Mirror Symmetry from Reflexive Polytopes

Chris Magyar
University of Wisconsin – Eau Claire

Motivations

There are two main theories used by physicists to explain the inner workings of the universe. *General relativity* is used to describe the very large, while *quantum mechanics* describes the very small. For decades physicists have sought after a so called *unified field theory* to combine these two models. Currently, the most widely accepted candidate for a unified field theory is known as *string theory*.

In order to reconcile general relativity with quantum mechanics, string theory extends our classical 4D model of space-time into extra dimensions. At every unique point in our known four dimensions, these extra dimensions have the structure of *Calabi-Yau varieties*, or 6D algebraic varieties. It turns out that there are always two Calabi-Yau varieties that produce a particular physical model. In mathematics we call this phenomenon *mirror symmetry*.

Reflexive Polytopes

The *polar duality* transformation takes a polytope with integer lattice points to its polar dual. Let Δ be a lattice polytope which contains the origin. The polar dual polytope Δ° is the polytope given by:

$$\{(m_1, \ldots, m_k) \mid (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \\ \forall\ (n_1, \ldots, n_k) \in \Delta\}$$

A lattice polytope is defined to be *reflexive* if its polar dual is also a lattice polytope. In addition, the polar dual of the polar dual of a reflexive polytope is the original polytope.

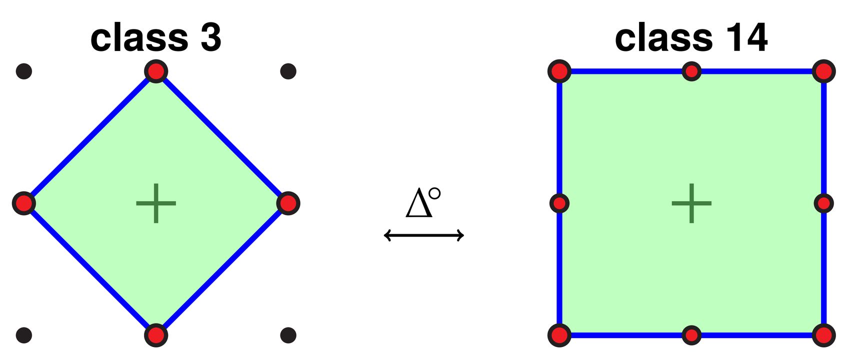

fig 1. Dual reflexive quadrilaterals.

There are an infinite number of reflexive polytopes in any given dimension ≥ 2, but they can all be classified into different equivalence classes if we consider the linear transformations of reflecting, rotating, and shearing as mapping to the same class. Reflexive polytopes have been classified in 2D, 3D, and 4D, with 16, 4 319, and 473 800 776 classes of equivalent polytopes respectively.

Dual Varieties over Finite Fields

Reflexive polytopes can be used to describe Calabi-Yau varieties. To construct a pair of varieties from dual reflexive polytopes we take the anticanonical hypersurfaces in the associated toric varieties. The relationship between these pairs of varieties exhibits some of the same properties observed in the mirror symmetry of string theory.

The dual varieties obtained from the the reflexive quadrilaterals in fig. 1 are:

$$\mathbb{V}\left(z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_4^2 + z_3^2z_4^2 + tz_1z_2z_3z_4\right)$$

and

$$\mathbb{V}\left(z_3^2z_4^2z_5^2z_6z_7 + z_1^2z_3^2z_5z_6^2z_8 + z_2^2z_4^2z_5z_7^2z_8 + z_1^2z_2^2z_6z_7z_8^2 + tz_1z_2z_3z_4z_5z_6z_7z_8\right)$$

where t is a parameter that is varied through our field.

When we count solutions of these dual varieties over various finite fields we discover that the solution count is equal for all fields tested.

It has previously been conjectured for all reflexive simplices that these dual corresponding varieties exhibit a *strong arithmetic mirror symmetry*, meaning they have an equivalent number of points mod q over the same finite field $\mathbb{F}_q$

In the case of the 2-dimensional reflexive polytopes, we realize our corresponding varieties are elliptic curves and are able to prove, using classical results, that the dual pair of quadrilaterals shown in fig. 1 give rise to the same strong arithmetic point counting relation as the simplices.

Picard-Fuchs Equations

Another way we can examine elliptic curves is through *Picard-Fuchs equations*. A Picard-Fuchs equation is an ordinary differential equation that describes how the value of a *period* of a family of varieties changes as we move through the family, where a period is the integral of a *differential form* with respect to a specified subspace. Picard-Fuchs equations encode variations of complex structure in the family.

