Computer Sciences Department

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Denis Xavier Charles

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SQUAREFREE INTEGERS WITHOUT LARGE PRIME FACTORS IN SHORT INTERVALS

DENIS XAVIER CHARLES

ABSTRACT. We show that for every $\epsilon > 0$ and $\delta > 0$ there are squarefree integers that are free of prime factors $> X^{\delta}$ in the interval $[X \cdots X + X^{\frac{1}{2}+\epsilon}]$ for all large enough X. The approach used is a simple variant of the methods used by Balog [Ba87] and by Harman [Har91] in their study of smooth integers in short intervals.

(Preliminary Version)

1. Introduction

The number of integers below x having no prime factors greater than y is denoted by $\Psi(x, y)$. Let $\Psi((x \cdots z], y) = \Psi(z, y) - \Psi(x, y)$. The behaviour of $\Psi((x \cdots x + x^{\epsilon}], y)$ is still largely a mystery for small ϵ. Friedlander and Lagarias considered this problem in [FL87] and showed that intervals of size $x^{1-2\alpha\left(1-2^{-\lfloor \frac{1}{\alpha} \rfloor}\right)}$ contain x^{α}-smooth integers. This result was later improved by Balog ([Ba87]) who showed the existence of x^{δ}-smooth integers in any interval of size $x^{\frac{1}{2}+\epsilon}$. Later Harman [Har91] went further by showing the existence of $\exp((\log X)^{\frac{2}{3}+\epsilon})$-smooth integers in the same interval. Here we consider the question of the existence of integers which are both squarefree and x^{δ}-smooth in intervals of size $x^{\frac{1}{2}+\epsilon}$. We show that indeed there are such integers in these intervals using analytic arguments. The approach is from Balog [Ba87] who considered a weighted sum of the smooth integers in the interval. This approach in turn was inspired by the work of Heath-Brown and Iwaniec [HI79]. We also employ the extra-averaging idea of Harman [Har91] in estimating this weighted sum. The following is known about the distribution of squarefree numbers without any smoothness restriction. We know that intervals of size $x^{\frac{1}{5}} \log x$ contain squarefree integers [FT92]. Recently Granville [Gr98] has shown the existence of squarefree integers in interval of size x^{ϵ} for every $\epsilon > 0$, if one assumes the ABC-conjecture.

2. Preliminary Results

Let

$$0 < \epsilon < \frac{1}{2},$$
$$Y = \frac{1}{2} X^{\frac{1}{2}+\epsilon},$$
$$u = \left\lfloor \frac{1}{\delta} \right\rfloor + 1,$$
$$\mathcal{I}(x) = [x \cdots x + Y].$$

Define L by

$$L^{u} = X^{1-\gamma}$$

where $0 < \gamma < \frac{1}{u}$ (we will impose further restrictions on γ later).

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Department of Computer Science, University of Wisconsin-Madison, Madison, WI - 53706. E-mail:cdx@cs.wisc.edu.

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Let

$$H(x) = \sum_{\substack{n \in \mathcal{I}(x) \\ n = mp_1p_2\cdots p_u \\ p_i \in (L, eL], 1 \le i \le u}} \mu(m)^2 \log p_1 \log p_2 \cdots \log p_u .$$

Note that by the choice of the parameters n is X^{δ}-smooth since $\gamma < \delta$ but the sum is over numbers some of which are not squarefree.

We will prove an asymptotic formula for the integral:

$$\int_X^{X+Y} H(x)dx.$$

Let $P(s) = \sum_{L<p\le eL} \frac{\log p}{p^s}$.

Lemma 2.1. *Let* $s = \sigma + it$. *Then for* $t \ge T_0 = \exp\{(\log X)^{\theta}\}$, $0 < \theta < 1$ *and* $1 - \frac{1}{\log^{\lambda} X} \le \sigma \le 1$. *If* λ *is sufficiently close to* 1 *then there is a* μ, $0 < \mu < 1$ *such that*

$$|P(s)| \ll \exp\{-(\log X)^{\mu}\}.$$

Proof : We use the effective Perron's formula to estimate partial sums of $\sum_{1\le n} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$. Now $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{1\le n} \frac{\log p}{p^s} + f(s)$, where $f(s)$ converges for $\sigma > \frac{1}{2}$. In particular since

$$\sum_{L\le p^m \le eL, m>1} \frac{\log p}{p^{ms}} \ll \frac{\log^2 L}{L^{\frac{3}{2}}}$$

so we can ignore the contribution from the higher prime powers in this interval.

