A COMPARATIVE ANALYSIS OF AP CALCULUS AB AND MATH 231 AT THE UNIVERSITY OF WISCONSIN–MILWAUKEE

by

Yan Yi He

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ABSTRACT

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The University of Wisconsin-Milwaukee, 2025
Under the Supervision of Professor Kevin McLeod

This study looks at how well the AP Calculus AB curriculum prepares students for college mathematics at the level of Calculus I or higher. It focuses on two main questions: (1) How the goals of AP Calculus AB compare to those of a college Calculus I course; (2) How closely AP Calculus AB matches with MATH 231 at The University of Wisconsin-Milwaukee. Recent changes to the AP Calculus AB exam, influenced by broader calculus reform efforts, aim to shift the focus from procedural steps to fostering deeper conceptual understanding among students. This study explores whether these changes have meaningfully aligned the AP curriculum with the rigor and expectations of college calculus.

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Chapter 1: Introduction

1.1 Overview of AP Calculus AB

Many high school students enroll in Advanced Placement (AP) classes to strengthen their college applications. Some even take as many as five AP courses during their senior year. Among these courses, AP Calculus is frequently regarded as a marker of college readiness. In recent years, enrollment in high school calculus, particularly Advanced Placement, has grown substantially. This is often because schools are pushing students into harder math classes earlier. Some students are even told to take Algebra 2 and Geometry at the same time to move ahead faster. This trend reflects national data—over 20% of high school graduates now study calculus before college, with roughly 800,000 students enrolled in some form of high school calculus (Bressoud, 2016). While enrollment in high school calculus has grown significantly, a large portion of students take the course for reasons other than genuine interest in mathematics, often due to external expectations or perceived admissions advantages. Some take it to improve their transcripts or avoid college math, while others are guided by school placement policies or peer pressure. These mixed motivations raise questions about how effectively AP Calculus AB supports college readiness, especially for those not planning to pursue STEM pathways (Rosenstein & Ahluwalia, 2014). Anecdotally, many college professors say that even students who took AP Calculus in high school still struggle with basic ideas and need to retake Calculus in college. Bressoud (2015) observed that many college faculty express concern that students who enter with AP Calculus experience may still have gaps in foundational skills, which can affect their ability to succeed in subsequent mathematics courses. Similarly, the push toward early calculus, particularly through high school AP programs, has revealed concerning outcomes for a subset of students. According to the Committee on the Undergraduate Program in

Mathematics Panel on Calculus Articulation, there is "a widespread and growing dissatisfaction with the performance in college calculus courses of many students who had studied calculus in high school" (Committee on the Undergraduate Program in Mathematics, P184).

The Advanced Placement (AP) program started as a way to let high school students try out college-level classes before they graduate. Developed by the College Board, the program was designed to offer academically strong students early access to challenging subjects—calculus among them. Flowers (2008) explains that the intent was to give students a way to earn college credit by demonstrating deep understanding in advanced topics. Bressoud (2015) also explains that AP Calculus was designed to let students who were strong in math take the course in high school, so they could move on to more advanced topics once they got to college. The idea took place in the 1950s, when there was a broader push to give high-achieving students more access to college-level learning.

1.2 History of the College Board AP Calculus AB

The AP Calculus AB course was created in 1969 to help address concerns about students' preparation for college calculus. It gives high school students the opportunity to take a course similar to the first semester of a college Calculus I course. The course was developed as part of a broader national initiative to improve mathematics education in the United States during the Cold War. The initiative was supported by the Ford Foundation and carried out in partnership with the College Board. The goal was to give academically strong students early access to the kind of material they would encounter in a college Calculus I course so they would be more prepared for university classes (Bennett, 2019). AP Calculus AB focuses on foundational topics including limits, derivatives, and basic integrals.

Over time, both the structure of the AP Calculus AB course and the content of the examination have changed. In the early years, many test questions did not require calculus at all and instead emphasized algebraic manipulation and memorized procedures. As Bennett (2019) describes the tests from 1969 through the 1980s were “extremely algebraic” and rarely included graphical or tabular data, reflecting a procedural rather than conceptual approach (Bennett, 2019). By the late 1980s, high failure rates and decreasing student engagement in AP Calculus AB led mathematics educators to reconsider how the course was being taught. Backed by early funding from the National Science Foundation and supported by organizations such as the Mathematical Association of America, the Calculus Reform Movement emerged as a response. This movement encouraged instructors to shift from rote procedural instruction to conceptual reasoning, multiple representations, and real-world applications. Lynn Arthur Steen, a key figure in the movement, summarized this new vision by stating that mathematics should serve “as a pump, not a filter,” meaning it should propel students forward in their academic journey rather than screen them out (Steen, 1988).

These reform ideas influenced changes in the AP Calculus AB examination. In earlier years, many questions were primarily algebra-based. Beginning in the 1980s, however, the exam began to shift in response to the reform movement. By the late 1990s, the questions were designed to assess whether students could apply calculus concepts using graphs, tables, equations, and real-world contexts (Bennett, 2019).

1.3 Purpose of the Study

Many college students in science, technology, engineering, or mathematics begin their studies with calculus. It is often the first challenging mathematics course they take, where they encounter new ideas and problem-solving methods. The course sets the foundation for more advanced college mathematics. For students in STEM fields, doing well in calculus is often essential, as many college programs require it. Success in calculus can help students move forward in these areas. But even though it is important, many students have a hard time with it in college. This makes people ask if high school classes, like AP Calculus AB, really help students get ready for the thinking and problem-solving they will need in a college Calculus I course or subsequent courses. This study looks at how AP Calculus AB is taught, what it covers, and if the skills students learn in high school match what colleges want them to know. It also looks at how the topics, goals, and work in AP Calculus AB compared to those in a college Calculus I course. This comparison is based on the intended design of the AP Calculus AB and MATH 231 curricula, not on how the courses are taught in classrooms.

To do this, I compare the AP Calculus AB Mathematical Practice Standards with those from the Common Core State Standards for Mathematical Practice and the course outline for Math 231: Calculus I at the University of Wisconsin–Milwaukee. I also examine existing research on student performance in college calculus after completing the AP course. The goal is to evaluate whether students develop both procedural fluency and conceptual understanding in a way that aligns with the expectations of a college Calculus I course.

By 2010, two-thirds of the students in the first mainstream college calculus course were repeating a class they had taken in high school (Bressoud, 2021). This raises further questions about whether AP Calculus AB prepares students not only to pass the course in high school but to succeed in college Calculus I without needing to repeat content.

1.4 Statement of the Problem

While many students take AP Calculus AB in high school, it remains unclear whether the course prepares them for success in college Calculus I. Conflicting research findings and the number of students who retake calculus in college suggest a gap between high school preparation and college expectations. To better understand this gap, this study compares the goals, content, and expectations of AP Calculus AB with those of a college Calculus I course, using the Common Core State Standards for mathematical practice and Math 231 at the University of Wisconsin–Milwaukee as benchmarks.

Chapter 2: Framework

2.1 AP Calculus AB Curriculum overview

The Advanced Placement (AP) Calculus AB course is "designed to be equivalent to a first semester college calculus course devoted to topics in differential and integral calculus" (College Board, 2020). It provides high school students with an opportunity to study foundational concepts of calculus at a college level before graduation. The course features "a multi-representational approach to calculus, with concepts, results, and problems expressed graphically, numerically, analytically, and verbally" (College Board, 2020). This structure allows students to develop a flexible understanding of calculus ideas and apply them across different representations.

AP Calculus AB is designed for students who already have a strong background in algebra, geometry, trigonometry, and functions. Before taking the course, students should understand how to work with different types of functions, such as linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and piecewise-defined functions. They should also know how to graph these functions and solve equations involving them. In addition, students should be familiar with function transformations, combinations, compositions, and inverses. These skills help prepare students for the main topics in calculus, such as limits, derivatives, and integrals (College Board, 2020).

The AP Calculus AB curriculum outlines the core topics students must master, including limits, continuity, derivatives, definite integrals, and the Fundamental Theorem of Calculus. According to the College Board, "the course frameworks define content students must know and skills students must master in order to earn transferable, long-term understandings of calculus" (College Board, 2020). In addition to computational skills, the curriculum emphasizes interpreting and applying results in a variety of contexts. The AP Calculus AB Exam, administered by the College Board, is approximately 3 hours and 15 minutes long. It is divided into two main sections: a multiple-choice section and a free-response section, each contributing 50% to the final score. The multiple-choice section includes Part A with 30 questions in 60 minutes without a calculator and Part B with 15 questions in 45 minutes that require a graphing calculator. The free-response section includes Part A with 2 questions in 30 minutes where a calculator is allowed, and Part B with 4 questions in 60 minutes without a calculator. (College Board, 2020).

