) if $a \oplus b = 1$, *then* $b = a'$

Proof

(a) By (ii), we have $0 \oplus 0' = 0'$, and by (iii) we have $0 \oplus 0' = 1$. So, $0' = 1$.

By (ii), we know $0 \perp 1$. Then, using (iii), $0 \perp (1 \oplus 1')$. By the previously proven result, this is equivalent to $0 \perp (0' \oplus 1'$. Finally, (iv) gives the result $1' = 0$.

(b) By (iii), $a \perp a$. Using (vi), (iii), (ii), and the first part of this proposition: $(a')' = a \oplus (a \oplus a')' = a \oplus 1' = a \oplus 0 = a$.

(c) If $a \oplus b = a \oplus c$, then $a \perp b$ and $a \perp c$. By (vi), $a \perp (a \oplus b)'$ and $b' = a \oplus (a \oplus b)'$. Likewise, $a \perp (a \oplus c)'$ and $c' = a \oplus (a \oplus c)'$. Using substitution, $c' = a \oplus (a \oplus b)'$. So, $b' = c'$. By the lemma below, $b = c$.

(d) By (iii), $a \oplus a' = 1$. So, if $a \oplus b = 1$, then $a \oplus b = a \oplus a'$. Then, by part (c) of the proposition, $b = a'$. ∎

Lemma

If $a' = b'$ then $a = b$.

Proof For all $a \in \mathcal{L}, a \perp a'$. Using (vi), $(a')' = a \oplus (a \oplus a')'$. By (iii) and using substitution with the given condition, $(b')' = a \oplus 1'$. Lastly, using parts (a) and (b) of the preceding proposition and (ii): $b = a \oplus 0 = a$. So, $a = b$ by (i). ∎

It's important to note that in Lock and Hardegree's definition, the orthogonal sum operation is not necessarily associative. [LH1] Some authors, including Wilce in "Test Spaces and Orthoalgebras" requires associativity in his definition of the orthogonal sum operator. [WA]

Definition 4.3. An orthoalgebra $\mathcal{L}$ is associative if, for all $a, b, c \in \mathcal{L}$: If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$ and $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

Definition 4.4. Let $\mathcal{L}$ be an orthoalgebra with $a, b \in \mathcal{L}$. Then $a \leq b$ if and only if there is a $c \in \mathcal{L}$ with $a \perp c$ and $a \oplus c = b$.

Proposition 4.5. $\leq$ *is reflexive and antisymmetric.*

Proof

By (iii), $a \oplus 0 = a$, so $a \leq a$.

If $a \leq b$ and $b \leq a$, then there exists $c, d \in \mathcal{L}$ such that $a \oplus c = b$ and $b \oplus d = a$. So, $a \perp c$ and $b \perp d$. By (vi), $c' = a \oplus (a \oplus c)'$. Using substitution; $c' = (b \oplus d) \oplus b'$. So, $(b \oplus d) \perp b'$. By (iv), $d = 0$. So, $a = b \oplus 0 = b$. ■

Note that while the relation $\leq$ is reflexive and antisymmetric, as we have just shown, it is not necessarily transitive. A counterexample illustrating this is given in Patricia Lock's Thesis "Categories of Manuals." [LP]

Proposition 4.6. *Let $\mathcal{L}$ be an orthoalgebra with $a, b \in \mathcal{L}$. Then; (1) $a \leq b$ if and only if $b' \leq a'$; (2) $a \leq b$ if and only if $a \perp b'$; (3) if $\mathcal{L}$ is associative and $a \perp b$, then $x \leq a, b$ implies $x = 0$.*

Proof

(1) If $a \leq b$, then there exists $c \in \mathcal{L}$ such that $a \oplus c = b$. So, $a \perp c$ and by (vi), $a' = c \oplus (a \oplus c)'$. Using substitution, $a' = c \oplus b'$. So, $b' \leq a'$. If $b' \leq a'$, then there exists $d \in \mathcal{L}$ such that $b' \oplus d = a'$. So, $b' \perp d$ and by (vi), $(b')' = d \oplus (b' \oplus d)'$. Using substitution and part (b) of the preceding proposition, $b = d \oplus (a')' = d \oplus a$. So, $a \leq b$.

