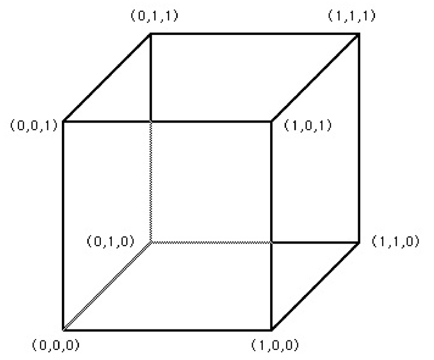
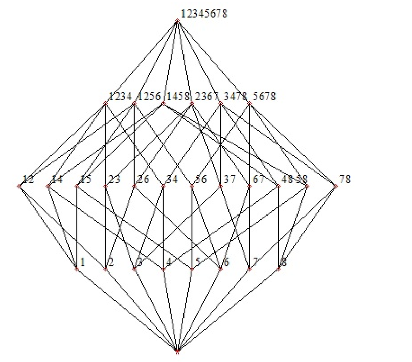


ALGEBRA ASSOCIATED WITH THE HASSE GRAPH OF THE HYPERCUBE

WAI SHAN CHAN, CARY SCHNEIDER, KELSEY FRANKO

Faculty Mentor: Colleen Duffy



THE PROBLEM

With every n-dimensional cube, there is a graph that is associated with it, a Hasse Diagram. Different levels of the Hasse graph represent how many vertices there are in the cube, how many edges, how many faces, and so on. From these graphs we can determine a set of polynomials such as:

$$28 - 62t + 48t^2 - 14t^3 - t^4$$

Where the power of t represents the dimension: The coefficient in front of t^0 is the number of vertices, the coefficient in front of t^1 is the number of connections of length 1 in the Hasse Diagram, the coefficient in front of t^2 is the number of connections of length 2, etc. To find a general equation of these polynomials, instead of tedious counting, we must find the Hasse graphs of different symmetries of the cube and find the relation between them.

BACKGROUND

We define a permutation $\equiv \sigma$. $\sigma = (123)$ implies that the value of dimension 1 goes to dimension 2, the value of dimension 2 goes to dimension 3, and dimension 3 goes to 1. A single group implies that dimension stays the same. Point $(1,0,0)$ under $\sigma = (1)(23)$ is fixed at $(1,0,0)$.

We define $\mathcal{L}_\sigma \equiv \{l_1, l_2, l_3, \cdots, l_n\}$ as a set of numbers representing the number of times a cycle of a certain length is used in permutation σ. We then define a new set $q = \{q_1, q_2, q_3, \cdots, q_n\}$ where q_i is a partition of a subpermutation. Set q must be a subset of $\mathcal{L}_\sigma$.

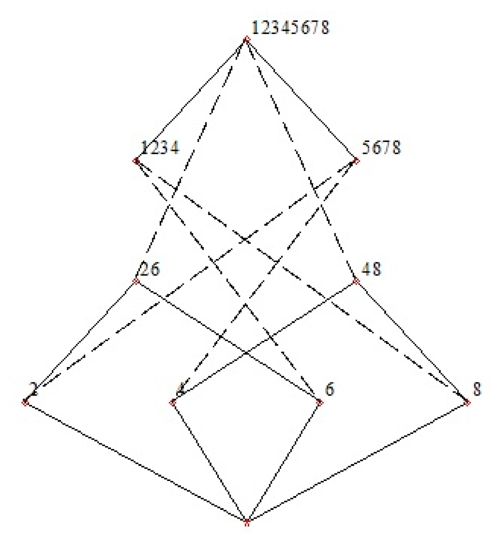
It is noted as a special case when $\sigma + [1]_2$, which implies $0 \leftrightarrow 1$, and there are separate solutions for that permutation.

WORKTHROUGH

We take the $n = 3$ cube and apply $\sigma = (12)(3)$.