The AP Calculus AB course is built around three big ideas that serve as the foundation of the course and allow students to create meaningful connections among concepts. These are:

Big Idea 1: Change (CHA) – Using derivatives to describe rates of change of one variable with respect to another or using definite integrals to describe the net change in one variable over an interval of another allows students to understand change in a variety of contexts. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus. (Units 1, 2, 4, 6, 8)

Big Idea 2: Limits (LIM) – Beginning with a discrete model and then considering the consequences of a limiting case allows us to model real world behavior and to discover and understand important ideas, definitions, formulas, and theorems in calculus, for example, continuity, differentiation, and integration. (Units 1, 2, 4, 6)

Big Idea 3: Analysis of Functions (FUN) – Calculus allows us to analyze the behaviors of functions by relating limits to differentiation, integration, and infinite series and relating each of these concepts to the others. (Units 1, 2, 3, 5, 6, 7)

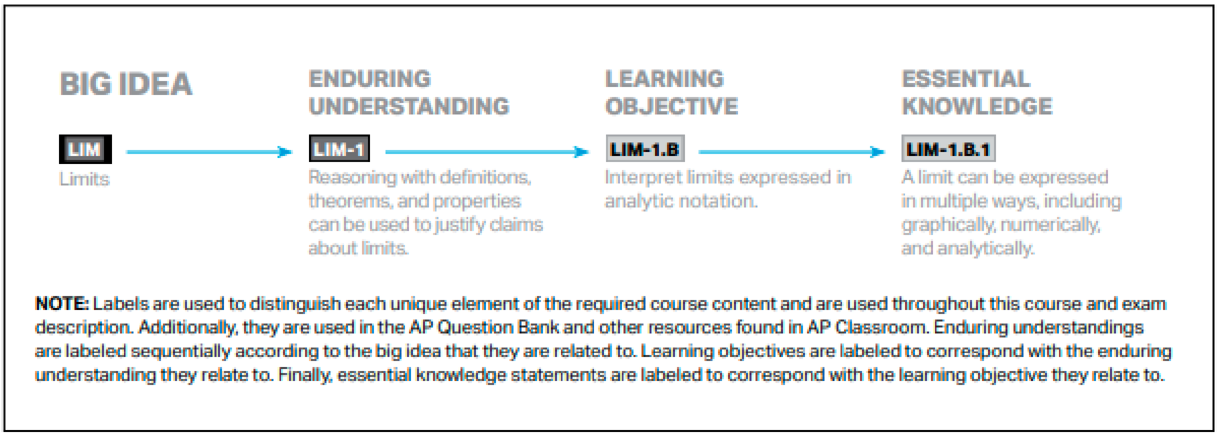


Table 1-Required AP Calculus AB Course Content Labeling System used in AP Calculus, showing the structure from big ideas to essential knowledge. Retrieved from *AP Calculus AB and BC Course and Exam Description* (p. 26), by College Board, 2020, https://apcentral.collegeboard.org. © 2020 College Board.

The course is organized into eight major units, beginning with limits and continuity and progressing through differentiation, integration, and their applications. A summary of the eight units is provided below to offer an overview of the course structure:

Unit	Class Periods	Exam Weighting
Unit 1: Limits and Continuity	22-23	10-12%
Unit 2: Differentiation: Definition and Fundamental Properties	13-14	10-12%
Unit 3: Differentiation: Composite. Implicit, and Inverse Functions	10-11	9-13%
Unit 4: Contextual Applications of Differentiation	10-11	10-15%
Unit 5: Analytical Application of Differentiation	15-16	15-18%
Unit 6: Integration and Accumulation of Change	18-20	17-20%
Unit 7: Differential Equations	8-9	6-12%
Unit 8: Applications of Integration	19-20	10-15%

Table 2-AP Calculus AB Units

In the AP Calculus AB course, the College Board's mathematical practice standards are a key part of instruction. Students are expected to regularly apply skills such as connecting multiple representations, justifying reasoning, constructing mathematical arguments, and communicating ideas with precision. As the College Board explains, "students should develop and apply the described skills on a regular basis over the span of the course" (College Board, 2020). These practices support students in developing both procedural fluency and deeper conceptual understanding.

2.2 Overview of MATH 231 Curriculum at UWM

Students at the University of Wisconsin–Milwaukee (UWM) begin their college mathematics coursework at different levels based on placement. UWM accepts Advanced

Placement (AP) Calculus for credit. Whether students receive credit depends on their AP Calculus exam score and the degree program they are pursuing. Students must also apply to UWM as degree-seeking to receive the credit. According to the UWM AP Credit Guide, a score of 3 on the AP Calculus AB exam gives students 4 credits for MATH 211, and a score of 4 or 5 gives students 4 credits for MATH 231 (Registrar's Office, 2024).

Students who do not have a qualifying AP score can qualify for MATH 231 by either placing high enough on the placement test or by passing earlier college mathematics courses. UWM currently uses the ALEKS PPL placement test to check if students are ready for college calculus. Students who pass college pre-calculus course with a good grade may also qualify.

MATH 231 is a four-credit, in-person course that covers limits, derivatives, graphs of algebraic, trigonometric, exponential, and logarithmic functions, antiderivatives, definite integrals, and the Fundamental Theorem of Calculus. The course is primarily taught in a traditional lecture format, with an emphasis on student-centered learning that promotes clear reasoning, accurate mathematical writing, and careful problem solving. In the following sections, the content and learning expectations of MATH 231 are described in more detail.

Math 231 at the University of Wisconsin-Milwaukee is taught in a similar way across all sections. Most sections are led by PhD students in mathematics. Instructors follow the same syllabus and course schedule, which keeps the content and pace consistent. In Spring 2025, there were six sections of MATH 231 offered and taught by five different PhD students. Students attended class twice a week for 110 minutes in person. The required textbook was *Essential Calculus* (2nd Edition) by James Stewart, and a copy is available in the campus library. Grades were based on four Gateway Exams, two midterm exams, one final exam, weekly quizzes, and written homework. All tests are completed on paper. Students are not allowed to use calculators or notes. Homework is checked for completion, clear mathematical steps, and correct notation, not just for the correct answers. It must be completed by hand, without the use of technology. Gateway exams are a key component of MATH 231**.** These timed assessments test students' ability to perform basic calculus skills: including limits, derivatives, graphing, and integration with a high level of accuracy and strong procedure understanding. The purpose of the Gateway Exams is to ensure that students can carry out standard calculations without error. Conceptual understanding of calculus topics is assessed separately through quizzes, homework, midterms, and the final exam. Students must score at least 80% on each Gateway Exam to earn a grade higher than C−, which allows students to proceed to Calculus II. (Note: a grade of C− does not qualify students to advance.) If needed, students can retake Gateway Exams once per day during the designated retake periods. Students who do not pass all four Gateway Exams can earn no higher than a C− in the course, regardless of their other grades. However, if a student scores 90% or higher on all four Gateway Exams within a week of the in-class exam and passes the final exam with a C or better, they are guaranteed at least a C for the course. The Gateway system ensures that students have mastered fundamental computational skills before advancing to higher-level mathematics.

The following table outlines the weekly topics, assessments, and key milestones for MATH 231, including the placement of quizzes, midterms, and Gateway Exams:

Week	Sections
Week 1	➔ **1.1** Functions and their Representations ➔ **1.2** A Catalog of essential Functions ➔ **1.3** The Limit of a Function
Week 2	➔ **1.4** Calculating Limits ➔ **1.5** Continuity ➔ **1.6** Limits Involving Infinity ➔ **Quiz 1**
Week 3	➔ **2.1** Derivatives and Rates of Change ➔ **2.2** The Derivative as a Function ➔ **2.3** Basic Differentiation Formulas ➔ **Gateway 1**
Week 4	➔ **2.4** The Product and Quotient Rules ➔ **2.5** The Chain Rule ➔ **2.6** Implicit Differentiation ➔ **Quiz 2**
Week 5	➔ **2.7** Related Rates ➔ **2.8** Linear Approximations and Differentials ➔ **Gateway 2 (Derivatives)**
Week 6	➔ **3.1** Maximum and Minimum Values ➔ **3.2** The Mean Value Theorem ➔ **3.3** Derivatives and the Shapes of Graphs ➔ **Quiz 3**
Week 7	➔ Review ➔ **Midterm Exam One**
Week 8	➔ **3.4** Curve Sketching ➔ **3.5** Optimization Problems
Week 9	➔ **3.7** Antiderivatives ➔ **4.1** Areas and Distances ➔ **4.2** The Definite Integral ➔ **Gateway 3 (Graphing)**
Week 10	➔ **4.3** Evaluating Definite Integrals ➔ **4.4** The Fundamental Theorem of Calculus ➔ **Quiz 4**
Week 11	➔ **4.5** The Substitution Rule ➔ **5.2** The Natural Logarithmic Function ➔ **Quiz 5**
Week 12	➔ **5.3** The Natural Exponential Functions ➔ **Gateway 4 (Integration)**
Week 13	➔ **5.4** General Logarithmic and Exponential Functions ➔ **5.1** Inverse Functions ➔ Review
Week 14	➔ **Midterm Exam Two** ➔ **5.6** Inverse Trigonometric Functions ➔ **5.7** Hyperbolic Functions

Table 3- MATH 231 Units

Table 3 compares the major units and instructional pacing of AP Calculus AB and MATH 231 at the University of Wisconsin–Milwaukee. The topics are grouped by theme and content area to show where the curricula align or differ. Where no college-level equivalent exists, the MATH 231 column is left blank.

AP Unit	AP Topics	MATH 231 Weeks	MATH 231 Topics
Unit 1: Limits and Continuity	Limits, Continuity, Infinity	Week 1–2	1.1–1.6: Functions, Limits, Continuity
Unit 2: Differentiation: Definition and Fundamental Properties	Derivative Concepts, Basic Formulas	Week 3	2.1–2.3: Derivatives, Rates of Change, Basic Rules
Unit 3: Composite, Implicit, and Inverse Functions	Chain Rule, Implicit, Inverse Functions	Week 4, Week 13	2.4–2.6 (Week 4), 5.1 (Week 13)
Unit 4: Contextual Applications of Differentiation	Related Rates, Linear Approximations, Differentials	Week 5–6	2.7, 2.8, 3.1–3.3: Applications, Mean Value Theorem
Unit 5: Analytical Applications of Differentiation	Curve Sketching, Optimization	Week 8	3.4–3.5: Curve Sketching, Optimization
Unit 6: Integration and Accumulation of Change	Antiderivatives, Definite Integrals, FTC	Week 9–10	3.7, 4.1–4.4: Antiderivatives, FTC
Unit 7: Differential Equations	Slope Fields, Separable DEs	—	
Unit 8: Applications of Integration	Area Under Curve, Volume of Solids	Week 10–11	4.3–4.5: Evaluating Integrals, Substitution, Area

Table 4- AP Calculus AB VS MATH 231

2.3 Mathematical Practice Standards

The AP Calculus AB course puts a strong focus on mathematical practices. These are skills students need to build throughout the course. According to the College Board, these skills are expected to be applied consistently across all units and topics in the course to support long-term understanding. The four major mathematical practices in AP Calculus AB are implementing mathematical processes, connecting representations, justification, and communication and notation. The following is a more detailed description of each practice standard.