(2) If $a \leq b$, then $a \oplus c = b$. So, $a \perp c$. By (vi), $a \perp (a \oplus c)'$. By substitution, $a \perp b'$.

If $a \perp b'$, then by (vi), $a \perp (a \oplus b')'$ and $a \oplus (a \oplus b')' = (b')' = b$ So, $a \leq b$.

(3) If $x \leq a$ and $x \leq b$, there exists $a_1, b_1 \in \mathcal{L}$ such that $x \oplus a_1 = a$ and $x \oplus b_1 = b$. Since $a \perp b$, using substitution yields $a \perp (x \oplus b_1)$. Since $\mathcal{L}$ is associative, $x \perp a$. Substitution gives the result $x \perp (x \oplus a_1)$. By (v), $x = 0$. ■

Now, it remains to examine the relationship between the test spaces from the previous chapter and orthoalgebras. The following theorem shows that associative orthoalgebras are exactly the logics of algebraic test spaces. Recall that $L(\mathfrak{A})$ is the set of all equivalence classes in $\mathfrak{A}$ under the perspectivity operation, and that $\oplus$ is the partial binary operator defined at the end of the section on test spaces. $(L, \oplus, ')$ is the logic of $\mathfrak{A}$. The following proposition is from Wilce. [WA]

Proposition 4.7. *(a) Let $\mathfrak{A}$ be an algebraic test space and $(L, \oplus, ')$ the logic of the test space as defined above. Then, $(L, \oplus, ')$ is an associative orthoalgebra with zero element $0 = p(\emptyset)$ and unit element $1 = p(E)$, $\forall E \in \mathfrak{A}$. The orthocomplement of $p = p(A) \in L$ is given by $p' = p(E \backslash A)$, where $A \subseteq E \in \mathfrak{A}$;*

(b) If $(L, \oplus)$ is an associative orthoalgebra, the set $\mathfrak{A}_L$ of all finite subsets $E = \{a_1, ..., a_n\}$ of L with $a_1 \oplus \cdots \oplus a_n = 1$ is an algebraic test space and the set of equivalence classes of events in $\mathfrak{A}_L$, $\Pi(\mathfrak{A}_L)$ is isomorphic to L.

Proof (a) Let $\mathfrak{A}$ be an algebraic test space, and $(L, \oplus, ')$ the logic of $\mathfrak{A}$. Let $p(A)$ be the equivalence class of $A \in L$ under perspectivity in L.

Associativity: Let $A, B, C \in L$ and let $p(A) \perp (p(A) \oplus p(B))$. If $p(A) \oplus p(B)$ is defined, then $p(A) \perp p(B)$ and $p(A) \oplus p(B) = p(A \cup B)$. Since $p(C) \perp p(A \cup B)$, $p(C) \perp p(A)$ and $p(C) \perp p(B)$. So, by the definition of the $\oplus$ operator, $(p(A) \oplus p(B)) \oplus p(C) = p(A \cup B) \oplus p(C) = p((A \cup B) \cup C) = p(A \cup B \cup C)$. Similarly, $p(A) \oplus (p(B) \oplus p(C)) = p(A \cup B \cup C)$, so $(L, \oplus)$ is associative.

Orthocomplement: Let p' be the orthocomplement of $p(A) \in L$, for $A \subseteq E \in \mathfrak{A}$. Then, $p(A) \cap p' = \emptyset$ and $p(A) \cup p' = 1$. So, $p' = p(E) \backslash p(A) = p(E \backslash A) = p(A')$.

Conditions for an orthoalgebra: (i) If $p(A) \perp p(B)$, then $p(A) \cap p(B) = \emptyset$ and $p(A) \cup p(B) \in E$ for some $E \in \mathfrak{A}$. So, $p(B) \perp p(A)$. (ii) For any $p(A) \in L$, $p(A) \perp \emptyset$ and $A \cup \emptyset = A$. (iii) Using the previous result, for any test $E \in \mathfrak{A}$, $(p(A))' = p(A')$, and $p(A) \oplus p(A') = p(A \cup A') = P(E) = 1$. (iv) If $p(A) \perp (p(A') \oplus p(B))$, then $p(A) \oplus (p(A') \oplus p(B)) = (p(A) \oplus p(A')) \oplus p(B) = p(A \cup A') \oplus p(B)$. Since $p(A \cup A') = 1$, $B = 0$. (v) Let $p(A) \perp (p(A) \oplus p(B))$. Then, $p(A) \oplus p(A \oplus B) = (p(A) \oplus p(A)) \oplus p(B)$. But, $p(A) \oplus p(A)$ is defined only if $p(A) \perp p(A)$. So, $p(A) \cap p(A) = \emptyset$ and $p(A) \oplus p(A) = 1$. Therefore, $p(A) = 0$. (vi) Let $p(A) \perp p(B)$. Then, $p(A) \oplus p(B) = p(A \cup B) \in p(E)$, $E \in \mathfrak{A}$. So, there exists $(p(A \cup B)') \in L$ such that $p(A \cup B) \oplus (p(A \cup p(B))' = p(E) = 1$. Then,

