Gross's Proof of Local Existence for the Coupled Maxwell-Dirac Equations

by

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ABSTRACT

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Abstract

The Maxwell-Dirac equations are a model for the interaction of a relativistic electron with an electromagnetic field. It is to be expected that the initial value problem will have an unique solution which exists for all time $t > 0$, for all appropriate initial conditions. This is not yet known, but in 1966, Leonard Gross proved a local existence theorem. In this thesis, we will present an overview of Leonard Gross's proof.

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Kimberly Harry

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Introduction

I.1 The Maxwell-Dirac Equations

The Maxwell-Dirac equations are a model for the interaction of a relativistic electron with an electromagnetic field. This arise from using the Dirac's wave function of the electron as the source term in the Maxwell equation. This gives a nonlinear system of partial differential equations (PDE's). The (coupled) Maxwell-Dirac equations, in units in which $c = \hbar = 1$, are

$$\Box A^{\mu} = q\overline{\psi}\gamma^{\mu}\psi \equiv s^{\mu} \tag{1.1}$$

$$\left[i\gamma^{\mu}(\partial_{\mu} + iqA_{\mu}) + m\right]\psi = 0 \tag{1.2}$$

where $\Box$ is the d'Alembertian (wave operator) such that $\Box = \partial_t^2 - \triangle_x$, $A^{\mu} : \mathbb{R}^3 \to \mathbb{R}^4$ $(\mu = 0, 1, 2, 3)$ is the electromagnetic 4-potential operator, $\psi : \mathbb{R}^3 \to \mathbb{C}^4$ is the Dirac wave function for a relativistic electron for mass (m) and charge of a single electron (q), and the γ^{μ}'s are 4×4 matrices.

Notice that if we fix ψ in equation 1.1, we get a linear equation for A^{μ}. On the other hand if we fix A^{μ} in equation 1.2, we get a linear equation for ψ. If we know either ψ or A^{μ}, we have a linear equation. These two parts interact with each other. However if we do not know either, it is a nonlinear system of equations.

The Maxwell equations describe the behavior of the electromagnetic fields E and B. They imply that existence of potential functions φ and $\vec{A} = \langle A^1, A^2, A^3 \rangle$, such that $B = \nabla_{\lambda}\vec{A}$ and

$\vec{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ where $(c = 1)$. In terms of the 4-potential, $A^\mu = (\vec{A}, \varphi)$ the Maxwell equations become

$$\Box A^\mu = s^\mu \tag{1.3}$$

(where s^μ is the change current density) with the auxiliary Lorenz gauge condition

$$\partial_\mu A^\mu = 0. \tag{1.4}$$

The Maxwell equations describe the electromagnetic fields produced by a source s^μ.

The Dirac equation for the wave function $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ of a relativistic electron moving in a 4-potential A^μ in (1.2) where γ^μ $(\mu = 0, 1, 2, 3)$ are 4×4 matrices. The charge-current 4-vector of the electron is taken to be

$$s^\mu = q\overline{\psi}\gamma^\mu\psi. \tag{1.5}$$

The Maxwell equations tell how the electron will move in a given field.

The Maxwell-Dirac equations are the result of using s^μ from (1.5) as the source term in the Maxwell equations.

Mainly, our goal in this paper is to show that the Maxwell-Dirac equations have a local solution for all appropriate initial conditions.

I.2 Description of Equations and Spaces

Before stating our main theory, we need to define a few equations and spaces.

First we are going to define the Sobolev Space based on L^2. Let $s \in \mathbb{R}$. Then

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}' : (1 + |\cdot|^2)^{\frac{s}{2}}\hat{f} \in L^2\} \tag{2.1}$$

$$H^{\{s\}}(\mathbb{R}^n) = \{f \in \mathcal{S}' : (1 + |\cdot|^2)^{\frac{s-1}{2}}|\cdot|\hat{f} \in L^2\}. \tag{2.2}$$

These are spaces that have s derivatives in the L^2 spaces and are defined in terms of the Fourier Transform. Also, I would like to point out that $\mathcal{S}'$ is the space of tempered distribution on $\mathbb{R}^n$, $H^0 = L^2$, and $H^s(\mathbb{R}^n;\mathbb{C}^n) = (H^s(\mathbb{R}^n))^n$.

Second, we want to define the Banach Spaces (actually Hilber spaces)

$$X = X_D \oplus X_M \tag{2.3}$$

where

$$X_D = H^0(\mathbb{R}^3;\mathbb{C}^4), \quad X_M = H^{\{\frac{1}{2}\}}(\mathbb{R}^3;\mathbb{R}^4) \oplus H^{-\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^4); \tag{2.4}$$

and

$$Y = Y_D \oplus Y_M \tag{2.5}$$

where

$$Y_D = H^1(\mathbb{R}^3;\mathbb{C}^4), \quad Y_M = H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R}^4) \oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^4). \tag{2.6}$$

We are setting up the X_M and Y_M to carry the complete A^μ (the electromagnetic potential, hence the M for Maxwell) and the X_D and Y_D for those of ψ (the Dirac wave equation).

I.3 Statement of Gross's Theorem

In a fixed reference frame, writing $t = x_0$ and introducing a new variable $\dot{A}^\mu = \frac{\partial A^\mu}{\partial t}$ and write $A^\mu = (\vec{A}, \varphi)$; the Maxwell-Dirac equations become

$$\begin{aligned} \frac{\partial A^\mu}{\partial t} - \dot{A}^\mu &= 0 && (3.1)\\ \frac{\partial \dot{A}^\mu}{\partial t} - \triangle A^\mu &= q\overline{\psi}\gamma^\mu\psi && (3.2)\\ \frac{\partial \psi(x,t)}{\partial t} &= (\vec{\alpha}\cdot\nabla + \beta m)\psi(x,t) + i(\vec{\alpha}\cdot\vec{A}^\mu(x,t) + \varphi)\psi(x,t). && (3.3) \end{aligned}$$

In (3.3), the α's are 4×4 matrices related to the γ's.

Considering equations 3.1, 3.2, and 3.3 as the initial value problem and these suitable initial conditions

$$\psi_0(x) \equiv \psi(x,0) \tag{3.4}$$

$$A_0^\mu(x) \equiv A^\mu(x,0) \tag{3.5}$$

$$\dot{A}_0^\mu(x) \equiv \frac{\partial}{\partial t}A^\mu(x,0); \tag{3.6}$$

it is expected that the initial value problem will have an unique solution which exists for all time, for all appropriate initial conditions. The goal to show that the existence of global solutions to the Maxwell-Dirac equations is still out of reach. However in Gross's paper [G], he proved the following local existence theorem.

Theorem 3.7. *Let ψ_0 be in $Y_D = H^1(\mathbb{R}^3;\mathbb{C}^4)$ and let the pair $(A_0,\dot{A}_0)$ be in $Y_M = H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R})$. Then there exists a number $T>0$ and unique continuous functions $\psi(t)$, $(A(t),\dot{A}(t))$, from $[0,T]$ into Y_D and Y_M, respectively, such that $\psi(0)=\psi_0$ and $(A(0),\dot{A}(0)) = (A_0,\dot{A}_0)$ and such that as functions into $X_D = L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $X_M = H^{\{\frac{1}{2}\}}(\mathbb{R}^3;\mathbb{R}^4) \oplus H^{-\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^4)$, respectively, they are strongly continuously differentiable and satisfy*

$$\frac{d\psi}{dt} = i(H+A)\psi \tag{3.8}$$

and

$$\frac{d}{dt}\begin{pmatrix} A \\ \dot{A} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix}\begin{pmatrix} A \\ \dot{A} \end{pmatrix} - \begin{pmatrix} 0 \\ 4\pi q\overline{\psi}\gamma^\mu\psi \end{pmatrix}. \tag{3.9}$$

The $H+A$, in (3.8) is a selfadjoint operator with domain Y_D and $(A(t)\psi)(x) = A(x,t)\psi(x)$, while the square matrix in equation (3.9) represents a skewadjoint operator on X_M with domain Y_M.