Practice 1: Implementing mathematical processes

- Identify the question to be answered or problem to be solved
- Identify key and relevant information to answer a question or solve a problem
- Identify an appropriate mathematical rule or procedure based on the classification of a given expression.
- Identify an appropriate rule or procedure based on the relationships between concepts or processes.
- Apply appropriate mathematical rules or procedures, with and without technology.
- Explain how an approximated value relates to the actual value.

Practice 2: Connecting representations

- Identify common underlying structures in problems involving different contextual situations.
- Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
- Identify a re-expression of mathematical information presented in a given representation.
- Identify how mathematical characteristics or properties of functions are related in different representations.
- Describe the relationships among different representations of functions and their derivatives.

Practice 3: Justification

- Identify an appropriate mathematical definition, theorem, or test to apply.

- Confirm whether hypotheses or conditions of a selected definition, theorem, or test have been satisfied.
- Apply an appropriate mathematical definition, theorem, or test.
- Provide reasons or rationales for solutions and conclusions.
- Explain the meaning of mathematical solutions in context.
- Confirm that solutions are accurate and appropriate.

Practice 4: Communication and notation

- Use precise mathematical language.
- Use appropriate units of measure.
- Use appropriate mathematical symbols and notation.
- Use appropriate graphing techniques.
- Apply appropriate rounding procedures.

MATH 231 at the University of Wisconsin–Milwaukee satisfies two sets of university-wide academic requirements. First, it fulfills the General Education Requirements (GER) for Natural Science. Second, it meets the Quantitative Literacy Part B (QL-B) requirement. These outcomes reflect UWM's broader goals for student learning in mathematics and science, including real-world applications, ethical reasoning, and critical thinking with numbers and models. (University of Wisconsin–Milwaukee, 2024)

General Education Requirements (GER) for Natural Science:

- Understand and apply the major concepts of a natural science discipline, including its breadth and relationship to other disciplines.
- Apply ethical reasoning to questions, concepts, and practices within a natural science discipline.

Quantitative Literacy Part B (QL-B) Learning Outcomes:

- Recognize and construct mathematical models or hypotheses that represent quantitative information.
- Evaluate the validity of mathematical models or hypotheses.
- Analyze and manipulate mathematical models using quantitative information.
- Reach logical conclusions, predictions, or inferences.
- Assess the reasonableness of conclusions and results.

While the General Education Learning Outcomes provides a broad framework, some mathematics courses at UWM also include the Common Core Mathematical Practice Standards (CCSS) in their syllabi. (Common Core State Standards Initiative [CCSSI], 2010) The CCSS were developed to establish clear and consistent learning goals across states and to ensure students are prepared for post-secondary success. Listing these standards helps clarify the mathematical practices students are expected to develop in high school and continue applying in college-level calculus courses at UWM.

Common Core State Standards for Mathematical Practice:

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They monitor and evaluate their progress and change course if necessary. They can understand the approaches of others to solving complex problems.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They decontextualize a situation and represent it symbolically, then recontextualize it when needed. They attend to the meaning of quantities, not just how to compute them.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students justify their conclusions and respond to the arguments of others. They reason inductively about data, making plausible arguments. They distinguish correct logic from flawed reasoning.

4. Model with mathematics.

Mathematically proficient students apply the mathematics they know to solve problems in everyday life. They interpret mathematical results in context and adjust models as needed. They reflect on whether the results make sense in context.

5. Use appropriate tools strategically.

Mathematically proficient students consider available tools when solving mathematical problems. They make sound decisions about when tools might be helpful. They are able to use technology to explore and deepen understanding.

6. Attend to precision.

Mathematically proficient students communicate precisely to others. They use clear definitions and specify units of measure. They calculate accurately and label axes appropriately in graphs.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern patterns or structure. They can see complicated things as single objects or as being composed of several objects. They use structural insight to simplify calculations.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice repeated calculations and look for general methods. They evaluate the reasonableness of their results. They monitor errors and adjust procedures as needed.

Note: The previous descriptions represent selected excerpts from the official Common Core State Standards for Mathematical Practice. They have been shortened to highlight key aspects most relevant to this study. The full list of standards provides more comprehensive explanations and examples.

To set the stage for the next chapter, we will compare the mathematical practice standards in AP Calculus AB with the expectations outlined by the College Board. It will also examine how MATH 231 reflects the University of Wisconsin system's General Education Requirements (GER) and UWM's Quantitative Literacy Part B (QL-B) outcomes. Finally, these frameworks will be compared with the Common Core State Standards to evaluate how similar or different they are.

The AP Calculus AB Mathematical Practices and the Common Core State Standards for Mathematical Practice share many goals in promoting strong mathematical thinking. Both emphasize the importance of understanding problems, selecting appropriate strategies, and clearly communicating mathematical reasoning. For instance, students are expected to justify their answers, use precise notation and language, and interpret mathematical relationships across different forms such as equations, graphs, and verbal descriptions.

Despite their similarities, the two sets of practices differ in scope and focus. The AP Calculus AB practices are more specific to the study of calculus, highlighting formal definitions, theorems, symbolic notation, and function analysis. In contrast, the Common Core practices are broader and designed to apply across all grade levels and areas of mathematics. They place a stronger emphasis on general problem-solving habits, such as using tools strategically and identifying patterns through repeated reasoning. While these ideas may be embedded in AP expectations, they are not explicitly outlined.

Chapter 3: Comparative Analysis of AP Calculus AB and MATH 231

3.1 Assumptions and Methodology

This comparison is based on the intended design of the AP Calculus AB and MATH 231 curricula, not on how the courses are taught in classrooms. For AP Calculus AB, the analysis uses official materials from the College Board, which are assumed to reflect the course's core content and expectations, even though teaching practices can vary widely from one high school to another. Likewise, the MATH 231 syllabus and assessments examined in this study are from a representative semester at the University of Wisconsin–Milwaukee and reflect the intended structure of the course. While instructional methods may differ from one instructor to another, MATH 231 is coordinated within a single institution and includes internal auditing processes to maintain consistency across sections. The College Board also conducts course audits nationally to verify that AP Calculus AB classes follow the required framework, though this is a far more complex task given the scale and diversity of AP programs across the country. In both cases, the analysis compares the curricula as designed, not as enacted in the classroom.

3.2 Data Sources and Scope of Analysis

This research investigates both the format and content of the AP Calculus AB exam, with a focus on the cognitive demands placed on students through its assessment questions. Due to confidentiality rules, the College Board does not release the full set of multiple-choice questions (MCQs) from recent AP Calculus AB exams. However, free-response questions (FRQs) are made publicly available each year. To conduct this analysis, two key sources were used:

1. **The 2016, 2021–2024 Free-Response Questions (FRQs):** These questions provide open-ended assessment tasks that reflect the current exam framework and content priorities.

2. **The 2012 Released Practice MCQ Exam:** This is the most recent complete set of multiple-choice questions made public by the College Board. Although it predates some curriculum updates, it remains useful for examining question format, topic distribution, and levels of cognitive demand.

These two sources provide a foundation for analyzing both selected-response and open-ended questions in the AP Calculus AB exam. The 2012 MCQ set, while somewhat outdated, allows for expanded comparative analysis and helps identify patterns in question types. Using both sources together offers insight into how the exam is structured and how different question types may influence student learning and instructional outcomes.

3.3 Item Analysis

Unit 1: Limits and Continuity

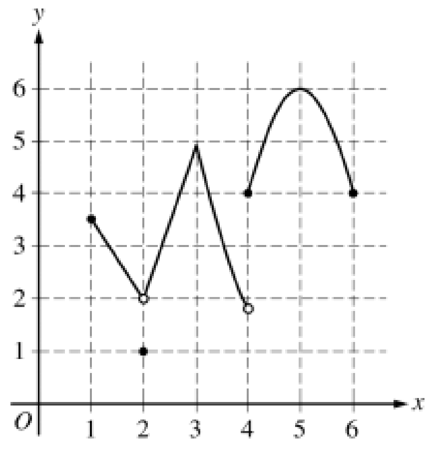
Topic 1.2-1.6: Reasoning with Definitions, Theorems, and Properties to Justify Limits

Topic 1.11 Defining Continuity at a Point

According to the AP Calculus AB Course and Exam Description, Topics 1.2 through 1.6 fall under Enduring Understanding LIM-1, which states: “Reasoning with definitions, theorems, and properties can be used to justify claims about limits.” Topic 1.11 fall under Enduring Understanding LIM-2, which states: “Reasoning with definitions, theorems, and properties can be used to justify claims about continuity.” (College Board, 2020, P36-45)

The sequence introduces students to formal limit notation, estimation methods, algebraic manipulations, and continuity. Instruction builds from intuitive understanding to rigorous analysis, supporting students’ ability to justify conclusions about function behavior using graphical, numerical, analytical, and verbal representations.

Let us look at the 2012 AP Calculus AB Exam multiple-choice question 5,



Graph of f

5. The graph of the function f is shown above. Which of the following statements is false?

(A) $\lim_{x \to 2} f(x)$ exists.

(B) $\lim_{x \to 3} f(x)$ exists.

(C) $\lim_{x \to 4} f(x)$ exists.