$(p(A) \oplus p(B)) \oplus (p(A) \oplus p(B))' = 1$. By associativity and commutativity, $p(A) \oplus (p(B) \oplus (p(A) \oplus p(B))') = p(B) \oplus (p(A) \oplus (p(A) \oplus p(B))') = 1$. So, $p(A) \perp (p(A) \oplus p(B))'$ and $p(B') = p(A) \oplus (p(A) \oplus p(B))'$. Therefore, $(L, \oplus)$ is an associative orthoalgebra. ■

(b) Let $(L, \oplus)$ be an orthoalgebra, and let $\mathfrak{A}_L$ be the set of finite subsets $E = \{a_1, ..., a_n\} \subseteq L$ such that $\bigoplus_{a \in E} = 1$.

First, show that $\mathfrak{A}_L$ is a test space: By the definition of orthoalgebras, $1 \in L$ so L is non-empty. Then, for all $A \in L$, $\bigoplus_{a \in A} = 1$. Now, suppose there exists $B = \{\emptyset\} \in L$. Then, $\bigoplus_{b \in B} = 0$, a contradiction. Therefore, E is nonempty, for all $E \in L$. Now, let $E = \{a_1, ..., a_n\} \in \mathfrak{A}_L$ and $F = \{a_{i_1}, ..., a_{i_k}\} \subseteq E$ $(1 \leq k \leq n)$. Then, $\bigoplus_{j=1}^{k} a_{i_j} = 1 = \bigoplus_{j=1}^{n} a_j = (\bigoplus_{j=1}^{k} a_{i_j}) \oplus (\bigoplus_{j=1}^{k} a_{i_j})'$. So, $\bigoplus_{j \in \{1,...,n\} \setminus \{i_1,...,i_k\}} a_j = 0$. This contradiction shows that $\mathfrak{A}_L$ is irredundant. Therefore, $\mathfrak{A}_L$ is a test space.

Now it remains to show that $\mathfrak{A}_L$ is algebraic. Let $A \sim B$, with $A, B \in \mathfrak{A}_L$. Then, there exists some $D \in \mathfrak{A}_L$ with A c D and B c D. So, $\bigoplus_{a \in A} a = (\bigoplus_{d \in D} d)' = \bigoplus_{b \in B} b$.

Now suppose A c C. Then, $A \cap C = \emptyset$ and $\bigoplus_{c \in C} c = (\bigoplus_{a \in A} a)' = (\bigoplus_{b \in B} b)'$. So, $(\bigoplus_{c \in C} c) \oplus (\bigoplus_{b \in B} b) = 1$. Since this orthosum is defined, $B \cap C = \emptyset$ and $B \cup C \in \mathfrak{A}_L$. Therefore, B c C and $\mathfrak{A}_L$ is an algebraic test space.

Last, it will be shown that there exists an isomorphism between the the set of all equivalence classes in an algebraic test space,$L(\mathfrak{A}_L)$ and the orthoalgebra L. Note that each equivalence class in $L(\mathfrak{A}_L)$ has a representative in L, i.e. $p(A) = A \in L$. Now, define a map $\varphi : L(\mathfrak{A}_L) \to L$ such that $p(A) \mapsto A$. To show that φ is surjective, let $p \in L$. Then, take A to be the singleton set $A = \{p\} \subseteq E$. Now, for any element in L, its orthocomplement exists, so $E = \{p, p'\}$. Then, $\bigoplus_{a \in A} a = p$. and so $\varphi : p(A) \mapsto p$. Therefore, φ is surjective. Now, suppose that $\varphi : p(A) \mapsto A$ and $\varphi : p(B) \mapsto A$. If $p(B) \mapsto A$, then $B \sim A$ and $p(A) = p(B)$. So, φ is injective. Therefore, $\varphi : L(\mathfrak{A}_L) \to L$ is a bijection.