The proof of this theorem will be done as a fixed point argument after working out a few needed details. We will show that give a suitable $\psi \in X = C([0,T];Y_D)$, we can define A^μ as the solution of the Maxwell equation (1.1) which is a linear system for fixed ψ. Then we will define $R\psi$ as the solution of the Dirac equation (1.2) with this A^μ and show that $R : C([0,T];Y_D) \to C([0,T];Y_D)$ and is a contraction for sufficiently small T. With this unique fixed point, we can get the solution to the full Maxwell-Dirac equation. A key component of this strategy is that we have to solve A^μ with suitable space. In the next section, we will analyze this.

Kato's Theorem

II.1 The Main Tool: Kato's Theorem

To work through Gross's Theorem I.3.7, we are mainly going to work with Kato's Theorem on linear evolution equations in Banach Space. An operator A in a Banach space X is said to have property S if

- A is a closed linear operator with domain $D(A)$ dense in X; and
- the resolvent set $\rho(A)$ contains all positive reals, and

$$\|(I - \lambda A)^{-1}\| \leq 1, \quad \forall \lambda > 0. \tag{1.1}$$

Before we get into Kato's Theorem, let's first state it. Note that in his paper [K], Kato states the theorem with seemingly different hypothesis. We prove the equivalence of the two versions of Kato's Theorem in the Appendix (A.2) following [SG]. A proof of Kato's Theorem as stated here can also be found in Theorem (5.3) from Pazy paper, [P]. Note that if A is a self-adjoint operator, then iA has the property of S.

Theorem 1.2. *Suppose that $A(t)$ has property S for each $t \in [a, b]$, and that the domain $D = D(A(t))$ is independent of t. Moreover, suppose that for every $x \in D$,*

$$t \mapsto A(t)x \textit{ is continuously differentiable} \tag{1.3}$$

in the norm of X. Then, if $x_\alpha \in D$ and if $f : (a,b) \mapsto D$ has the property that $(A(r) - I)f(t)$ is strongly continuous in t for some fixed r, the differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad a \leq t \leq b, \tag{1.4}$$

has a unique solution $x = x(t)$ with $x(t) \in D$ for all t, which is strongly differentiable in t and satisfies the initial condition $x(a) = x_\alpha$. Moreover, $t \mapsto \frac{\partial x}{\partial t}$ and $t \mapsto A(t)x(t)$ are strongly continuous.

Moreover, there is a family of evolution operators $U(t,s)$ such that $0 \leq s \leq t$, depending only on $A(t)$, such that the solution of the Dirac equation is given by

$$x(t) = U(t,0)x_a + \int_0^t U(t,s)f(s)ds. \tag{1.5}$$

The solution of (1.5) is also unique and the operators $U(t,s)$ have the following properties:

1. $U(t,s)$ is an unitary operator in L^2 that maps $X \to X$ for $0 \leq s \leq t \leq T$. This implies $\|U(t,s)\|_{X \to X} = 1$
2. Strongly continuous in (t,s) jointly and $U(t,t) = I$ (identity). Thus, $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in X$ (pointwise continuity)
3. $U(t,s)\big|_D : D \to D$
4. $U(t,s)\big|_D$ is strongly continuous in D

The Dirac Operator is the constant-coefficient differential operator

$$H_0 = -i\vec{\alpha} \cdot \nabla + \beta m \tag{1.6}$$

where α_j, $j = 1,2,3$ and β are anti-commuting 4×4 hermitian matrices satisfying $\alpha_j^2 = \beta^2 = I$.

Here H_0 is a combination of derivatives. Let $X_D = L^2(\mathbb{R}^3;\mathbb{C}^4)$, with scalar product

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}^3} \sum_{j=1}^{4} \overline{\phi_j(x)} \psi_j(x)\, dx, \tag{1.7}$$

and recall that $Y_D = H^1(\mathbb{R}^3;\mathbb{C}^4)$. It is clear that $H_0\psi$ can be defined for all $\psi \in Y_D$. We will show that this is in fact the "right" domain for this operator (these results on the free Dirac hamiltonian are taken from [T]). This is saying that H_0, as an operator in X_0, is self-adjoint in this domain.

In the actual Maxwell-Dirac equation, H_0 is not the operation. It is

$$\frac{d\psi}{dt} = i(H_0 + \mathbf{A})\psi. \tag{1.8}$$

$\mathbf{A}$ is the evolution generator in Kato's Theorem. In order to make things easier, we are going to split $\mathbf{A}$ into two parts by notating them as $A^{1,\mu}$ and $A^{2,\mu}$. Then our equation becomes

$$\frac{d\psi}{dt} = i(H_0 + A^{1,\mu})\psi + i(A^{2,\mu})\psi. \tag{1.9}$$

Making sure our spaces are comparable for out work, we let $X = X_D$ and $D = Y_D$. Now we can see this equation, (1.9), looks like (1.4) in Kato theorem where

$$A(t) = i(H_0 + A^{1,\mu}(t)) \quad and \quad f(t) = iA^{2,\mu}(t)\psi(t) \tag{1.10}$$

and where $(H_0 + A^{1,\mu})$ and $A^{2,\mu}(t)\psi(t)$ satisfy the following conditions.

$$\begin{cases} \Box A^{1,\mu} = 0 \\ A^{1,\mu}(0) = A^{\mu}{}_0 \\ \dfrac{\partial}{\partial t} A^{1,\mu}(0) = \dot{A}^{\mu}{}_0 \end{cases} \tag{1.11}$$

In the application to the Maxwell-Dirac equation, $A(t)$ is not $H_0 + A^{\mu}(t)$, but $H_0 + A^{1,\mu}$

when it satisfies these conditions. Thus, $H_0 + A^{1,\mu}$ represents A(t) in Kato's Theorem. The conditions for $A^{2,\mu}(t)\psi(t)$ are

$$\begin{cases} \Box A^{2,\mu} = q\overline{\psi}\gamma^{\mu}\psi \\ A^{2,\mu}(0) = 0 \\ \dfrac{\partial}{\partial t}A^{2,\mu}(0) = 0 \end{cases} \tag{1.12}$$

And $A^{2,\mu}$ represents f(t) in Kato's Theorem.

Analysis of $A(t)$

In this sections we are going to talk about the Dirac Operator.

III.1 H_0 is Self-adjoint

For the Dirac Operator, we will show that H_0 is self-adjoint on Y_D by showing that it is unitarily equivalent to a multiplication operator. In order to do this, we need the definition of a multiplication operator.

Definition 1.1. *Let (X, μ) be a measure space, and let $f : X \mapsto \mathbb{R}$ be finite almost everywhere. The multiplication operator M_f in $L^2 \equiv L^2(X, \mu)$ is defined by*

$$\begin{aligned} D(M_f) &= \{g \in L^2 \mid fg \in L^2\} \\ M_f g &= fg. \end{aligned} \tag{1.2}$$

Now that we have defined what a multiplication operator is, we can state this theorem.

Theorem 1.3. *If μ is a σ-finite measure, the multiplication operator M_f is self-adjoint.*

Because this theorem is well known, we are omitting the proof. However, one thing that we want to take away is that immediately following the proof an operator which is unitarily equivalent to a multiplication operator must be self-adjoint.

Theorem 1.4. *The operator* H_0 *in* $L^2(\mathbb{R}^3;\mathbb{C}^4)$ *defined by*

$$D(H_0) \;=\; H^1(\mathbb{R}^3;\mathbb{C}^4) \tag{1.5}$$

$$H_0\psi \;=\; (-i\vec{\alpha}\cdot\nabla + \beta m)\psi \tag{1.6}$$

is self-adjoint.