(D) $\lim_{x \to 5} f(x)$ exists.

(E) The function f is continuous at $x = 3$.

Figure 1. AP Calculus AB Exam 2012 MCQ 5 (College Board, 2012)

In the AP question, students are asked to determine whether two-sided limits exist and whether the function is continuous at a given point. The graph of the piecewise defined function is provided, and students must extract and apply information from it. This task aligns with Essential Knowledge LIM-1. B.1, which states: "A limit can be expressed in multiple ways, including graphically, numerically, and analytically." Learning objective LIM-1.C, "Estimate limits of functions." Answer choice E requires students to understand Essential Knowledge LIM-2.A.2, which states: "A function f is continuous at $x = c$ provided that $f(c)$ exits, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$."

Let us look at UWM MATH 231 Fall 2018 Exam 1, Question 3,

3. Consider the following piecewise defined function: $f(x) = \begin{cases} x+3 & x \leq -1 \\ 1-x^2 & -1 < x < 1 \\ \sqrt{x-1} & x > 1 \\ 3 & x = 1 \end{cases}$

(a) Sketch the graph of $f(x)$.

(b) Evaluate each expression:

$\lim_{x \to -1^-} f(x) =$ $\lim_{x \to -1^+} f(x) =$ $\lim_{x \to -1} f(x) =$

$\lim_{x \to 1^-} f(x) =$ $\lim_{x \to 1^+} f(x) =$ $\lim_{x \to 1} f(x) =$

$f(-1) =$ $f(0) =$ $f(1) =$

$f'(-2) =$ $f'(0) =$ $f'(0.5) =$

(c) On what intervals is $f(x)$ continuous?

Figure 2. UWM MATH 231 Fall 2018 EXAM 1 Question 3

The MATH 231 question asks students to work with a piecewise defined function by first sketching its graph and then answering questions about limits, function values, and derivatives. The function includes four different expressions, each applying to a different part of the domain. Students are asked to find one sided and two-sided limits at the points where the expressions change. They also need to evaluate the function and its derivative at specific input values. This task connects to several areas of the AP Calculus framework, including one part that focuses on expressing limits using correct notation and another that emphasizes estimating limits, including one-sided limit, and using graphs to do so.

Unlike the typical AP question, which provides a graph and focuses mainly on interpreting it, this question asks students to build the graph themselves using equations before they can evaluate limits. This adds an extra step and requires students to move between different forms of representation. While this can help students build a deeper understanding, it can also make the task more challenging, especially for those who are less confident working from symbolic information. Some students may be able to figure out the limits using just the equations, but others may rely on seeing the graph to understand how the function behaves. If they struggle to create an accurate graph, it could make the rest of the question much harder. This raises an important point about whether the task is asking students to do too many things at once, and whether that might prevent some students from showing what they know.

Mathematical Practice Standards – Topics 1.2–1.6, 1.11

The AP Calculus AB exam question aligns with Practice 2B, which states "Identify mathematical information from graphical, numerical, analytical, and or verbal representations." Students are required to examine a graph and interpret key features to determine the existence of limits and continuity at specific points. This task involves translating visual features such as jump discontinuities, open and closed points, and behavior from the left and right into symbolic conclusions about limits. The question also engages Practice 1C, which states "Identify an appropriate mathematical rule or procedure based on the classification of a given expression." To determine whether a limit exists, students must decide whether the discontinuity is removable or not based on graphical structure. This recognition process demands strategic classification and selection of the correct conceptual rule. Furthermore, the task involves Practice 1E, which states

"Apply appropriate mathematical rules or procedures with and without technology." Here, students apply the formal properties of limits without computation or calculator use, relying instead on conceptual reasoning based on directional behavior. The problem requires the integration of visual interpretation, rule identification, and conceptual limit application, demonstrating both procedural fluency and reasoning.

This same question also reflects four Common Core State Standards for Mathematical Practice: MP2 (Reason abstractly and quantitatively), MP3 (Construct viable arguments and critique the reasoning of others), MP6 (Attend to precision), and MP7 (Look for and make use of structure). MP2 is seen in how students must move between the visual representation of a function and the symbolic understanding of its limit behavior. The question asks students to determine which of several statements is false, requiring them to analyze mathematical claims and justify conclusions using limit laws and definitions, which reflects MP3. To do so accurately, students must apply MP6 by attending to the distinction between a function's value and its limit at a point. Finally, MP7 is evident as students examine the structural components of the graph, such as endpoint behavior and piecewise definitions, to understand how they influence limit conditions.

The MATH 231 exam question demonstrates alignment with the General Education Requirements (GER) for Natural Sciences and the Quantitative Literacy Part B (QL-B) outcomes. The GER emphasizes applying foundational disciplinary ideas and considering their connections across fields of knowledge. In this case, the question engages students in analyzing a piecewise-defined function by examining continuity and limits at specific points. The QL-B outcomes focus on interpreting mathematical models, evaluating their effectiveness, and drawing conclusions based on quantitative evidence. Students are asked to reason about the function's behavior using symbolic rules, visual representations, and analytical techniques. This involves understanding how different function rules interact at transition points and justifying whether the function is continuous or differentiable at those locations. Although this task aligns with UWM's broader educational goals, the GER and QL-B standards are intentionally broad and apply across disciplines. For that reason, this will be the only instance where these outcomes are mentioned. The remainder of the chapter will rely on the AP Calculus AB framework and the Common Core State Standards, which provide more specific guidance aligned with secondary and early college-level calculus instruction.

The MATH 231 exam question aligns with several Common Core State Standards for Mathematical Practice, including MP2 (Reason abstractly and quantitatively), MP3 (Construct viable arguments and critique the reasoning of others), MP6 (Attend to precision), and MP7 (Look for and make use of structure). This question requires students to analyze both symbolic expressions and graphical features of a piecewise-defined function to evaluate limits and determine continuity. MP2 is evident as students must move between symbolic rules and visual representations, reasoning about function behavior in both abstract and contextualized forms. MP3 is reflected in how students justify whether a limit exists at a particular point and whether the function is continuous, using formal properties and logical reasoning. MP6 is demonstrated through accurate limit computations and the careful interpretation of function values, emphasizing clarity and precision. Finally, MP7 plays a critical role as students examine the overall structure of the function, including how the pieces fit together at defined intervals, to determine whether limit conditions are satisfied.

Unit 2: Differentiation: Definition and Fundamental Properties

Topics 2.1–2.2

In the AP Calculus AB curriculum, Topics 2.1 and 2.2 introduce students to the foundational idea of derivatives. These topics fall under Enduring Understanding CHA-2, which states: “Derivatives allow us to determine rates of change at an instant by applying limits to knowledge about rates of change over intervals.” (College Board, 2020, p.57).

Topic 2.1 Learning Objective CHA-2.A states: “Determine average rates of change using difference quotients.” Students are expected to determine average rates of change of a function over an interval using expressions such as $\frac{f(a+h)-f(a)}{h}$and $\frac{f(x)-f(a)}{x-a}$ as outlined in Essential Knowledge CHA-2.A.1. Graphical interpretations, numerical estimation, and verbal reasoning are often used alongside algebraic techniques to reinforce the connection between change over an interval and instantaneous rate of change.

Topic 2.2 formalizes this understanding by introducing the definition of the derivative of a function and using derivative notation. Learning Objective CHA-2.B states: “Represent the derivative of a function as the limit of a difference quotient. This is explained in Essential Knowledge CHA-2.B.2, which states that the definition of the derivative is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

This shift in reasoning is difficult for many students, especially those who initially think of slope only as a single value. The idea that the derivative itself is a function can help them recognize patterns in change over time or space. These topics form a key bridge between students’ prior experiences with rate of change and their ability to apply differentiation rules later in the course.

In Question 6 of the official AP sample questions, students are given a piecewise-linear function and asked to evaluate three statements: (I) the left-hand limit of the difference quotient at x=3, (II) the right-hand limit, and (III) the existence of the derivative at that point. This question directly engages the limit definition of the derivative and checks whether both one-sided limits match.

$$f(x)=\begin{cases}2x-2 & \text{for } x<3\\ 2x-4 & \text{for } x\ge 3\end{cases}$$

6. Let f be the piecewise-linear function defined above. Which of the following statements are true?

I. $\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=2$

II. $\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=2$

III. $f'(3)=2$

(A) None

(B) II only

(C) I and II only

(D) I, II, and III

Figure 3 — AP Calculus AB sample Question (2016), Question 6

The following comparison comes from Question 4 on Exam 1 of the Fall 2018 MATH 231 course. Students are asked to use the definition of the derivative to calculate $g'(x)$ for a given function without using known derivative rules. This question requires expanding $g(x+h)$, simplifying the expression, and taking the limit as h approaches zero. It aligns closely with CHA-2.B and provides practice with applying the definition in an algebraic way.

4. Use the definition of the derivative to calculate $g'(x)$ for the function $g(x) = \frac{1}{3-2x}$

Figure 4. UWM MATH 231 Fall 2018 EXAM 1 Question 4

Mathematical Practice Standards – Topics 2.1-2.2

This AP question aligns with 2.B and 1.D. According to 2.B, students should "identify mathematical information from graphical, numerical, analytical, and/or verbal representations."

This skill is evident as students interpret the symbolic structure of a piecewise-defined function to evaluate one-sided limits at a point. They must identify the correct piece of the function on either side of $x = 3$ and use this information to assess differentiability. The question also reflects 1.D, which states that students should “identify an appropriate mathematical rule or procedure based on the relationship between concepts or processes to solve problems.” In this case, the derivative exists at a point only if both one-sided limits exist.