Finally, to prove that φ creates an isomorphism between the two sets, it remains to show that three following conditions hold: (i) $p \perp q \iff \varphi(p) \perp \varphi(q)$; (ii) $\varphi(p') = (\varphi(p))'$; (iii) $\varphi(p \oplus q) = \varphi(p) \oplus \varphi(q)$.

(i) $p(A) \perp p(B) \iff A \perp B \iff \bigoplus_{a \in A} a \perp \bigoplus_{b \in B} b \iff \varphi(p(A)) \perp \varphi(p(B))$. (ii) Let $A = \{a_1, ..., a_m\} \in \mathcal{E}(\mathfrak{A}_L)$, $A \subseteq E = \{a_1, ..., a_m, b_1, ..., b_n\} \in \mathfrak{A}_L$. Let $B = \{b_1, ..., b_n\} = E \backslash A$. Then $p(A) \oplus p(B) = p(A \oplus B) = p(E) = 1$. So, $(p(A))' = p(B)$. Then, $\varphi(p(A))' = \varphi(p(B)) = \bigoplus_{b \in B} b = [\varphi(p(A))]'$ since $(\bigoplus_{a \in A} a) \oplus (\bigoplus_{b \in B} b) = \bigoplus_{e \in E} e = 1$, $1 \in L$. (iii) If $A \oplus B$ is define, there exists $E \in \mathfrak{A}_L$ such that $A \cup B \subseteq E$ and $A \cap B = \emptyset$. Let $E = \{a_1, ..., a_m, b_1, ..., b_n, c_1, ..., c_k\}$, $A = \{a_1, ..., a_m\}$, and $B = \{b_1, ..., b_n\}$. Then, $\varphi(p(A) \oplus p(B)) = \varphi(p(A \oplus B)) = \bigoplus(A \oplus B) = (\bigoplus_{a \in A} a) \oplus (\bigoplus_{b \in B} b) = \varphi(p(A)) \oplus \varphi(p(B))$. Therefore, $L(\mathfrak{A}_L) \simeq L$. ■

Thus we have shown that if we start with an associative orthoalgebra, we can construct from it the set of equivalence classes of events under perspectivity, which is itself an algebraic test space. Then, this construction is isomorphic to the orthoalgebra that we started with.

REFERENCES

[B] Buss, S. (2023). *Introduction to Mathematical Logic (Version: Draft B.2.3)*. Downloaded from `https://mathweb.ucsd.edu/~sbuss/IntroMathLogic/Fullbook_Draft.pdf`

[FR] Foulis, D. and Randall, C. (1981). What Are Quantum Logics and What Ought They to Be?, *Current Issues in Quantum Logic*. Springer, Boston, MA.

[F] Frege, G. (1879, 1972). *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*, Halle: L. Nebert. Translated as *Conceptual Notation: a formula language of pure thought modeled upon the formula language of arithmetic* in T. W. Bynum (Trans.), *Conceptual Notation, and Related Articles*, Oxford: Clarendon Press.

[H] Halmos, P. (1956). The basic concepts of algebraic logic, *The American Mathematical Monthly*, *63*(6), pp.363–387.

[LH1] Lock, P. and Hardegree, G. (1984). Connections Among Quantum Logics Part 1: Quantum Propositional Logics, *International Journal of Theoretical Physics*, Vol. 24, No. 1, 43-53

[LP] Lock, P. (1981). "Categories of Manuals," [Doctoral Dissertation, University of Massachusetts].

[L] Łukasiewicz, J. (1970). Selected Works, North-Holland.

[WA] Wilce, A. (2000). Test Spaces and Orthoalgebras, *Current Research in Operational Quantum Logic*, pp. 81-114.

[WR] Wright, R. (1978). The State of the Pentagon: A Nonclassical Example, in *Mathematical Foundations of Quantum Theory,* A.R. Marlow (ed.), Academic Press, New York, pp. 255-274