Proof. Under the Fourier transform (operating unitarily on X_D), H_0 transforms to a multiplication operator: $\mathcal{F}H_0\mathcal{F}^{-1} = M_h$, where

$$h(\xi) = \begin{pmatrix} mI & \vec{\sigma}\cdot\xi \\ \vec{\sigma}\cdot\xi & -mI \end{pmatrix}, \tag{1.7}$$

and when α and β are chosen a certain way we can arrange it that $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the matrix-valued vector whose components are the (hermitian) **Pauli matrices,**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.8}$$

which satisfy

$$(\sigma_j)^2 = 1, \qquad \sigma_j\sigma_k = -\sigma_k\sigma_j, \quad (j \neq k; j,k = 1,2,3). \tag{1.9}$$

The 4×4 matrix $h(\xi)$ is therefore hermition for each $\xi \in \mathbb{R}^3$, with eigenvalues

$$\lambda_1(\xi) = \lambda_2(\xi) = -\lambda_3(\xi) = -\lambda_4(\xi) = \sqrt{m^2+\xi^2} \equiv \lambda(\xi). \tag{1.10}$$

For each ξ, let $U(\xi) : \mathbb{C}^4 \to \mathbb{C}^4$ be the unitary transformation which diagonalizes $h(\xi)$, so that $U(\xi)h(\xi)U(\xi)^{-1}$ is diagonal. ($U(\xi)$ can be computed explicitly, if desired.) Then the unitary transformation $U\mathcal{F}$ converts H_0 into multiplication by the diagonal matrix function

$$\Delta(\xi) \equiv \operatorname{diag}(\lambda(\xi), \lambda(\xi), -\lambda(\xi), -\lambda(\xi)). \tag{1.11}$$

Now, since the unitary operator $U(\xi)$ does not change the domain of any multiplication operator,

$$D(H_0) = \mathcal{F}^{-1}U^{-1}D(M_\Delta) = \mathcal{F}^{-1}D(M_\Delta) = H^1(\mathbb{R}^3;\mathbb{C}^4), \tag{1.12}$$

for any $m \neq 0$. (If $m = 0$, the domain is $\dot{H}^1(\mathbb{R}^3;\mathbb{C}^4) = H^{\{1\}}(\mathbb{R}^3;\mathbb{C}^4)$.)

The statements about $\sigma(H_0)$ now follow easily from the corresponding statements about $\sigma(M_\Delta)$. □

III.2 $H_0 + A^{1,\mu}(t)$ is Self-adjoint

We are going to treat $H_0 + A^{1,\mu}$ as a perturbation of H_0 using the Kato-Rellich Theorem. The proof of the Kato-Rellich Theorem in turn depends on this definition and theorems.

Definition 2.1. *Let A and B be operators in a Hilbert space $\mathcal{H}$. We say that B is relatively bounded with respect to A, or A-bounded, if $D(B) \supseteq D(A)$ and there are positive constants α and β such that*

$$\|B\phi\| \leq \alpha\|A\phi\| + \beta\|\phi\| \qquad \forall\phi \in D(A). \tag{2.2}$$

The infimum of possible constants α for which there is such a β is called the A-bound of B.

Theorem 2.3. (The Fundamental Criterion for Self-Adjointness) *Let A be a symmetric, densely-defined, operator in a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent:*

1. *A is self-adjoint;*
2. *A is closed, and $\mathrm{Ker}(A^* \pm i) = \{0\}$;*
3. *$\mathrm{Ran}(A \pm i) = \mathcal{H}$.*

Theorem 2.4. *Let A be self-adjoint, and $\lambda \in \mathbb{R}$. Then $(A \pm i\lambda)^{-1}$ and $A(A \pm i\lambda)^{-1}$ are bounded, and*

$$\|(A \pm i\lambda)^{-1}\| \leq |\lambda|^{-1} \text{ and } \|A(A \pm i\lambda)^{-1}\| \leq 1. \tag{2.5}$$

Theorems 2.3 and 2.4 are taken from [RS] where the proofs to each can be found.

Theorem 2.6. (Kato-Rellich) *Suppose that A is self-adjoint, B is symmetric, and B is A-bounded with A-bound less than 1. Then $A+B$ is self-adjoint on $D(A)$.*

Proof. We will show that $Ran(A+B\pm i\mu)=\mathcal{H}$ for some $\mu>0$. Since A is self-adjoint, we can use (2.5) to estimate $\|B(A\pm i\mu)^{-1}x\|$ for $x\in\mathcal{H}$, leading to

$$\|B(A-i\mu)^{-1}x\| \leq \left(\alpha+\frac{\beta}{\mu}\right)\|x\|. \tag{2.7}$$

Since $\alpha<1$, the estimate (2.7) shows that $C:=B(A\pm i\mu)^{-1}$ has norm less than 1 for μ sufficiently large, so $I+C$ is invertible. In particular, $\mathrm{Ran}(I+C)=\mathcal{H}$. Since A is self-adjoint, $\mathrm{Ran}(A\pm i\mu)=\mathcal{H}$ also. Now the equation

$$(I+C)(A\pm i\mu)x=(A+B\pm i\mu)x, \qquad \forall\phi\in D(A) \tag{2.8}$$

shows that $\mathrm{Ran}(A+B\pm i\mu)=\mathcal{H}$. Thus $A+B$ is self-adjoint by the fundamental criterion. □

In the application to the Maxwell-Dirac equations, we wish to consider the operator $H_0+A^{1,\mu}(t)$, where for each $t\in\mathbb{R}$ and $A^{1,\mu}(t)$ represents the operator of multiplication by the function $A^{1,\mu}(x,t)$ whose Fourier transform is

$$\hat{A}^{1,\mu}(\xi,t)=\hat{A}_0(\xi)\cos(|\xi|t)+\hat{\dot{A}}(\xi)\frac{\sin(|\xi t|)}{|\xi|}. \tag{2.9}$$

Here this equation is obtained from the the definition of $A^{1,\mu}$. To show that $H_0+A^{1,\mu}(t)$ is also self-adjoint on $D(H_0)$, it suffices by the Kato-Rellich Theorem to show that $A^{1,\mu}(t)$ is H_0-bounded, with H_0-bound less than 1. In fact, we will show the H_0-bound is equal to 0.

Lemma 2.10. *Let $s\geq 0$, and suppose $(f_0,g_0)\in Z:=H^{\{s\}}(\mathbb{R}^3)\oplus H^{s-1}(\mathbb{R}^3)$. Define $f(x,y)$ by*

$$\hat{f}(\xi,t)=\hat{f}_0(\xi)\cos(|\xi|t)+\hat{g}_0(\xi)\frac{\sin(|\xi|t)}{|\xi|}. \tag{2.11}$$

Then $(f(\cdot,t), \partial_t f(\cdot,t)) \in Z$ for all $t \in \mathbb{R}$, and its Z-norms is independent of t.

Proof. Since $|\hat{f}|^2 = (Re(\hat{f}))^2 + (Im(\hat{f}))^2$, we can assume that $\hat{f}$ is real-valued. In this case, the norm in question is

$$\begin{aligned} & \|\nabla f(\cdot,t)\|^2_{H^{s-1}} + \|\partial_t f(\cdot,t)\|^2_{H^{s-1}} && (2.12)\\ = & \int (1+|\xi|^2)^{s-1}|\xi|^2|\hat{f}(\xi,t)|^2 d\xi + \int (1+|\xi|^2)^{s-1}|\partial_t \hat{f}(\xi,t)|^2 d\xi && (2.13)\\ = & \int (1+|\xi|^2)^{s-1}(|\xi|^2 \hat{f}_0(\xi)^2 + \hat{g}_0(\xi)^2) d\xi && (2.14) \end{aligned}$$

using (2.11). This is clearly independent of t and is finite if $(f_0, g_0) \in Z$. □

Applying this result to the definition of $A^{1,\mu}$, we see that if $(A_0, \dot{A}_0) \in X_M$, then $(A^{1,\mu}(\cdot,t), \frac{\partial}{\partial t}A^{1,\mu}(\cdot,t)) \in X_M$ for all time. Also, if $(A_0, \dot{A}_0) \in Y_M$, then $(A^{1,\mu}(\cdot,t), \frac{\partial}{\partial t}A^{1,\mu}(\cdot,t)) \in Y_M$ for all time. Even though we are looking at $(A_0, \dot{A}_0)$ and want to know what happens for both parts, we are more interested in A_0 than in $\dot{A}_0$ (the time derivative).