From [TH86] (p. 60-63) we have

$$\sum_{n\le eL} \frac{\Lambda(n)}{n^s} = \frac{-1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta'(s+w)}{\zeta(s+w)} \frac{x^w}{w} dw + O\left(\frac{L^c}{Tc}\right) + O\left(\frac{\log^2 L}{T}\right).$$

In this case we take $c = \frac{h}{\log^{\lambda} X}$, where $h > 1$. Shifting the contour of integration to: $\{\sigma + iT \mid -f < \sigma < c\} \cup \{-f + it \mid |t| \le T\}$ where f is picked such that no other poles apart from the one at $w = 0$ and at $w = 1 - s$ are introduced.

We get

$$\sum_{L<p\le eL} \frac{\log p}{p^s} = \frac{(eL)^{1-s} - L^{1-s}}{1-s} + O\left(\frac{L^c}{Tc}\right) + \left(\frac{\log^2 L}{T}\right) + O\left(\frac{L^c \log^9 T}{T}\right) + O\left(\frac{\log^{10} T}{L^f}\right)$$

and now the lemma follows if $T = \exp\{\log^{\frac{1}{10}} X\}$ and $\lambda > \frac{9}{10}$. □

We will also make use of the following theorem from the theory of Mean-values of Dirichlet polynomials see [Mon71] Chapters 6 and 9.

Theorem 2.2. *Let* b_n *be any sequence of complex numbers, and* $S(s) = \sum_{1\le n\le N} \frac{b_n}{n^s}$. *Then for any integer* $k \ge 0$*:*

$$\int_0^T \left| \sum_{1\le n\le N} \frac{b_n}{n^{it}} \right|^{2k} dt \ll (T + N^k) \left(\sum_{1\le n\le N^k} |b_n(k)|^2 \right)$$

where $b_n(k)$ *is defined by:*

$$S(it)^k = \sum_{1\le n\le N^k} \frac{b_n(k)}{n^{it}}.$$

3. Proof of the theorem

Theorem 3.1. *Let* X, Y *and* H(x) *be the quantities defined earlier. Then*

$$\int_X^{X+Y} H(x)dx = bY^2 + o(Y^2)$$

for some constant $b > 0$.

Proof : Using Perron's formula we have

$$H(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta(s)}{\zeta(2s)} P^u(s)\left\{\frac{(x+Y)^s - x^s}{s}\right\} ds.$$

Thus we have

$$\int_X^{X+Y} H(x)dx = \frac{1}{2\pi i}\int_{2-i\infty}^{2+\infty} \frac{\zeta(s)}{\zeta(2s)} P^u(s)A(s)ds$$

where

$$A(s) = \frac{(X+2Y)^{s+1} - 2(X+Y)^{s+1} + X^{s+1}}{s(s+1)}$$

by a justifiable interchange of the integrals. We note that

$$A(s) \ll \min\{Y^2X^{\sigma-1}, X^{1+\sigma}|t|^{-2}\}$$

this is display (11) in [Har91].
We now shift the contour of integration to $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Let $T_0 = \exp(\log X)^\theta$, $a = \frac{d}{\log^\lambda X}$, $d > 0$, where $\frac{9}{10} < \lambda < 1$, d is a constant and

$$\mathcal{C}_1 = \{1 + it \mid |t| \geq T_0\}$$
$$\mathcal{C}_2 = \left\{\sigma + it \;\middle|\; |t| = T_0, 1 - a \leq \sigma \leq 1\right\}$$
$$\mathcal{C}_3 = \left\{1 - a + it \;\middle|\; |t| \leq T_0\right\}.$$

The only pole encountered in the shifting is at $s = 1$. Thus by the theorem of residues:

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} = \frac{Y^2}{\zeta(2)}P^u(1) + \frac{1}{2\pi i}\left\{\int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3}\right\}.$$

Since $\frac{1}{\zeta(2)}P^u(1)$ is a constant, we are done if we show that the integral over the contour is $o(Y^2)$.