The *Griffiths-Dwork technique* is a standard method for calculating the Picard-Fuchs equation for a given family. We calculate the Picard-Fuchs equation of our variety $V = \mathbb{V}\left(z_1^2z_2^2 + z_1^2z_3^2 + z_2^2z_4^2 + z_3^2z_4^2 + tz_1z_2z_3z_4\right)$ from fig. 1 by first recognizing that V is an elliptic curve so it must have a single *holomorphic 1-form* $\in H^{1,0}$. Let $\Omega \in H^{1,0}$, then $\mathcal{P} = \int \operatorname{Res}\left(\frac{\Omega_0}{Q}\right)$ is a period of the holomorphic 1-form. We differentiate $\mathcal{P}$ with respect to t a couple times to get:

$$\mathcal{P}' = \int \operatorname{Res}\left(-z_1z_2z_3\frac{\Omega_0}{Q^2}\right) = \int \operatorname{Res}\left(-\frac{dQ}{dt}\frac{\Omega_0}{Q^2}\right)$$

$$\mathcal{P}'' = \int \operatorname{Res}\left(2z_1^2z_2^2z_3^2\frac{\Omega_0}{Q^3}\right) = \int \operatorname{Res}\left(2\left(\frac{dQ}{dt}\right)^2\frac{\Omega_0}{Q^3}\right)$$

To find a linear combination of $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}''$ in terms of our parameter t that equals 0, we first need to use the formula:

$$\frac{\Omega_0}{Q^{k+1}}\sum_{i=0}^{n} A_i\frac{\partial Q}{\partial x_i} = \frac{1}{k}\frac{\Omega_0}{Q^k}\sum_{1=0}^{n}\frac{\partial A_i}{\partial x_i} + d(\cdots)$$

to reduce the order the pole until it is expressed in terms of lower derivatives. This part is done quickly in Magma mathematics software. Finally, we are left with solving:

$$\begin{bmatrix} 1 & -\frac{1}{t} & \frac{2t^2-16}{t^4-16t^2} \\ 0 & \frac{2}{t}(Z_1+Z_2) & -\frac{6t^2-32}{t^4-16t^2}(Z_1+Z_2) \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $Z_1 = z_2^2z_4^2$, $Z_2 = z_3^2z_4^2$.

To get our Picard-Fuchs equation:

$$t^2\mathcal{P} + \left(3t^3 - 16t\right)\mathcal{P}' + \left(t^4 - 16t^2\right)\mathcal{P}'' = 0$$

When we calculate the Picard-Fuchs equations for all our 2D reflexive polytopes, we notice that those polytopes that exhibit a strong arithmetic mirror symmetry also have the same Picard-Fuchs equation.

3D Reflexive Polytopes

We continue our search for strong arithmetic mirror symmetry over the 4319 reflexive polytopes in 3 dimensions. When counting points on both sides of 3D reflexive polytopes, we observe an interesting phenomenon: there appear to be only five non-simplex, non-self-dual class pairs that exhibit strong arithmetic mirror symmetry.

Upon further investigation, we notice that these five class pairs experimentally exhibiting strong arithmetic symmetry are the only non-simplex, non-self-dual pairs with the same kernel of the matrix of vertices. Then, because the corresponding hypersurfaces are strictly determined by the vertices of the polytopes, these five pairs of toric varieties are isogenous. We can further classify these into two groups of reflexive pairs that exhibit vertex matrices with the same null space:

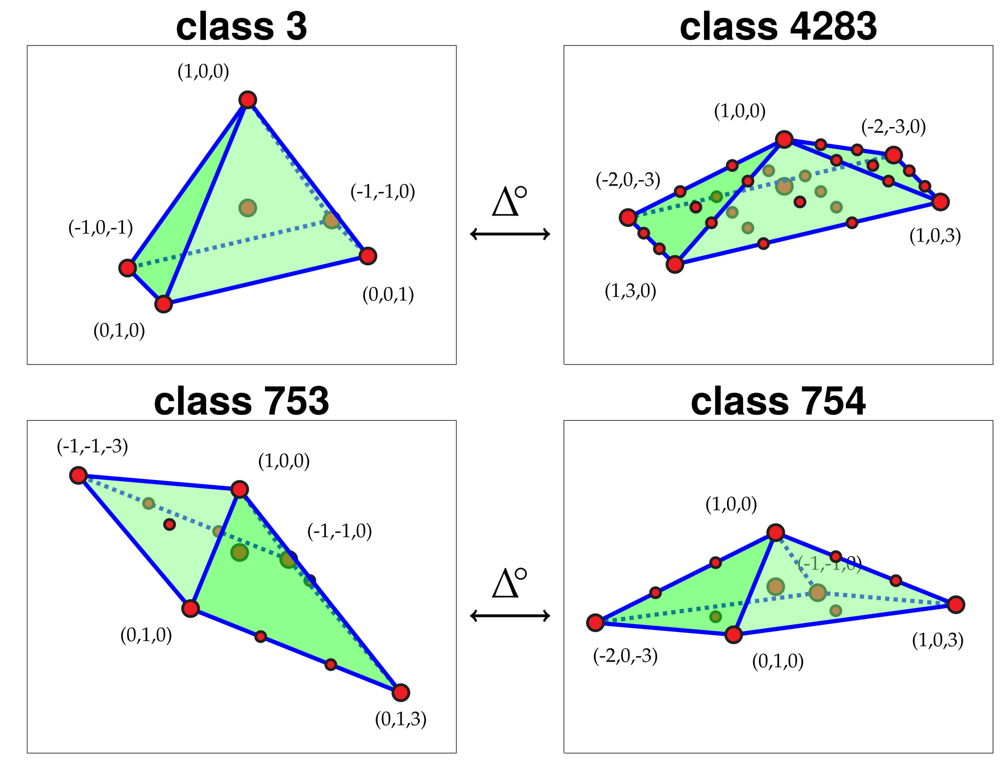

fig 2. Group I.