Lemma 2.15. *Let $f \in H^{\{s\}}(\mathbb{R}^n)$ for some $s \geq 0$. Then f is a function (defined almost everywhere on $\mathbb{R}^n$), and in fact $f \in C_0(\mathbb{R}^n) + H^s(\mathbb{R}^n)$.*

Proof. From the definition of $H\{s\}$

$$\begin{aligned} \infty \ > \ & \int (1+|\xi|^2)^{s-1}|\xi|^2|\hat{f}(\xi)|^2 d\xi && (2.16)\\ \geq \ & C\left(\int_{|\xi|\leq 1} |\xi|^2|\hat{f}(\xi)|^2 d\xi + \int_{|\xi|\geq 1} (1+|\xi|^2)^s|\hat{f}(\xi)|^2 d\xi\right) && (2.17)\\ \approx \ & C\left(\int |\xi|^2|\hat{f}_1(\xi)|^2 d\xi + \int (1+|\xi|^2)^s|\hat{f}_2(\xi)|^2 d\xi\right). && (2.18) \end{aligned}$$

Let $f = f_1 + f_2$ where $f \in H^{\{s\}}$ and f_1 and f_2 can be defined by $\hat{f}_1(\xi) = \lambda_B \hat{f}(\xi)$ and $\hat{f}_2(\xi) = (1-\lambda_B(\xi))\hat{f}(\xi)$.

Now, $f_2 \in H^s \subset L^2$ since $s \geq 0$. So then looking at f_1, for $r > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int |\hat{f}_1(\xi)|^r d\xi = \int |\xi|^{-r}|\xi \hat{f}_1(\xi)|^r d\xi \leq \left(\int_{supp(\hat{f}_1)} |\xi|^{-rp} d\xi\right)^{\frac{1}{p}} \left(\int |\xi \hat{f}_1(\xi)|^{rq} d\xi\right)^{\frac{1}{q}} < \infty. \quad (2.19)$$

If $rp < n$ and $rq = 2$, which implies

$$1 = \frac{1}{p} + \frac{1}{q} > \frac{r}{n} + \frac{r}{2}, \tag{2.20}$$

or $r < \frac{2n}{n+2}$. In particular, $\hat{f}_1 \in L^1$, and so $f_1 \in C_0$. $\square$

Using Calderon's Theorem A.1.1 in the appendix, we see that $A^{1,\mu}(t) \in H^{\frac{1}{2}} \subset L^\infty + H^{\frac{1}{2}} \hookrightarrow L^\infty + L^3$. It is clear from the definition that $f_1 \in L^\infty$ is H_0-bounded with H_0-bound 0, and so our proof is complete if we show that $f_2 \in L^3$ is also H_0-bounded with H_0-bound 0.

Theorem 2.21. *If $V \in L^3(\mathbb{R}^3; \mathbb{R}^4)$, then V is H_0-bounded with H_0-bound 0.*

Proof. We will show that given any $\epsilon > 0$, there is a constant $K_\epsilon > 0$ such that

$$\|Vf\|_{L^2} \leq \epsilon \|H_0 f\|_{L^2} + K_\epsilon \|f\|_{L^2} \tag{2.22}$$

for all $f \in D(H_0)$.

For any $K > 0$, let $T_K = \{x \in \mathbb{R}^3 \mid V(x) \geq K\}$. Then $\int_{T_K} |V(x)|^3\, dx \to 0$ as $K \to \infty$, so we may choose K_ϵ so that

$$\int_{T_K} |V(x)|^3\, dx \leq \epsilon^3. \tag{2.23}$$

Let $V_1(x) = \chi_{T_{K_\epsilon}}(x)V(x)$ and $V_2(x) = V(x) - V_1(x)$. Then $\|V_2(x)\|_{L^\infty} \leq K_\epsilon$, while $\|V_1\|_{L^3} \leq \epsilon$. Hence,

$$\begin{aligned} \|Vf\|_{L^2} &\leq \|V_1 f\|_{L^2} + \|V_2 f\|_{L^2} && (2.24) \\ &\leq \|V_1\|_{L^3}\, \|f\|_{L^6} + K_\epsilon \|f\|_{L^2} && (2.25) \\ &\leq \epsilon \|H_0 f\|_{L^2} + K_\epsilon \|Vf\|_{L^2}, && (2.26) \end{aligned}$$

as required. $\square$

So we have $A^{1,\mu}(\cdot, t) \in L^\infty + L^3$. After this proof, we can now say that $A^{1,\mu}$ is H_0-bounded with H_0 bounded 0, and thus the multiplication operator $A^{1,\mu}(t)$ is H_0-bounded.

We will now show that $H_0 + A^{1,\mu}(t)$ satisfies the final condition for the operator family $A(t)$ in Kato's Theorem.

Lemma 2.27. *Let* $(A_0, \dot{A}_0) \in Y_M \equiv H^{\{3/2\}}(\mathbb{R}^3; \mathbb{R}^4) \oplus H^{1/2}(\mathbb{R}^3; \mathbb{R}^4)$*, and define* $A^{1,\mu}(t)$ *by*

$$\hat{A}^{1,\mu}(\xi, t) = \hat{A}_0(\xi) \cos(|\xi| t) + \hat{\dot{A}}_0(\xi) \frac{\sin(|\xi t|)}{|\xi|}. \tag{2.28}$$

Then, for all $x \in Y_D \equiv H^1(\mathbb{R}^3; \mathbb{C}^4)$*,*

$$t \mapsto (H_0 + A^{1,\mu}(t))x \tag{2.29}$$

is continuously differentiable in the norm of $X_D = L^2(\mathbb{R}^3; \mathbb{C}^4)$*.*

Proof. It is clearly enough to show that, for all $x \in X_D$, $t \mapsto A^{1,\mu}(t)x$ is strongly C^1. Moreover, since $H_0^{-1} : L^2 \to H^1$ is bijective, this last statement is equivalent to the statement that $t \mapsto A^{1,\mu}(t) H_0^{-1}$ is strongly C^1 on L^2.

There is an obvious candidate for the derivative of $t \mapsto A^{1,\mu}(t) H_0^{-1}$, namely $t \mapsto \dot{A}^{1,\mu}(t) H_0^{-1}$, where $\dot{A}^{1,\mu}(t)$ is the function on $\mathbb{R}^3$ whose Fourier transform is obtained by differentiating (2.28) with respect to t:

$$\hat{\dot{A}}^{1,\mu}(\xi, t) = -|\xi| \hat{A}_0(\xi) \sin(|\xi| t) + \hat{\dot{A}}_0(\xi) \cos(|\xi t|). \tag{2.30}$$

From $(A_0, \dot{A}_0) \in Y_M$, we conclude that $\hat{\dot{A}}^{1,\mu}(t) \in H^{\frac{1}{2}}$ for all t; moreover, it is clearly a continuous function of t into $H^{\frac{1}{2}}$ and hence (by Sobolev imbedding by Case 2 in Appendix A.1) into L^3.

Now, since $H_0^{-1} : L^2 \to H^1$ is continuous, and H^1 imbeds in L^6 (by Sobolev imbedding by Case 1 in A.1), the operator norm of $\dot{A}^{1,\mu}(t) H_0^{-1}$ in $\mathcal{L}(L^2)$ is bounded by

$$\begin{aligned} \|\dot{A}^{1,\mu}(t) H_0^{-1}\|_{L^2 \to L^2} &\leq \|\dot{A}^{1,\mu}(t)\|_{L^6 \to L^2} \, \|H_0^{-1}\|_{L^2 \to L^6} \\ &\leq C \|\dot{A}^{1,\mu}(t)\|_{L^6 \to L^2}. \end{aligned} \tag{2.31}$$

Since $\dot{A}^{1,\mu}(t) \in L^3$ and $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$, we have for any $f \in L^6$ that $\dot{A}^{1,\mu}(t)f \in L^2$ and $\|\dot{A}^{1,\mu}(t)f\|_{L^2}^2 \leq \|\dot{A}^{1,\mu}(t)\|_3 \, \|f\|_6$. Hence

$$\begin{aligned} \|\dot{A}^{1,\mu}(t)H_0^{-1}\|_{L^2\to L^2} &\leq C\|\dot{A}^{1,\mu}(t)\|_3 \\ &\leq C\|\dot{A}^{1,\mu}(t)\|_{H^{1/2}} \leq C\|(A_0, \dot{A}_0)\|_{Y_M}. \end{aligned} \tag{2.32}$$

Similarly, from $A^{1,\mu}(t) \in H^{\{3/2\}} \subset L^\infty + H^{3/2}$, it follows that

$$\|\dot{A}^{1,\mu}(t)H_0^{-1}\|_{L^2\to L^2} \leq C\|(A_0, \dot{A}_0)\|_{Y_M}. \tag{2.33}$$

The proof of the Lemma is completed by showing that $\dot{A}^{1,\mu}(t)H_0^{-1}$ is indeed the time derivative of $A^{1,\mu}(t)H_0^{-1}$ in the norm of $\mathcal{L}(L^2)$. This is equivalent to saying that

$$(1+|\xi|^2)^{\frac{1}{4}} \left[\frac{\hat{A}^{1,\mu}(\xi, t+h) - \hat{A}^{1,\mu}(\xi, t)}{h} - \frac{\partial}{\partial t}\hat{A}^{1,\mu}(t) \right] \to 0 \tag{2.34}$$

in L^2 as $h \to 0$, which follows easily from the definition of $\hat{A}^{1,\mu}(t)$ and dominated convergence.