The AP question aligns with several Common Core State Standards for Mathematical Practice, specifically MP2 (Reason abstractly and quantitatively), MP3 (Construct viable arguments and critique the reasoning of others), and MP6 (Attend to precision). The task requires students to analyze the behavior of a function near a point, interpreting symbolic expressions and graphical behavior to determine properties related to limits, continuity, and differentiability.
The standard MP2 is reflected in the need to translate between graphical representations and abstract limit concepts. Students must reason quantitatively about one-sided function behavior while connecting these observations to formal definitions of the limit. MP3 is engaged as students evaluate the validity of multiple statements regarding continuity and the existence of derivatives. This involves applying precise logical reasoning rooted in the definitions of one-sided limits, two-sided limits, and pointwise continuity. MP6 is evident in the requirement for careful distinction between function values and derivative values, particularly in cases involving discontinuities or corners, where imprecision could lead to incorrect conclusions.

The MATH 231 exam question aligns with several Common Core State Standards for Mathematical Practice, particularly MP2 (Reason abstractly and quantitatively), MP6 (Attend to precision), and MP7 (Look for and make use of structure). This question requires students to work with a rational function and apply the limit definition of the derivative using symbolic reasoning. MP2 is demonstrated as students must interpret the algebraic form of the function $g(x) = \frac{1}{3-2x}$ within the structure of a difference quotient and reason about the function's behavior as the variable approaches zero. The task challenges students to move fluidly between the symbolic form of the expression and its conceptual interpretation as a derivative. MP6 is reflected in the precise algebraic manipulations required to simplify the difference quotient, particularly the need to eliminate complex fractions without error. The accuracy of each step directly affects the validity of the derivative calculation. MP7 is evident in how students must recognize and make use of the function's structure, including identifying a common denominator and applying limit rules to complete the simplification.

Topics 2.5–2.7: Building Procedural Fluency Through Derivative Rules

In the AP Calculus AB curriculum, Topics 2.5 through 2.7 focus on the development of procedural fluency in differentiation. These topics fall under Enduring Understanding FUN-3, which states: "Recognizing opportunities to apply derivative rules can simplify differentiation" (College Board, 2020, p. 61). Together, these topics provide students with the basic tools they need to compute derivatives of polynomial, trigonometric, radical, exponential, and logarithmic functions. While earlier units emphasize conceptual understanding of limits and the derivative as instantaneous rate of change, this unit focuses more on helping students develop procedural fluency using different derivative rules.

Topic 2.5, Applying the Power Rule, begins with Learning Objective FUN-3.A, which asks students to "calculate derivatives of familiar functions". According to Essential Knowledge FUN-3.A.1, students learn that the derivative of functions in the form $f(x) = x^r$ can be computed using the power rule for integer, fractional, and negative exponents. This rule serves as a gateway to more advanced techniques and helps students build recognition of derivative patterns.

Topic 2.6, Derivative Rules for Sums, Differences, and Constant Multiples, extends these skills under the same learning objective. As outlined in FUN-3.A.2 and FUN-3.A.3, students apply rules for constant multiples and linear combinations of functions, enabling them to simplify more complex expressions by applying derivative rules term by term. These skills support fluency with polynomial and multi-term algebraic functions.

Topic 2.7 expands the set of functions students are expected to differentiate by introducing trigonometric, exponential, and logarithmic rules. According to Essential Knowledge FUN-3.A.4, students learn how to compute the derivatives of $\sin(x)$, $\cos(x)$, e^x, $\ln(x)$. These rules are important because they allow students to solve problems that involve common functions found in real-world applications. This is also the first point in the course where students apply rules to both algebraic and non-algebraic functions together, helping them build more flexible and complete differentiation skills.

4. If $f(x) = 7x - 3 + \ln x$, then $f'(1) =$

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Figure 5. AP Calculus AB exam 2012 MCQ 4 (College Board, 2012)

The AP question asks students to differentiate a function that includes both logarithmic and polynomial terms and evaluate the result at a specific point. This task aligns with Topic 2.7, as it requires applying derivative rules to composite functions. The problem reflects the type of symbolic fluency developed through repeated practice in the AP course sequence.

Let us look at the UWM MATH 231 Fall 2018 exam question 6,

6. Find the derivative of each function.

(a) $f(x) = \dfrac{6x - 2\sqrt{x^3} - 8}{\sqrt{x}}$

(b) $g(x) = (x^5 + \tan(2x))(7\csc(x) - 4x^{3/2})$

(c) $h(t) = \dfrac{8t^3 - \cos(2t)}{2t^5 - \sin(t+1)}$

(d) $j(\theta) = \sin^3(\tan(\theta^2 - 12\theta))$

(e) $k(\theta) = \sin^2(\theta) + \sin(\theta^2)$

Figure 6. UWM MATH 231 Fall 2018 EXAM 1 Question 6

The MATH 231 question requires students to compute the first derivative of multiple composite functions with respect to the variable t or x, depending on the expression. Each part of the problem presents a different symbolic structure, combining a variety of function types. The tasks involve differentiating rational expressions, products of functions, and compositions involving trigonometric, exponential, radical, and polynomial terms. For example, part (a) requires differentiating a rational expression involving square roots and polynomials, while part (d) asks students to differentiate a nested composite expression of the form $\sin^3(\tan(t^2 - 12t))$. To solve these problems, students must correctly apply rules such as the power rule, product rule, quotient rule, and chain rule, along with derivatives of trigonometric and radical functions. The breadth of function types and rule combinations assessed in this question requires students to demonstrate symbolic fluency and procedural versatility across multiple representations.

Mathematical Practice Standards – Topics 2.5-2.7

The AP question aligns with Practice 1E, which states: "Apply appropriate mathematical rules or procedures, with and without technology." In solving the problem, students must apply the limit definition of the derivative as a formal procedure to evaluate whether the derivative exists at a specific point for a piecewise function. This involves substituting expressions, simplifying, and calculating limits from both sides. No technology is required, making it a direct application of a fundamental calculus procedure by hand. The question reinforces procedural fluency and accuracy, which are central to this standard. This question also supports the development of precision and fluency, connected with MP6 (Attend to precision). Students must use accurate mathematical language and notation when applying the limit definition of the derivative, clearly expressing left-hand and right-hand limits and justifying whether the derivative exists at the point in question. Additionally, the problem engages MP2 (Reason abstractly and quantitatively), as students must move between the concrete expressions defining each piece of the function and the abstract concept of differentiability. They must decontextualize the function's behavior into symbolic reasoning using limits and then recontextualize their findings to interpret differentiability. Altogether, the question reflects the depth and rigor expected in both AP Calculus AB and the Common Core Mathematical Practices, promoting conceptual understanding, procedural skill, and thoughtful reasoning.

The MATH 231 question aligns with several Common Core Mathematical Practice Standards, particularly MP2 (Reason abstractly and quantitatively), MP3 (Construct viable arguments and critique the reasoning of others), MP6 (Attend to precision), and MP7 (Look for and make use of structure). MP2 is evident as students interpret the behavior of a piecewise-defined function near a point and connect symbolic expressions with conceptual understanding to determine continuity. MP3 is reflected in how students assess the truth of statements based on formal definitions of one-sided limits and differentiability, requiring logical reasoning and justification. MP6 is important as students must distinguish between a function's value and its derivative at a point, emphasizing accuracy and clarity in interpretation. MP7 plays a key role as students examine how each piece of the function shapes its overall behavior, especially near points where the definition changes.

Topic 2.8-2.9: The Product and Quotient Rule

In the AP Calculus AB curriculum, Topics 2.8 and 2.9 introduce the product and quotient rules, essential techniques for differentiating expressions involving the multiplication or division of functions. These topics fall under Enduring Understanding FUN-3, which emphasizes recognizing opportunities to apply derivative rules. Students are expected to identify when to use these rules and apply them correctly to different types of function combinations.

Let us look at the 2023 AP Calculus AB exam Free Response Question 5, I included the scoring guidelines for this free response question and a model solution to show how student responses are evaluated.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

	Model Solution	Scoring	
(a)	Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.		
	$h'(x) = f'(g(x)) \cdot g'(x)$ $h'(7) = f'(g(7)) \cdot g'(7)$	Chain rule	**1 point**
	$= f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $h'(x) = f'(g(x)) \cdot g'(x)$ or $h'(7) = f'(g(7)) \cdot g'(7)$.
- If the first point is earned, the second point is earned only for an answer of 12 (or equivalent).
- If the first point is not earned, the second point can be earned only for a response of either $f'(0) \cdot 8 = 12$ or $\frac{3}{2} \cdot 8$.
- A response of 12 with no supporting work does not earn either point.

Total for part (a) 2 points

(b)	Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.		
	$k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$	Product or chain rule	**1 point**

$k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$	
$= 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2 = -72 + 32 = -40$	$k''(4)$ **1 point**
The graph of k is concave down at the point where $x = 4$ because $k''(4) < 0$ and k'' is continuous.	Answer with reason **1 point**

Scoring notes:

- The first point is earned for either $k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$.
- The first point is also earned by any of the following incorrect expressions, each of which has a single error in the application of the product rule or the chain rule:
  - $2f(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $2f(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $2f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $2f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $2f(x) \cdot f'(x) \cdot g'(x)$ or $2f(4) \cdot f'(4) \cdot g'(4)$
  - Note: A response that presents one of these expressions cannot earn the second point.
- To earn the second point a response must correctly find $k''(4) = -40$ (or equivalent) with supporting work.
- The third point is earned for an answer and reason that are consistent with any declared nonzero value of $k''(4)$.