1. **Fixed Vertices ***
   (a) **(0, 0, 0) → (0, 0, 0)***
   (b) (1, 0, 0) → (0, 1, 0)
   (c) (0, 1, 0) → (1, 0, 0)
   (d) **(1, 1, 0) → (1, 1, 0)***
   (e) **(0, 0, 1) → (0, 0, 1)***
   (f) (1, 0, 1) → (0, 1, 1)
   (g) (0, 1, 1) → (1, 0, 1)
   (h) **(1, 1, 1) → (1, 1, 1)***
2. **Determine Fixed Edges**
   (a) (0, 0, 0), (0, 0, 1)
   (b) (1, 1, 0), (1, 1, 1)
3. **Determine Fixed Faces**
   (a) (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)
   (b) (0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1)

We have 4 fixed vertices, 2 fixed edges, 2 fixed faces, and because each vertex is occupied, 1 fixed cube. With this information, we can build the Hasse Graph:



We can then count the paths to get the polynomial:

$$10 - 10t - 4t^2 + 6t^3 - t^4 = (3 - 2t^2)(3 - 2t) - t(2 - t^2)(2 - t) + 1$$

RESULTS

Looking at the polynomials for different permutations and higher dimensional cubes, we determined a function that counts fixed k-faces for σ:

$$\sum_{\substack{q \subseteq L \\ \sum iq_i = k}} 2^{r - \sum q_i} \prod_{1 \le i \le n} \binom{l_i}{q_i} \quad (1)$$

We can use formula (1) to determine a larger sample of polynomials to determine a general equation from.

1. $n = 4 \; \sigma + 0_2$
   (a) (1)(2)(3)(4): $82 - 232t + 248t^2 - 120t^3 + 24t^4 - t^5 = (3 - 2t)^4 - t(2 - t)^4 + 1$
   (b) (12)(3)(4): $28 - 44t + 2t^2 + 26t^3 - 12t^4 + t^5 = (3 - 2t^2)(3 - 2t)^2 - t(2 - t^2)(2 - t)^2 + 1$
   (c) (123)(4): $10 - 10t + 2t^2 - 6t^3 + 6t^4 - t^5 = (3 - 2t)(3 - 2t^3) - t(2 - t^3)(2 - t) + 1$
   (d) (12)(34): $10 - 4t - 12t^2 + 4t^3 + 4t^4 - t^5 = (3 - 2t^2)^2 - t(2 - t^2)^2 + 1$
   (e) (1234): $4 - 2t - 2t^4 + t^5 = (3 - 2t^4) - t(2 - t^4) + 1$
2. $n = 4 \; \sigma + 1_2$
   (a) (1)(2)(3)(4): $2 - t^5$
   (b) (12)(3)(4): $4 - 2t^2 - 2t^3 + t^5 = (2 - t^2)(2 - t^3)$

With this information we were able to factor the polynomials into a general series solution that depends purely on l_i, which depends on σ:

$$1 + \prod (3 - 2t^i)^{l_i} - t \prod (2 - t^i)^{l_i} \quad (2)$$

For the special case $\sigma + 1$, we were able to determine the counting function:

$$\sum_{\substack{q \subseteq L \\ \sum iq_i = k \\ l_i - q_i = 0, \forall \text{ odd } i}} 2^{r - \sum q_i} \prod_{1 \le i \le n} \binom{l_i}{q_i} \quad (3)$$

Where we can get the array of polynomials for $\sigma + 1$ like above, and factor that into a series solution:

$$1 + \prod_{\substack{i=2 \\ i \text{even}}}^{n} (3 - 2t^i)^{l_i} - t \prod_{\substack{i=2 \\ i \text{even}}}^{n} (2 - t^i)^{l_i} \quad (4)$$

REFERENCES

[1] Duffy, Colleen
Graded traces and irreducible representations of $\mathrm{Aut}(A(\Gamma))$ acting on gr $A(\Gamma)$ and gr $A(\Gamma)^!$
Rutger's University, 2008

FUTURE GOALS

We can now use equations (2) and (4) to find the representations of the Algebra.
We want to extend these methods and results to other polyhedra.

ACKNOWLEDGEMENTS

- Colleen Duffy
- UWEC Mathematics Dept.
- ORSP