□

We know for every $(A_0, \dot{A}_0) \in Y_M$ the equation $\frac{dx}{dt} = (H_0 + A^{1,\mu}(t))x + f(t)$ has a unique solution $t \to x(t) \in Y_M$ for every $f(t)$ satisfying Kato's conditions.

Analysis of $f(t)$

IV.1 Gross's First Lemma

In Lemma 1.1 of [G], Gross's states a formula for a function he calls $A_\mu(x,t)$. He gives no motivation for the formula. However, it comes from the method of Spherical means and Duhamel's Principal. Gross's A in his lemma is in part our $A^{2,\mu}$ where we recall is the solution of

$$\begin{cases} \Box A_\mu = \langle \alpha_\mu \psi_1, \psi_2 \rangle \\ A_\mu(x,0) = \partial_t A_\mu(x,0) = 0. \end{cases} \tag{1.1}$$

By Duhamel's Principle $A_\mu(x,0) = \int_0^t U(x,t,s)ds$ where $U(x,t,s)$ for each fixed $s \geq 0$ is the solution of

$$\begin{cases} \Box U(x,t,s) = 0 \\ U(x,s,s) = 0 \\ U_t(x,s,s) = \langle \alpha_\mu \psi_1, \psi_2 \rangle(s). \end{cases} \tag{1.2}$$

By using the method of Spherical Means, (note that $U(x,t,s) = U(x,t-s,0)$)

$$U(x,t,s) = \frac{1}{4\pi c^2(t-s)} \int_{|y-x|=c|t-s|} \langle \alpha_\mu \psi_1(y), \psi_2(y) \rangle dS_y. \tag{1.3}$$

We will use change of variables by $y = x + c|x - s|\xi$ (so $dS_y = c^2|t-s|^2 dS_\xi$). After changing the variables we get

$$U(x,t,s) = \frac{1}{4\pi c^2(t-s)} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + c|t-s|\xi, s), \psi_2(x + c|t-s|\xi, s)\rangle c^2(t-s)^2 dS_\xi \quad (1.4)$$

$$= \frac{t-s}{4\pi} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + c|t-s|\xi, s), \psi_2(x + c|t-s|\xi, s)\rangle dS_\xi. \quad (1.5)$$

So then,

$$A_\mu(x,t) = \int_0^t \frac{t-s}{4\pi} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + c|t-s|\xi, s), \psi_2(x + c|t-s|\xi, s)\rangle dS_\xi ds. \quad (1.6)$$

Now let's use the change of variables again and let $s = t - s$.

$$A_\mu(x,t) = \int_0^t \frac{t-(t-s)}{4\pi} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + c|t-(t-s)|\xi, s), \psi_2(x + c|t-(t-s)|\xi, s)\rangle dS_\xi ds \quad (1.7)$$

$$= \int_0^t \frac{s}{4\pi} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + cs\xi, s), \psi_2(x + cs\xi, s)\rangle dS_\xi ds, \quad (1.8)$$

which is Gross's formula. This solution of the wave equation carries the right hand side of the Maxwell equation. This function that is being integrated is the kind of function appearing as the Dirac source term on the right hand side of the wave equation for A^μ. From this point on, we are going to call Gross's $A_\mu(x,t)$ our $A^{2,\mu}(x,t)$.

IV.2 Gross's Remaining Lemmas

Remember that we need to prove that $A^{2,\mu}(t)\psi(t)$ is a continuous function of t into $H^1(\mathbb{R}^3;\mathbb{C}^4)$. We will accomplish this by a series of lemmas Gross provides in his work.

Lemma 2.1. *Let $T > 0$ and let $\psi_k(t)$, $k = 1, 2$, be continuous functions from $[0, T]$ into the Hilbert space $H^1(\mathbb{R}^3, \mathbb{C}^4)$.*

$$A^{2,\mu}(x,t) = \int_0^t \frac{s}{4\pi} \int_{|\xi|=1} \langle \alpha_\mu \psi_1(x + cs\xi, s), \psi_2(x + cs\xi, s) \rangle dS_\xi ds \tag{2.2}$$

where $\alpha_4 = 1$. Then, for $0 \le t \le T$,

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^3} |A^{2,\mu}(x,t)| \le a \int_0^t \|\psi_1(s)\|_{H^1} \|\psi_2(s)\|_{H^1} ds, \tag{2.3}$$

where a is a constant. Moreover for $\mu = 1, ..., 4$, the maps $t \to A^{2,\mu}(x,t)$ from $[0, T]$ into $L^\infty(\mathbb{R}^3)$ are continuous.

Proof. Let $u \in C_c^1(\mathbb{R}^3)$ be a continuous differentiable function on $\mathbb{R}^3$ with compact support. Taking

$$u(x + s\xi) = -\int_s^\infty \frac{\partial u(x + \rho\xi)}{\partial \rho} d\rho, \tag{2.4}$$

and putting $\hat{y} = y/|y|$, we get

$$\begin{aligned} s \int_{|\xi|=1} u(x + s\xi) d\xi &= -s \int_{|\xi|=1} \int_s^\infty \frac{\partial u(x + \rho\xi)}{\partial \rho} d\rho d\xi && (2.5) \\ &= -s \int_{|\xi|=1} \int_s^\infty \frac{\xi \cdot \nabla u(x + \rho\xi)}{\rho^2} \rho^2 d\rho d\xi && (2.6) \\ &= -s \int_{|y| \ge s} \frac{\hat{y} \cdot \nabla u(x + y)}{|y|^2} dy. && (2.7) \end{aligned}$$

Using Holder's inequality we get

$$\begin{aligned} \left| s \int_{|\xi|=1} u(x + s\xi) d\xi \right| &\le s \left(\int_{|y| \ge s} \frac{1}{|y|^6} dy \right)^{\frac{1}{3}} \times \left(\int_{|y| \ge s} |\nabla u(x + y)|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} && (2.8) \\ &\le s \left(\frac{4\pi}{3s^3} \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |\nabla u(x + y)|^{\frac{3}{2}} dy \right)^{\frac{2}{3}} && (2.9) \\ &\le \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} \|\nabla u\|_{L^{\frac{3}{2}}}. && (2.10) \end{aligned}$$

Let $B = \{f \in L^1(\mathbb{R}^3) | \nabla f \in L^{\frac{3}{2}}\}$ be the Banach space and $\|u\|_B = \|u\|_{L^1} + \|\nabla u\|_{L^{\frac{3}{2}}}$. The functions of u with compact support are dense in B. Then let the map $u \to v$,

$$v(x) = s \int_{|\xi|=1} u(x + s\xi) d\xi. \tag{2.11}$$

Since C_c^1 is dense in B, this map has a continuous function. Let a map $T : u \to v$ where $B \mapsto L^\infty$ and T is continuous. Thus

$$\begin{aligned}
\|v\|_{L^\infty} &\leq C\|u\|_B & (2.12)\\
&\leq C(\|u\|_{L^1} + \|\nabla u\|_{L^{\frac{3}{2}}}) & (2.13)\\
&\leq C\|\nabla u\|_{L^{\frac{3}{2}}} & (2.14)\\
&\leq C\|H_0\psi_1\|_2 + \|H_0\psi_2\|_2 & (2.15)\\
&\leq C\|\psi_1\|_{H^1}. & (2.16)
\end{aligned}$$

This also gives us $|v| \leq \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \|\nabla u\|_{L^{\frac{3}{2}}}$. When integrating $v(x)$ against a continuous function with compact support, the extension is given almost everywhere.