Total for part (b) 3 points

Figure 7. AP ***Calculus AB 2023 FRQ 5 (College Board, 2023)***

The AP question presents students with a table of values and requires them to apply derivative rules using the given information. In part (a), students differentiate a composite function by applying the chain rule and substituting values from the table. In part (b), they analyze the concavity of a function defined as a product, using the product rule and second derivative to determine whether the graph is concave up or down. While the chain rule is also used in part (b), the emphasis is on selecting and applying appropriate rules based on the structure of the expression. Because part (b) involves the multiplication of two functions, it is most closely aligned with the product rule. Although the quotient rule does not appear in this question, it remains an essential part of Topic 2.9 and is commonly assessed when students are asked to differentiate functions written as quotients.

Let us look at UWM Fall 2018 Final Exam 2018 Question 8:

8. Consider the table of values for $f(x), g(x), f'(x)$ and $g'(x)$ as shown.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	-9	7	4	1
0	5	-2	9	-3
$\frac{\pi}{4}$	-4	3	2	6
2	5	-8	-3	0

(a) Let $h(x) = f(x)\cos(x)$. What is $h'(\frac{\pi}{4})$?

(b) Let $k(x) = [g(x) + x^3]^2$. What is $k'(-1)$?

(c) Let $p(x) = \dfrac{\sin x}{g(x)}$. What is $p'(0)$?

Figure 8. UWM MATH 231 Fall 2018 Final Exam Question 8

The MATH 231 question engages students in applying fundamental differentiation rules using information provided in a table. It's very similar to the AP question. The question requires students to identify whether the product, chain, or quotient rule is appropriate based on the structure of the expressions and then carry out the differentiation using the corresponding table values. Students are expected to demonstrate conceptual understanding of how different rules apply in various contexts and how to interpret derivative expressions symbolically.

Mathematical Practice Standards – Topic 2.8-2.9

The AP question aligns with Mathematical Practice 1E: "Apply appropriate mathematical rules or procedures, with and without technology." In part (a), students apply the chain rule to differentiate the equation using values from the table. In part (b), students analyze the concavity of a function. This requires evaluating the behavior of the function by applying the product rule and reasoning about the second derivative using data from the table. Both parts assess the student's ability to select and apply appropriate derivative rules based on structure and given values. This skill also aligns with several Common Core Standards for Mathematical Practice. Specifically, it reflects MP1 (Make sense of problems and persevere in solving them) by requiring students to interpret a composite function $h(x) = f(g(x))$ and apply the chain rule appropriately, even when values are given in a table rather than a symbolic form. MP2 (Reason abstractly and quantitatively) is also engaged, as students must contextualize and decontextualize numerical values and function compositions to compute derivatives. The problem embodies MP6 (Attend to precision), as it demands careful attention to notation, values from the table, and differentiation rules to compute $h'(7)$ accurately. Furthermore, the analysis in part (b) encourages MP3 (Construct viable arguments and critique the reasoning of others), as students must justify whether the graph of k is concave up or down by computing a second derivative and interpreting its sign. MP7 (Look for and make use of structure) is evident as students must recognize and utilize the nested structure of composite and product functions in their differentiation processes.

The skills assessed in MATH 231 Question align with several Common Core State Standards for Mathematical Practice. The task exemplifies MP1 (Make sense of problems and persevere in solving them), as students must interpret a table of function values and derivatives to determine how to apply the appropriate calculus rules including: the product, quotient, and chain rules without given the specific function to differentiate. It also supports MP2 (Reason abstractly and quantitatively), requiring students to connect numerical data with symbolic expressions to compute derivative values without explicit function formulas. Evidence of MP6 (Attend to precision) is found in the need for accurate substitution and careful application of derivative rules.

Unit 3: Differentiation: Composite, Implicit, and Inverse Functions

Topic 3.2 Implicit Differentiation

In the AP Calculus AB curriculum, Topic 3.2 introduces the technique of implicit differentiation. This topic is part of Enduring Understanding FUN-3, which states: "Recognizing opportunities to apply derivative rules can simplify differentiation" (College Board, 2020, p.73). The Learning Objective is FUN-3.D, which requires students to "calculate derivatives of implicitly defined functions." According to Essential Knowledge FUN-3.D.1, "the chain rule is the basis for implicit differentiation." Together, these standards establish a foundation for interpreting and differentiating equations that define y implicitly in terms of x, an essential skill that becomes increasingly relevant in multivariable question contexts.

Let us look at the 2023 AP Calculus AB exam Free Response Question 6(a):

6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.

	Model Solution	Scoring	
(a)	Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.		
	$\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$	Implicit differentiation	**1 point**
	$\Rightarrow 2y = \frac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$	Verification	**1 point**

Scoring notes:

- The first point is earned only for the correct implicit differentiation of $6xy = 2 + y^3$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y'.
- The second point cannot be earned without the first point.
- It is sufficient to present $2y = \frac{dy}{dx}\left(y^2 - 2x\right)$ to earn the second point, provided there are no subsequent errors.

Total for part (a) **2 points**

Figure 9. AP ***Calculus AB 2023 FRQ 6 (College Board, 2023)***

This question offers an opportunity for students to demonstrate their understanding of implicit differentiation through differentiation rules. The equation $6xy = 2 + y^3$ define y implicitly in terms of x, requiring students to differentiate both sides of the equation with respect to x. Doing so accurately involves applying the product rule to the left-hand side of the equation and the chain rule to the right-hand side of the equation. Once both sides are differentiated, the task shifts to isolating $\frac{dy}{dx}$. Students need to rearrange and simplify the expression.

Let us look at MATH 231 Fall 2018 Exam 1, Question 9:

9. Use implicit differentiation to find a Linearization of the curve defined by $y^2 - x^2y^3 + 2x^3 = \cos(x^3y) + 1$. at the point $(1, 0)$.

Figure 10. UWM MATH 231 Fall 2018 EXAM 1 Question 9

The corresponding MATH 231 question advances this concept by explicitly requiring students to perform implicit differentiation on an equation involving both trigonometric and polynomial expressions. The task specifies a point of tangency and instructs students to construct a linearization at that point. To solve the question, students must differentiate both sides of the equation with respect to x, apply the chain rule when differentiating terms containing y, and solve for $\frac{dy}{dx}$. Once the slope is found, students must use it along with the given point to write the equation of the tangent line, completing the linearization process. This exercise merges the procedural objective of FUN-3.D with the conceptual application of linear approximation. Although both questions involve local linearization, the MATH 231 question requires a broader range of skills.

Mathematical Practice Standards – Topic 3.2

This AP Calculus AB question assesses a student's ability to apply implicit differentiation to verify a derivative expression, aligning directly with Mathematical Practice 1E: "Apply appropriate mathematical rules or procedures, with and without technology." The problem requires selecting and executing a derivative technique, in this case the chain rule, within the context of an implicitly defined relationship. This skill also aligns with several Common Core Standards for Mathematical Practice. Specifically, it reflects MP1 (Make sense of problems and persevere in solving them) by requiring students to interpret and reframe the given relationship between x and y to differentiate it correctly. It engages MP6 (Attend to precision), as students must correctly apply differentiation rules and algebraic manipulations to justify that the given expression is valid. Furthermore, the verification step calls on MP3 (Construct viable arguments and critique the reasoning of others), as students must demonstrate that both sides of the equation match through sound reasoning. Lastly, MP7 (Look for and make use of structure) is evident in how students must recognize the structure of a composite function embedded within an implicitly defined expression.

The MATH 231 question requires students to use implicit differentiation to find the linearization of a curve defined by an equation involving trigonometric and polynomial expressions. The task aligns with Common Core State Standards for Mathematical Practice in multiple ways. It exemplifies MP1 (Make sense of problems and persevere in solving them), as students must understand the structure of the equation and determine a strategy to differentiate both sides implicitly. MP2 (Reason abstractly and quantitatively) is involved when students manipulate symbolic representations and interpret the meaning of derivatives in the context of linearization. The problem also reflects MP6 (Attend to precision), as students must carefully compute derivatives and apply the point (1, 0) to obtain a correct tangent approximation. Additionally, MP7 (Look for and make use of structure) is essential as students identify function compositions and patterns, such as the product of x cubed and y inside the cosine function.

Unit 4 Contextual Applications of Differentiation

Topic 4.1 Interpreting the Meaning of the Derivative in Context

Topic 4.2 Straight-Line Motion: Connecting Position, Velocity, and Acceleration

In the AP Calculus AB curriculum, Topics 4.1 and 4.2 develop students' understanding of derivatives in real-world contexts. These topics fall under Enduring Understanding CHA-3, which states: "Derivatives allow us to solve real-world problems involving rates of change" (College Board, 2020, p. 84). Topic 4.1 focuses on Learning Objective CHA-3.A, which requires students to interpret the meaning of a derivative in context. According to Essential Knowledge statements CHA-3.A.1 through CHA-3.A.3, students learn that a derivative represents the instantaneous rate of change, can be applied to real-world contexts, and carries units based on the relationship between variables. Topic 4.2 builds on this by introducing Learning Objective CHA-3.B: “Calculate rates of change in applied contexts.” Essential Knowledge CHA-3.B.1 clarifies that: “The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.

Let us look at the AP Calculus AB 2024 exam free response question 6,

2. A particle moves along the x-axis so that its velocity at time $t \geq 0$ is given by $v(t) = \ln\left(t^2 - 4t + 5\right) - 0.2t$.

   (a) There is one time, $t = t_R$, in the interval $0 < t < 2$ when the particle is at rest (not moving). Find t_R. For $0 < t < t_R$, is the particle moving to the right or to the left? Give a reason for your answer.

   (b) Find the acceleration of the particle at time $t = 1.5$. Show the setup for your calculations. Is the speed of the particle increasing or decreasing at time $t = 1.5$? Explain your reasoning.