Additionally, when we calculate the Picard-Fuchs equations of these five pairs, we observe that each group gives rise to the same differential equation.

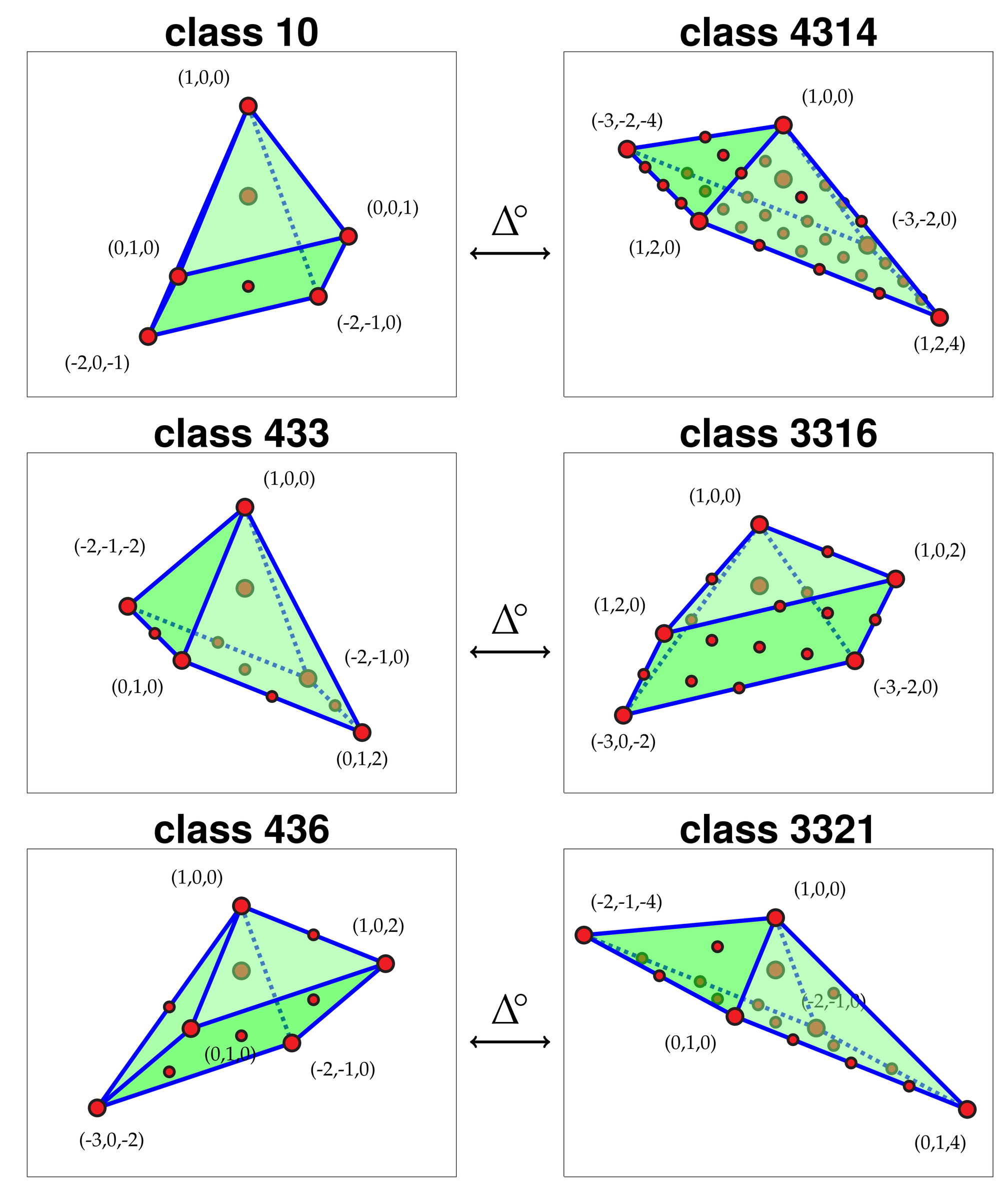

fig 3. Group II.

With experimental evidence confirming a point counting relation for these five 3D dual pairs over fields $\mathbb{F}_p$ of order 5, 7, or 11 where t is varied over the entire field, we make the following hypothesis:

Conjecture. *Let Δ and Δ° be polar dual polytopes with the same null spaces of their vertex column matrices. If X_t and Y_t are the respective hypersurfaces defined from their vertices, then for any rational t where X_t and Y_t are smooth, strong arithmetic symmetry holds:*

$$\#X_t(\mathbb{F}_q) \equiv \#Y_t(\mathbb{F}_q) \pmod{q}$$

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