Let

$$u(x) = \langle \alpha_\mu \psi_1(x + cs\xi, s), \psi_2(x + cs\xi, s) \rangle. \tag{2.17}$$

Take the gradient of both sides and use Sobolev's embedding, we will get

$$\|\psi\|_6 \leq \|H_0\psi\|_2. \tag{2.18}$$

Then use the triangle inequality, Calderon's Theorem A.1.1, and Holder's inequality to get

$$\begin{aligned}
\|\nabla u\|_{\frac{3}{2}} &\leq \||\nabla\psi_1||\psi_2|\|_{\frac{3}{2}} + \||\psi_1||\nabla\psi_2|\|_{\frac{3}{2}} & (2.19)\\
&\leq \|\nabla\psi_1\|_2\|\psi_2\|_6 + \|\psi_1\|_6\|\nabla\psi_2\|_2 & (2.20)\\
&\leq C\|H_0\psi_1(t-s)\|_2 + \|H_0\psi_2(t-s)\|_2 & (2.21)
\end{aligned}$$

where C is a constant.

Based off of this, the inequality (2.3) follows and the asserted continuity of $A^{2,\mu}(x,t)$. When $t > t'$, we get

$$\begin{aligned} \|A^{2,\mu}(t) &- A^{2,\mu}(t')\|_\infty \leq C\int_{t'}^{t} \|\psi_1(t-s)\|_{H^1}\|\psi_2(t-s)\|_{H^1}ds && (2.22)\\ &+ C\int_0^{t'} [\|\psi_1(t-s)-\psi_1(t'-s)\|_{H^1}\|\psi_2(t-s)\|_{H^1} && (2.23)\\ &+ \|\psi_1(t'-s)\|_{H^1}\|\psi_2(t-s)-\psi_2(t'-s)\|_{H^1}]ds && (2.24) \end{aligned}$$

where C is a constant. □

Lemma 2.25. *Under the same hypotheses as Lemma* (2.1)*,* $A^{2,\mu}(x,t)$ *defined there, with* $\psi_1(t) = \psi_2(t) = \psi(t)$*, defines an element of* $Y_M = H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R})$ *and*

$$\|(A(t), \partial A/\partial t)\|_{Y_M} \leq C\int_0^t \|\psi(s)\|^2_{H^1}ds. \quad (2.26)$$

Moreover, the map $t \to (A(t), \partial A/\partial t)$ *from* $[0,T]$ *into* $H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R})\oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R})$ *is continuous.*

Proof. Let $J_\mu(x,t) = \langle \alpha_\mu\psi(x,t), \psi(x,t)\rangle$ where $\psi(t) \in H^1(\mathbb{R}^3;\mathbb{C}^4)$ for each t and for the three-dimensional Fourier transform of $\hat{J}_\mu(k,t)$ it is

$$\hat{A}_\mu(k,t) = 4\pi\int_0^t \frac{sin|k|\tau}{|k|}\hat{J}_\mu(k,t-\tau)d\tau. \quad (2.27)$$

The first partial spatial derivatives of $J_\mu(s)$ are in $L^{\frac{3}{2}}(\mathbb{R}^3)$. For example

$$\frac{\partial}{\partial x_1}J_\mu(x,s) = \left\langle \alpha_\mu\frac{\partial\psi(x,s)}{\partial x_1}, \psi(x,s)\right\rangle + \left\langle \alpha_\mu\psi(x,s), \frac{\partial\psi(x,s)}{\partial x_1}\right\rangle. \quad (2.28)$$

These partial derivatives of $J_\mu(s)$ are a sum of products of an L^2 function with an L^6 function by Sobolev's inequality.

Now we want to know what space the product of L^2 and L^6 is in. By Holder's Inequality, if $f \in L^p$ and $g \in L^q$ such that $\frac{1}{p} + \frac{1}{p} = 1$ then $fg \in L^1$. Therefore, if $f \in L^p$ and $g \in L^q$ then

$fg \in L^r$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ $\left(\frac{r}{p} + \frac{r}{q} = 1\right)$. Assume that f and g are nonnegative. Then

$$\int (fg)^r = \int (|f|^r)^{\frac{p}{r}} (|g|^r)^{\frac{p}{r}} \tag{2.29}$$

$$\leq \left(\int (|f|^r)^{\frac{p}{r}}\right)^{\frac{r}{p}} \left(\int (|g|^r)^{\frac{q}{r}}\right)^{\frac{r}{q}} = \left(\int |f|^p\right)^{\frac{r}{p}} \left(\int |g|^q\right)^{\frac{r}{q}} < \infty. \tag{2.30}$$

Thus $f \in L^2$ and $g \in L^6$ implies that $fg \in L^{\frac{3}{2}}$, since $\frac{1}{2} + \frac{1}{6} = \frac{1}{r}$ implies $r = \frac{3}{2}$.

Therefore, $J_\mu \in L^{\frac{3}{2}}(\mathbb{R}^3)$. So, $J_\mu(s)$ is in everything between L^1 and L^3 since $\psi(s)$ is in L^2 and L^6. Then by Calderon Theorem A.1.3, $(1 + |k|^2)^{\frac{1}{2}} \hat{J}_\mu(k, s)$ is the Fourier transform of a function in $L^{\frac{3}{2}}(\mathbb{R}^3)$. Then by using Calderon's Theorem A.1.1, we get that $(1+|k|^2)^{\frac{1}{4}} \hat{J}_\mu(k, s)$ is the Fourier transform of a function in $L^2(\mathbb{R})$. These statements of location of functions are accompanied by inequalities and for each s we get

$$\|(1 + |k|^{\frac{1}{2}}) \hat{J}_\mu(k, s)\|_{L^2} \leq C\|\psi(s)\|_{H^1}^2. \tag{2.31}$$

Thus $(|k| + |k|^{\frac{3}{2}}) \hat{A}_\mu(k, t)$ is in $L^2(\mathbb{R}^3)$. Now replace τ by $t - \tau$ in (2.27) to get

$$\left(\frac{\partial \hat{A}_\mu}{\partial t}\right)(k, t) = 4\pi \int_0^t cos|k|(t - \tau) \hat{J}_\mu(k, \tau) d\tau. \tag{2.32}$$

By both 2.31 and 2.32 and the definition of $H^{\{\frac{3}{2}\}}(\mathbb{R}^3; \mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R})$, the map $t \to (A(t), \partial A/\partial t)$ from $[0, T]$ into $H^{\{\frac{3}{2}\}}(\mathbb{R}^3; \mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R})$ is continuous. □

Lemma 2.33. *Let F be the Banach space of continuous functions on $[0, T]$ with values in $H^1(\mathbb{R}^3; \mathbb{C}^4)$. For ψ in F, define $A^{2,\mu}$ as in equation (2.2) with $\psi_1 = \psi_2 = \psi$. Then the map $\psi(x, t) \to \sum_{\mu=1}^4 \alpha_\mu A^{2,\mu}(x, t)\psi(x, t)$ is continuous from F into F and is Lipschitz on bounded sets in F.*

For this proof, it is sufficient enough to show the scalar case that if F is $X = C([0, T]; H^1)$ then the map $\psi(x, t) \to A^{2,\mu}(x, t)\psi(x, t)$ is continuous from F to F.