Figure 11. AP ***Calculus AB 2024 FRQ 6 (College Board, 2024)***

This AP question requires students to analyze motion along a line by interpreting and differentiating a given velocity function. In part (a), students are asked to determine when the particle is at rest by solving $v(t) = 0$, and then assess the direction of motion immediately before that time by analyzing the sign of the velocity function. This requires conceptual understanding of how velocity influences direction and movement. In part (b), students must compute the acceleration by differentiating the velocity function and evaluating the result at $t = 1.5$. To determine whether the particle’s speed is increasing or decreasing, students must reason about the relationship between the signs of velocity and acceleration at that moment.

. Let us look at the MATH 231 exam 1 question 6,

11. The position of an object is given as $p(t) = 36t - t^3$ where p is in meters and t is in seconds.

(a) Write an expression for the velocity as a function of time.

(b) Write an expression for the acceleration as a function of time.

(c) Find the velocity at $t = 3$ seconds.

(d) Carefully draw the position, velocity, and acceleration functions on the same graph. Discuss the interesting features of the graphs!

Figure 12. UWM MATH 231 Fall 2018 EXAM 1 Question 11

The MATH 231 question offers a direct application of derivative concepts to model and analyze motion along a straight line. The task begins with a position function $p(t) = 36t - t^3$, and students are asked to compute the first and second derivatives to obtain expressions for velocity and acceleration. Part (c) requires evaluating the velocity at a specific time, assessing the function's behavior at a point. Part (d) integrates graphical interpretation, prompting students to compare position, velocity, and acceleration in a more visual presentation.

Mathematical Practice Standards – Topic 4.1 and 4.2

The AP question aligns with the AP Mathematical Practice Standards 1.D and 1.E by requiring students to identify and apply appropriate calculus techniques in context. In part (a), students must determine when the particle is at rest by solving $v(t) = 0$, demonstrating their ability to select a suitable procedure based on the relationship between motion and the sign of the velocity function (1.D). In part (b), students differentiate the given expression for velocity to find the acceleration and evaluate it at a specified time. They must then reason about whether the particle's speed is increasing or decreasing by comparing the signs of velocity and acceleration.

This reflects Standard 1.E, which involves applying correct mathematical procedures, such as differentiation and sign analysis, to interpret motion. The problem integrates symbolic computation with interpretation of physical behavior, reinforcing connections between derivative rules and their applications.

The MATH 231 question aligns closely with several Common Core Standards for Mathematical Practice. It reflects MP1 (Make sense of problems and persevere in solving them), as students must interpret a position function and determine how it changes over time through successive derivatives. MP4 (Model with mathematics) is evident in the requirement to graph and compare the position, velocity, and acceleration functions, translating calculus concepts into visual models. MP2 (Reason abstractly and quantitatively) is demonstrated as students move between the symbolic representations of functions and their interpretations in physical motion. MP6 (Attend to precision) is necessary throughout the problem as students compute exact expressions for derivatives and evaluate them accurately at specific time points. Lastly, MP7 (Look for and make use of structure) is engaged when identifying the consistent patterns among derivatives and interpreting key features such as increasing/decreasing behavior and concavity through their graphs.

Topic 4.3 Rates of Change in Applied Contexts Other Than Motion

Topic 4.4 Introduction to Related Rates

Topic 4.5 Solving Related Rates Problems

In the AP Calculus AB curriculum, Topics 4.3 through 4.5 build students' ability to apply derivatives in real-world contexts beyond motion, focusing on rates of change and related rates. These topics fall under Enduring Understanding CHA-3, which states: "Derivatives allow us to solve real-world problems involving rates of change" (College Board, 2020, p. 86). Collectively, these topics transition students from computing derivatives to interpreting and applying them in applied situations, strengthening their understanding of functional relationships and change over time.

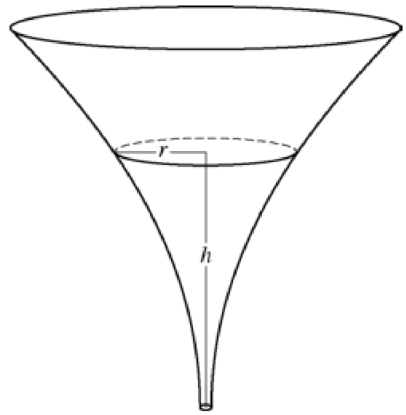
Topic 4.3, Rates of Change in Applied Contexts Other Than Motion, is grounded in Learning Objective CHA-3.C, which requires students to "interpret rates of change in applied contexts." According to Essential Knowledge CHA-3.C.1, students learn how to apply derivative concepts to problems involving economic, biological, or physical systems where motion is not the primary focus. This topic expands students' appreciation of the derivative as a modeling tool in varied scenarios.

Topic 4.4, Introduction to Related Rates, deepens this understanding by introducing problems where multiple variables change simultaneously. This topic is associated with Learning Objective CHA-3.D. Essential Knowledge statements CHA-3.D.1 and CHA-3.D.2 highlight that students must use the chain rule to differentiate implicitly with respect to a single independent variable, and they may also need to apply product and quotient rules to differentiate all changing quantities correctly. This topic emphasizes procedural accuracy in identifying and differentiating relationships among multiple variables.

Topic 4.5, Solving Related Rates Problems, requires students to interpret the meaning of related rates in applied contexts. Under Learning Objective CHA-3.E and Essential Knowledge CHA-3.E.1, students learn to compute unknown rates of change by connecting them to known rates through algebraic relationships. This topic marks the culmination of related rates instruction by requiring students to both construct and interpret differential equations that describe changing quantities. Together, Topics 4.3 through 4.5 support students in developing flexible reasoning strategies and applying calculus tools to multi-variable, real-world contexts.

Let us look at the AP Calculus AB 2016 exam free response question 5,

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

Figure 13. AP ***Calculus AB 2016 FRQ 5 (College Board, 2016)***

The AP question involves determining how one quantity changes in relation to others that are also changing, consistent with the AP Calculus learning objectives for related rates (CHA-3.D and CHA-3.E). The scenario describes a funnel with circular cross sections and changing height and radius. The question explores average value, volume, and related rates in the context of a draining liquid. This geometric relationship is modeled using the Pythagorean Theorem. Differentiating both sides of the equation with respect to time applies the chain rule, as outlined in standard CHA-3.D.1. The resulting equation relates the rates of change of the individual distances to the rate of change of the total distance. Known values are then substituted to solve for the unknown rate. This method demonstrates how derivatives are used to interpret and solve real-world problems involving rates of change, in line with the essential knowledge described in standards CHA-3.C.1 and CHA-3.E.1.

Let us look at the MATH 231 Fall 2018 exam 1 question 6,

12. The Brewers and the Packers met in Milwaukee for a pep rally at noon. When it finished, the Pack headed straight south towards Houston at a rate of 62 miles/hour. The Brewers headed straight west towards LA at a rate of 70 miles/hr. Use Calculus to determine how quickly the distance between the teams is increasing 3 hours later. Interesting fact: $210^2 + 186^2 \approx 280^2$.

Figure 14. UWM MATH 231 Fall 2018 EXAM 1 Question 12

The MATH 231 question demonstrates how derivatives can be applied to analyze how distances between moving objects change over time. By modeling the positions of two teams moving in perpendicular directions using functions of time, we relate their individual rates of motion to the rate at which the distance between them changes. Recognizing this as a geometric situation, we use the Pythagorean Theorem to express the total distance as a function of two right-angled legs representing their individual paths. Differentiating this relationship with respect to time and applying known motion rates, we find the rate at which the total distance is increasing. This approach relies on understanding how to relate multiple changing quantities through differentiation and interpret their rates of change in context. By structuring the solution in this way, we build both a conceptual and procedural understanding of how instantaneous rates reflect real-world motion.

Mathematical Practice Standards – Topic 4.3 – 4.5

The AP question aligns with multiple AP Calculus Mathematical Practice Standards. It requires students to "identify common underlying structures in problems involving different contextual situations" (2.A), as the motion of two objects is modeled using the familiar geometric structure of a right triangle. To solve the question, students must "apply appropriate mathematical rules or procedures, with and without technology" (1.E), specifically using implicit differentiation and the chain rule to relate the changing quantities. Finally, the result is interpreted in the context of the situation, fulfilling the expectation that students “explain the meaning of mathematical solutions in context” (3.F). This question also aligns with several Common Core Standards for Mathematical Practice. Specifically, it reflects MP4 (Model with mathematics), as students must represent a real-world motion scenario using a geometric model and express it algebraically using the Pythagorean Theorem. It engages MP2 (Reason abstractly and quantitatively), since students must interpret variables and rates in a dynamic context, translating physical motion into mathematical relationships. MP6 (Attend to precision) is evident as students apply the chain rule carefully, differentiate with respect to time, and substitute accurate values to find the correct rate. Finally, MP7 (Look for and make use of structure) plays a key role, as students must recognize the structure of the relationship among distance, time, and velocity within the triangle to relate the changing quantities effectively.

This MATH 231 question aligns with several Common Core Standards for Mathematical Practice. It reflects MP4 (Model with mathematics), as students must translate a real-world situation involving two teams moving in perpendicular directions into a geometric model that can be analyzed mathematically. MP2 (Reason abstractly and quantitatively) is evident as students interpret the rates of movement and the resulting change in distance between the teams using variables and functions. MP6 (Attend to precision) is required when applying differentiation accurately, including the chain rule, and substituting exact numerical values to compute the correct rate of change. Lastly, MP7 (Look for and make use of structure) is demonstrated through recognition of the right triangle formed by the teams' paths and the use of the Pythagorean Theorem to relate their distances in a structured way.