Contour $\mathcal{C}_1$: $1 + it, |t| > X$: We use the following estimates:

$$A(1+it) \ll \frac{X^2}{|t|^2}$$
$$P(s) \ll 1,$$
$$\zeta(1+it) \ll \log|t|$$
$$\frac{1}{\zeta(2+it)} \ll 1.$$

Thus

$$\int_{1+it\,:\,|t|\geq X} \ll \frac{X^2 \log X}{X} \ll Y^{2-2\epsilon}\log X.$$

Contour C_1 : $1+it, \frac{X}{Y} < |t| \leq X$: Assume that $u = 2k+1$.

$$\begin{aligned}
\int_{1+it\,:\,\frac{X}{Y}<|t|\leq X} &\ll X^2 \log X \int \frac{P(1+it)^{u-1}}{t^2} dt \\
&\ll X^2 \log X \int \frac{P(1+it)^{2k}}{t^2} \exp(-(\log X)^{\mu}) dt \text{ Lemma (2.1)}, \\
&\ll X^2 \log X \exp(-(\log X)^{\mu}) \left[\frac{1}{T^2} \int_0^T P_1(1+it)^{2k} dt \right]_{\frac{X}{Y}}^{X} \text{ by partial integration} \\
&\ll Y^2 \log X \exp(-(\log X)^{\mu}) \left[\left(\frac{X}{Y} + L^k \right) \frac{\log^{2k} X}{L^k} \right] \text{ by Theorem (2.2)} \\
&\ll Y^2 \log^{1+2k} X \exp(-(\log X)^{\mu}) (X^{-\epsilon + \frac{\gamma}{2} + \frac{1}{2u}}).
\end{aligned}$$

Now if u is large enough and γ is small this term is $o(Y^2)$, but we can assume this without loss of generality since the existence of a $X^{\frac{1}{u}}$-smooth integer for large u certainly implies the existence of "rougher" integers. Suppose $u = 2k$ then we split the product $P(s)^u$ into two parts one which has a square term and another with $P(s)^{2(k-1)}$ and proceed as above.

Contour C_1 : $1+it, \exp(\log X)^{\theta} < |t| < \frac{X}{Y}$: In this case we use the upper bound $A(1+it) \ll Y^2$ and proceed as in the previous region of the contour.
Thus

$$\begin{aligned}
\int_{\exp(\log X)^{\theta}<|t|<\frac{X}{Y}} &\ll Y^2 \log X \exp(-(\log X)^{\mu}) \left[(T + L^k) \frac{\log^{2k} X}{L^k} \right]_{T_0}^{\frac{X}{Y}} \\
&\ll Y^2 \log^{1+2k} X \exp(-(\log X)^{\mu}).
\end{aligned}$$

Contour C_2 : $\sigma + iT_0, 1 - a \leq \sigma \leq 1$: In this region we use $\zeta(\sigma + it) = O(\log t)$ and $\frac{1}{\zeta(1+it)} = O(\log t)$. Also $A(s) \ll Y^2 X^{\sigma-1}$ and $P(s) \ll \exp\{-\log^{\mu} X\}$, because if X is large enough we can use Lemma (2.1).

$$\int_{1-a+iT_0}^{1+iT_0} \ll Y^2 \log^2 X \ \exp\{-u \log^{\mu} X\}.$$

Contour C_3 : $1 - a + it, |t| \leq T_0$:
Here we use the $A(s) \ll Y^2 X^{\sigma-1}$, and $P(s) \ll L^a$, and the same bounds on the zeta function as in C_2. Thus we get

$$\int_{C_3} \ll \frac{Y^2}{X^a} L^{au} \log^2 X.$$

Now $L^{au} = X^{a(1-\gamma)}$, so this term is also $o(Y^2)$.
□

Corollary 3.2. *There is a squarefree* X^{δ}*-smooth integer in the interval* $[X \cdots X + X^{\frac{1}{2}+\epsilon}]$.

Proof : Let $Y = X^{\frac{1}{2}+\epsilon}$. By theorem 3.1 we know that there is an interval $I(x)$ with $X \leq x \leq X + Y$ such that $H(x) \gg Y$. Since the maximum weight given to any integer in this interval is $O(\log^u X)$ we immediately infer that the number of integers of the form $mp_1 \cdots p_u$, where $p_i \in (L \cdots eL]$ and $m \leq X^{\gamma}$ is squarefree is $\gg \frac{Y}{\log^u X}$. Now the number of integers in the interval $[X \cdots X + Y]$ that are divisible by a *square* of a prime

$p \in [L \cdots eL]$ is at most

$$\sum_{p \in [L \cdots eL]} \left\lfloor \frac{Y}{p^2} \right\rfloor \ll \frac{YL}{L^2} + O(L)$$
$$\ll X^{\frac{1}{2}+\epsilon-\frac{1-\gamma}{u}}$$
$$= o(Y).$$

Thus the number of integers involved in the sum $H(x)$ that are also squarefree is $\Omega\left(\frac{Y}{\log^u X}\right)$. □

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