Proof. Since

$$\|A^{2,\mu}(t)\psi(t)\|_{H^1}^2 = m^2\|A^{2,\mu}(t)\psi(t)\|_{L^2} + \sum_{j=1}^{3}\|\frac{\partial A^{2\mu}(t)\psi(t)}{\partial x_j}\|_{L^2}^2. \tag{2.34}$$

By Lemma 2.1 , $A^{2,\mu}(t)$ is bounded and $A^{2,\mu}(t)\psi(t)$ is in $H^1(\mathbb{R}^3;\mathbb{C}^4)$ for each t if

$$\frac{\partial(A^{2,\mu}(t)\psi(t))}{\partial x_j} = \frac{\partial A^{2,\mu}(t)}{\partial x_j}\psi(t) + A^{2,\mu}(t)\frac{\partial\psi(t)}{\partial x_j} \tag{2.35}$$

is in $L^2(\mathbb{R}^3)$ for $j = 1,2,3$. $\frac{\partial\psi(t)}{\partial x_j} \in L^2(\mathbb{R}^3)$ and the second term of the equation is square integrable. However, the first term is the product of $\frac{\partial A^{2,\mu}(t)}{\partial x_j}$ with $\psi(t) \in L^6(\mathbb{R})$ by Sobolev's inequality. It is enough to show that $\frac{\partial A^{2,\mu}(t)}{\partial x_j} \in L^3(\mathbb{R}^3)$. By Lemma 2.25 and the definition of $H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R})$, the components of $\frac{\partial A^{2,\mu}(t)}{\partial x_j} \in L^2_{1/2}$. Thus $\frac{\partial A^{2,\mu}(t)}{\partial x_j} \in L^3(\mathbb{R}^3)$ by Sobolev's inequality for fractional derivatives (Calderon Theorem A.1.1). When taking a combination of 2.3, 2.26, and 2.34 we get the inequality

$$\|A^{2,\mu}(t)\psi(t)\|_{H^1} \leq C\int_0^t \|\psi(s)\|_{H^1}^2 ds\|\psi(t)\|_{H^1}. \tag{2.36}$$

This inequality is from the fact that $A^{2,\mu}(t) \in L^\infty$ and $A^{2,\mu}(t) \in H^{\{\frac{3}{2}\}}(\mathbb{R}^3;\mathbb{R}) \oplus H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R})$ and $\psi(t) \in H^1(\mathbb{R}^3;\mathbb{C}^4)$. By both previous lemmas, these functions of t are continuous into the respective spaces. Thus $A^{2,\mu}(t)\psi(t)$ is a continuous function of t into $H^1(\mathbb{R}^3;\mathbb{C}^4)$. Furthermore, $A^{2,\mu}(\cdot)\psi(\cdot)$ is a trilinear function of $\psi(\cdot)$. By the polarization of (2.36), $A^{2,\mu}(\cdot)\psi(\cdot)$ is a Lipschitz function of $\psi(\cdot)$ on bounded sets in F. □

With these lemmas, we can see that we can use $A^{2,\mu}(t)$ as the $f(t)$ in Kato's Theorem.

Completion of the Proof

With all of the information leading up to this point, we have all that we need to complete Gross's proof. Remember that the proof of Gross's Theorem is going to use the fixed point argument.

Proof. Let $\psi_0 \in Y_D$, $(A_0, \dot{A}_0) \in Y_M$. We know $H_0 + A^{1,\mu}(t)$ is self-adjoint. We can take $A(t)$ in Kato's Theorem to be $i(H_0 + A^{1,\mu}(t))$ and get the evolution operator $U(t, s)$. For $a > 0$, $T > 0$, define $B_{a,T} = \{\psi \in C([0, t]; Y_D) | a > 0 \text{ and } T > 0\}$. If $\psi \in B_{a,T}$. Define $S\psi = A^{2,\mu}$ by Gross's IV.2.2. Define $R\psi$ by

$$(R\psi)(t) = U(t, 0)\psi_0 + \int_0^t U(t, s)(S\psi)(s)\psi(s)ds \tag{2.37}$$

$$= U(t, 0)\psi_0 + \int_0^t U(t, s)A^{2,\mu}(s)\psi(s)ds. \tag{2.38}$$

Note that $A^{2,\mu} = S\psi$ for some fixed operator S.

In the proof, we will be using C to denote various constants which may depend on ψ_0 and a but are independent of T.

Step 1: For all $a > 0$, there exists a T such that $R : B_{a,T} \to B_{a,T}$

$$\|(R\psi)(t) - \psi_0\|_{Y_D} \leq \|(U(t, 0) - I)\psi_0\|_{Y_D} + \int_0^t \|U(t, s)A^{2,\mu}(s)\psi(s)\|_{Y_D} ds. \tag{2.39}$$

So we want to pick T small enough so that the map can take $B_{a,T}$ into itself. Then

$$\begin{aligned}
(R\psi)(t) - \psi_0 &= U(t,0)\psi_0 + \int_0^t U(t,s)A^{2,\mu}(s)\psi(s)ds - \psi_0 && (2.40)\\
&= [U(t,0) - I]\psi_0 + \int_0^t U(t,s)A^{2,\mu}(s)\psi(s)ds && (2.41)\\
\|(R\psi) - \psi_0\| &= \|[U(t,0) - I]\psi_0 + \int_0^t U(t,s)A^{2,\mu}(s)\psi(s)ds\| && (2.42)\\
&\leq \|U(t,0) - U(0,0)\| + \int_0^t \|U(t,s)A^{2,\mu}(s)\psi(s)\|ds. && (2.43)
\end{aligned}$$

We know that $\|U(t,0) - U(0,0)\| \to 0$ as $t \to 0$. Now we want to show that integral part does the same. By Kato's Property 4 stated in II.1, the operators $U(t,s)$ are strongly continous from $Y_D \to Y_D$. Thus for each $x \in Y_D$, $U(t,s)x$ is bounded in Y_D for $0 \leq s \leq t \leq T$. By the Uniform Boundedness Principle, the $U(t,s)$ are norm-bounded as operators from $Y_D \to Y_D$. Thus we get,

$$\int_0^t \|U(t,s)A^{2,\mu}(s)\psi(s)\|_{Y_D} ds \leq C\int_0^t \|A^{2,\mu}(s)\psi(s)\|_{Y_D} ds. \tag{2.44}$$

Using Gross's Lemma IV.2.33, we use the Lipschitz continuity to bound the norm of $A^{2,\mu}(t)\psi(t)$ in terms of $\psi(t)$ to get

$$C\int_0^t \|A^{2,\mu}(s)\psi(s)\|_{Y_D} ds \leq CL\int_0^t \|\psi(s)\|_{Y_D} ds \tag{2.45}$$

where L stands for the Lipschitz constant. Just in case $\|\psi(s)\|_{Y_D} = 0$, we will write

$$CL\int_0^t \|\psi(s)\|_{Y_D} ds = CL\int_0^t (1 + \|\psi(s)\|_{Y_D}) ds. \tag{2.46}$$

From Lemma IV.2.33 if $\psi \in B_{a,T}$, then $\|\psi(t)\|_{Y_D} \leq \|\psi_0(t)\|_{Y_D} + a$. So $\psi(t)$ lies in a bounded subset. This means $\|\psi(s)\|_{Y_D}$ is in the ball $B_{a,T}$. Thus we get

$$CL\int_0^t (1 + \|\psi(s)\|_{Y_D}) ds \leq Ct. \tag{2.47}$$

This is bounded by a multiple of t. Therefore, $\int_0^t \|U(t,s)A^{2,\mu}(s)\psi(s)\|_{Y_D}ds \to 0$ as $t \to 0$.

Step 2: We want to show that for all $a > 0$, there exists $T > 0$ such that $R : B_{a,T} \to B_{a,T}$ is a contraction.

Given $\psi, \widetilde{\psi} \in B_{a,T}$, it is enough to show $\|(R\psi)(t) - (R\widetilde{\psi})(t)\|_{Y_D} \leq \lambda \|\psi(t) - \widetilde{\psi}(t)\|_{Y_D}$ for $\lambda < 1$. So

$$\|(R\psi)(t) - (R\widetilde{\psi})(t)\|_{Y_D} \leq \int_0^t \|U(t,s)\| \|A^{2,\mu}(s)\psi(s) - \widetilde{A}^{2,\mu}(s)\widetilde{\psi}(s)\|_{Y_D}ds. \tag{2.48}$$

Similar to Step 1, we will use Lemma IV.2.33 and Kato's Property 4 to get

$$\begin{aligned}
&\leq C \int_0^t L\|\psi(s) - \widetilde{\psi}(s)\|_{Y_D}ds && (2.49)\\
&\leq CL \int_0^t \|\psi - \widetilde{\psi}\|_{B_{a,T}}ds && (2.50)\\
&\leq Ct\|\psi(t) - \widetilde{\psi}(t)\|_{B_{a,T}}. && (2.51)
\end{aligned}$$

Since $t < T$ we get

$$\leq CT\|\psi(t) - \widetilde{\psi}(t)\|_{B_{a,T}}. \tag{2.52}$$

Then take the supremum over t:

$$\sup_{t\in[0,T]} \|R\psi(t) - R\widetilde{\psi}(t)\|_{B_{a,T}} \leq CT\|\psi(t) - \widetilde{\psi}(t)\|_{B_{a,T}}. \tag{2.53}$$

Because C depends on ψ_0 and a and does not depend on T, let $T < \frac{1}{C}$ and our condition is satisfied. Thus $R : B_{a,T} \to B_{a,T}$ is a contraction.

Therefore R is a contraction and has a fixed point, and this fixed point is the solution to the Maxwell-Dirac Equation.

Step 3: Now we want to show the Uniqueness. Notice that even though the fixed point ψ or R is unique, this is not enough to give the uniqueness of the solution pair (ψ, A^μ) for the Maxwell-Dirac system.

Suppose $(\psi, A), (\widetilde{\psi}, \widetilde{A})$ are both solutions. Write $\widetilde{A} = \widetilde{A}^1 + \widetilde{A}^2$ with $\widetilde{A}^1 = A^1$ (Since A^1 does not depend on ψ) and $\widetilde{A}^2 = S\widetilde{\psi}$. Since $\widetilde{A}^1 = A^1$ has not changed, $\widetilde{U}(t,s) = U(t,s)$. Now we want to show that $\widetilde{\psi}$ satisfies

$$\frac{d\widetilde{\psi}}{dt} = i(H_0 + \widetilde{A}^1)\widetilde{\psi} + (S\widetilde{\psi})\widetilde{\psi} \tag{2.54}$$

and therefore satisfies the integral equation

$$\widetilde{\psi}(t) = U(t,0)\psi_0 + \int_0^t U(t,s)(S\widetilde{\psi})(s)\widetilde{\psi}(s)ds = (R\widetilde{\psi})(t). \tag{2.55}$$

These two equations are equivalent. So by such $\widetilde{\psi}$ of (2.55), $\widetilde{\psi}$ is a fixed point of R. Therefore, $\widetilde{\psi} = \psi$ and $\widetilde{A}^2 = S\widetilde{\psi} = S\psi = A^2$. So $\widetilde{A} = A$ which satisfies the uniqueness.

Thus with all three steps, this completes the proof of Gross's Theorem. □

Appendix

A.1 Sobolev Imbedding

In [C], Calderon proves the following versions of Sobolev imbedding. He works with a slightly different notation for the L^p space. It is L^p_u where u represents the number of derivatives in the L^p space.

Theorem 1.1. *Let $p > 1$, $u \geq v$ and $\frac{1}{q} = \frac{1}{p} - \frac{u-v}{n} > 0$; then $L^p_u \subset L^q_v$ and the inclusion map is continuous. Let $1 > u - \frac{n}{p} > 0$; then every function in L^p_u coincides almost everywhere with a function $\overline{f}$ in $Lip(u - \frac{n}{p})$. Furthermore*

$$|\triangle_h \overline{f}| = |\overline{f}(x+h) - \overline{f}(x)| \leq C_{p,u}\|f\|_{p,u}|h|^{u-n/p}; \qquad |\overline{f}| \leq C_{p,u}\|p,u\|. \tag{1.2}$$

Mainly we want to know that $\frac{1}{q} = \frac{1}{p} - \frac{u-v}{n} > 0$ implies $L^p_u(\mathbb{R}^n) \hookrightarrow L^q_v(\mathbb{R}^n)$.

With this theorem, we get three cases.

Case 1: Let $n = 3$, $p = 2$, $u = 1$, and $v = 0$; where $H^s = L^2_s$. So $\frac{1}{q} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ thus $q = 6$. Then $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Case 2: Let $n = 3$, $p = 2$, $u = \frac{1}{2}$, and $v = 0$. So $\frac{1}{q} = \frac{1}{2} - \frac{1/2}{3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} > 0$ thus $q = 3$. Then $H^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$.

Case 3: Let $n = 3$, $p = \frac{3}{2}$, $u = 1$, and $v = 0$. So $\frac{1}{q} = \frac{1}{3/2} - \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} > 0$ thus $q = 3$. Then $L^{\frac{3}{2}}_1 \hookrightarrow L^3$.

Theorem 1.3. *Let u be a positive integer and $1 < p < \infty$. Then $f \in L^p_u$ if and only if f has* *derivatives (in the sense of distributions) of orders* $\leq u$ *in* L^p*. There is a constant* $C = C_{p,u}$ *such that*

$$C^{-1}\|f\|_{p,u} \leq \sum_{0\leq|\alpha|\leq u} \| \left(\frac{\partial}{\partial x}\right)^{\alpha} f\|_p \leq C\|f\|_{p,u}. \tag{1.4}$$

Let u *be a negative integer and* $1 < p < \infty$*. Then* $f \in L^p_u$ *if and only if* $f = \sum_{0\leq|\alpha|\leq u}(\partial/\partial x)^{\alpha} g_{\alpha}$, $g_\alpha \in L^p$ *and there exists a constant* $C = C_{p,u}$ *and a choice of the functions* g_α *such that*

$$C^{-1}\|f\|_{p,u} \leq \sum \|g_\alpha\|_p \leq C\|f\|_{p,u}. \tag{1.5}$$

A.2 Connection to Kato's Statement of His Theorem

In Kato's 1953 paper [K], he does not include his theorem in the form in which we have stated it. The form we are using can be found in Pazy's book [P]. The equivalence of Kato's original statement and the form we use is well known; we will reproduce a quick proof of the equivalence that can be found in a paper by Schmid and Griesemer [SG].

Kato made the following assumption:

1. $B(t, s) = (1 - A(t))(1 - A(s))^{-1}$ is uniformly bounded on $I \times I$

2. $B(t, s)$ is of bounded variation in t in the sense that there is an $N \geq 0$ such that

$$\sum_{j=0}^{n-1} \|B(t_{j+1}, s) - B(t_j, s)\| \leq N < \infty \tag{2.1}$$

   for every partition $0 = t_0, t_1 < \cdots < t_n = 1$ of I, at least for some s.

3. $B(t, s)$ is weakly differentiable in t and $\partial_t B(t, s)$ is strongly continuous in t, at least for some $s \in I$

Note that the statement (1) and (2) hold for all $s \in I$, if there are satisfied for some s. This follows from $B(t, s) = B(t, s_0)B(s_0, s)$. In the proof of the Proposition 2.2, below, we will see that conditions (1) and (2) follow from condition (3), and that (3) is equivalent to the C^1-condition (II.1.3). In 1953, Kato did not seem to be aware of that. But from remarks in later writings, it becomes clear that he know it by 1956.

Proposition 2.2. *Suppose that for each $t \in I$ the linear operator $A(t) : D \subset X \to X$ is closed and that $1 - A(t)$ has a bounded inverse. Then the above assumption is satisfied if and only if the C^1-contrition (II.1.3) holds.*

Proof. From (3) its follows (first in the weak, then in the strong sense) that

$$B(t,s)x - B(t',s)x = \int_{t'}^{t} \partial_\tau B(\tau,s)x d\tau. \tag{2.3}$$

This equation shows that $t \mapsto B(t,s)x$ is of class C^1, which is equivalent to the C^1-condition (II.1.3). Hence (3) is equivalent to the condition (II.1.3) and it remains to derive (1) and (2) from (3). By the strong continuity of $\tau \mapsto \partial_\tau B(\tau,s)$ and by the principle of uniform boundedness,

$$\sup_{\tau \in I} \|\partial_\tau B(\tau,s)\| < \infty. \tag{2.4}$$

Combining (2.3) with (2.4), we see that $B(t,s)$ is of bounded variation as a function of t, which is statement (2), and that $t \mapsto B(t,s)$ is continuous in norm. Therefore, the inverse $t \mapsto B(t,s)^{-1} = B(s,t)$ is continuous as well as $B(t,s) = B(t,0)B(0,s)$ is uniformly bounded for $t, s \in I$.

□

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