Unit 5: Analytical Applications of Differentiation

Topic 5.3 Determining Intervals on Which a Function Is Increasing or Decreasing

Topic 5.4 Using the First Derivative Test to Determine Relative (Local) Extrema

Topic 5.6 Determining Concavity of Functions over Their Domains

Topic 5.7 Using the Second Derivative Test to Determine Extrema

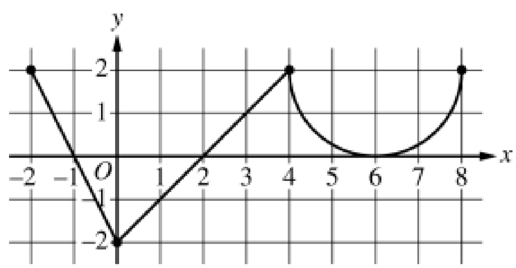
In the AP Calculus AB curriculum, Topics 5.3, 5.4, 5.6, and 5.7 help students deepen their understanding of how derivatives describe the behavior of functions, particularly how they inform function shape, increasing and decreasing behavior, and local/global extrema. These topics align with Enduring Understanding FUN-4, which states: "A functions derivative can be used to understand some behaviors of the function." (College Board, 2020, p. 99). These topics build students' analytical skills by teaching them to connect derivative information with graphical and contextual features of functions.

Topic 5.3, Determining Intervals on Which a Function Is Increasing or Decreasing, corresponds to Learning Objective FUN-4.A which states: "Justifying conclusions about the behavior of a function based on the behavior of its derivatives." Students apply the first derivative to identify where functions are increasing or decreasing by finding critical points and analyzing the sign of the derivative. According to Essential Knowledge FUN-4.A.1, the first derivative of a function can provide information about where the function is increasing or decreasing.

Topic 5.4, Using the First Derivative Test to Determine Relative (Local) Extrema, builds on Topic 5.3 and aligns with Learning Objective FUN-4.A which states: "Justify conclusions about the behavior of a function based on the behavior of its derivatives." Essential Knowledge FUN- 4.A.2 emphasize using derivative information to justify whether a function has a relative extremum.

Topic 5.6, Determining Concavity of Functions over Their Domains, supports the same Learning Objective from 5.4. The essential knowledge explains that the second derivative shows where a function is concave up or down. If the first derivative is increasing, the function is concave up; if it is decreasing, the function is concave down. Inflection points occur where the second derivative changes sign.

Let us look at the AP Calculus AB 2023 exam free response question 4,



Graph of f'

4. The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

Figure 15. AP ***Calculus AB 2023 FRQ 4 (College Board, 2023)***

This AP question aligns with several core skills outlined in the AP Calculus framework. In part (a), students are required to determine whether the function f has a relative minimum, maximum, or neither at x=6 by analyzing the behavior of its derivative f'. This tests their ability to interpret the sign changes of f' to justify conclusions about local extrema, demonstrating understanding of how increasing and decreasing behavior relates to relative maxima and minima. In part (b), students are asked to identify where the graph of f is concave down, which involves recognizing where the derivative f' is decreasing, indicating that the second derivative f'' is negative. This demonstrates the ability to use graphical information about the derivative to determine concavity of the original function.

Let us look at the MATH 231 exam 2 question 5,

5. Consider the function $f(x) = \dfrac{x}{1 - x^2}$.

(e) State the intervals where $f(x)$ is increasing and/or decreasing.

(f) State the locations (x coordinates) of the local maxima and/or minima of $f(x)$.

(g) State the intervals where $f(x)$ is concave up or down, and the locations of any inflection points. Note: $f''(x) = \dfrac{-2x(x^4 + 2x^2 - 3)}{(1 - x^2)^2}$

Figure 16. UWM MATH 231 Fall 2018 EXAM 2 Question 5

The MATH 231 question exemplifies how derivatives can be used to interpret the behavior of a rational function across its domain. By expressing the function $\frac{x}{1-x^2}$ and examining its first and second derivatives, connect algebraic expressions to graphical and conceptual insights about the function's growth and curvature. Recognizing this as a problem involving rates of change and function behavior, we use derivative rules to identify where the function increases, determine concavity, and locate inflection points. Differentiating the function and analyzing signs and critical values allows us to interpret how the function changes and how its shape reflects those changes. By structuring the solution around derivative-based reasoning, we develop both a conceptual and procedural mastery of how calculus models real-world function behavior through instantaneous change.

Mathematical Practice Standards – Topic 5.3, 5.4, 5.6, 5.7

The AP Calculus AB question aligns with several key Mathematical Practice Standards, specifically emphasizing graphical analysis and justification. It prompts students to "describe the relationships among different representations of functions and their derivatives” (2.E), as students must interpret the graph of f', and the properties of the original function f. The problem also requires students to “apply an appropriate mathematical definition, theorem, or test” (3.D), using the First Derivative Test to assess whether f has a relative extrema and determining concavity based on the sign and behavior of f'. This reinforces students’ ability to synthesize graphical and numerical information to make informed conclusions about a function’s behavior. This question also aligns with several Common Core Standards for Mathematical Practice. It reflects MP4 (Model with mathematics), as students must use the graphical model of f' to describe characteristics of the original function f, such as relative extrema and concavity. MP2 (Reason abstractly and quantitatively) is central, since students must interpret the meaning of the derivative graph.

The MATH 231 question aligns with several Common Core Standards for Mathematical Practice. It reflects MP2 (Reason abstractly and quantitatively) as students must interpret and manipulate the function’s algebraic form to determine increasing/decreasing behavior and local extrema using the first derivative. MP4 (Model with mathematics) is demonstrated as students apply mathematical tools like the first and second derivatives to analyze the behavior of a rational function, which models change in a general context. MP6 (Attend to precision) is required in performing accurate derivative computations and in correctly identifying domain restrictions and critical points.

Unit 6 Integration and Accumulation of Change

Topic 6.9 Integrating Using Substitution

In the AP Calculus AB curriculum, Topic 6.9 develops students’ ability to apply integration techniques strategically, specifically through substitution. This topic falls under

Enduring Understanding FUN-6, which states: “Recognizing opportunities to apply knowledge of geometry and mathematical rules can simplify integration” (College Board, 2020, p. 123). Topic 6.9 emphasizes Learning Objective FUN-6.D, which requires students to evaluate both indefinite and definite integrals using algebraic manipulation, especially substitution.

Let us look at the AP Calculus AB 2012 exam free response question 12,

12. Using the substitution $u = \sqrt{x}$, $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ is equal to which of the following?

(A) $2\int_1^{16} e^u\,du$ (B) $2\int_1^{4} e^u\,du$ (C) $2\int_1^{2} e^u\,du$ (D) $\frac{1}{2}\int_1^{2} e^u\,du$ (E) $\int_1^{4} e^u\,du$

Figure 17. AP ***Calculus AB 2012 MCQ question 12 (College Board, 2012)***

The AP question provides a clear application of the substitution method for definite integrals, as outlined in Topic 6.9 of the course framework. Students are instructed to evaluate the integral using the substitution $u = \sqrt{x}$, a common and effective strategy for simplifying integrals involving composite functions. According to Learning Objective FUN-6.D, students must be able to determine definite integrals through substitution or algebraic manipulation. In this case, the substitution simplifies the radical exponent and eliminates the square root in the denominator, making the integral easier to evaluate.

The substitution process demonstrates Essential Knowledge FUN-6.D.1, which identifies substitution of variables as a key technique for finding antiderivatives. By letting $u = \sqrt{x}$, we also have $x = u^2$, and $dx = 2u\,du$.When substituted into the integral, students also need to change the bound.

This transformation illustrates Essential Knowledge FUN-6.D.2, which emphasizes the importance of changing the limits of integration when performing substitution in definite integrals.

Let us look at the MATH 231 Fall 2018 Exam 2 question,

12. Evaluate. $\int 8y^3\sqrt{1-y^4}\,dy$

Figure 18. UWM MATH 231 Fall 2018 EXAM 2 Question 12

This MATH 231 question requires students to evaluate the integral. which is a clear example of applying substitution to simplify a complex expression. The question helps students strengthen their algebraic manipulation and substitution skills, encouraging them to recognize structure within expressions and apply strategies to reduce an integrand to a more manageable form. It supports conceptual understanding and procedural fluency with integration techniques used throughout the course.

Mathematical Practice Standards – Topic 6.9

This AP question aligns well with Mathematical Practice Standard 1.E, which asks students to "Apply appropriate mathematical rules or procedures, with and without technology." In this problem, students are given a definite integral involving an exponential function with a square root in the exponent and are instructed to apply the substitution $u = \sqrt{x}$. To solve the integral correctly, they must carry out a series of algebraic steps, including rewriting the integrand, finding the differential dx using the chain rule, and updating the limits of integration to reflect the new variable. This question also aligns with several Common Core Standards for Mathematical Practice. It reflects MP4 (Model with mathematics), as students must apply the substitution method to transform a complex integral into a simpler one, demonstrating how mathematical procedures can model and solve abstract problems efficiently. MP2 (Reason abstractly and quantitatively) is evident, as students must interpret the structure of the original integrand, recognize the relationship between variables, and reason through the change of bounds and differentials with precision.

The MATH 231 question aligns with several Common Core Standards for Mathematical Practice. It reflects MP2 (Reason abstractly and quantitatively), as students must interpret the expression $\int 8y^3 \sqrt{1-y^4}\, dy$ and recognize the underlying structure that allows for substitution. This involves moving fluidly between the symbolic representation and the reasoning behind the substitution process. It also supports MP7 (Look for and make use of structure) because students must identify the composite nature of the integrand and realize that setting $u = 1 - y^4$ simplifies the integral. Additionally, MP6 (Attend to precision) is critical as students must correctly differentiate, substitute, and manipulate expressions to arrive at a valid antiderivative.

Chapter 4: Conclusion

AP Calculus AB and MATH 231 at the University of Wisconsin–Milwaukee cover many of the same topics. Both include limits, derivatives, and integrals, which are the core parts of a first-semester college calculus course. The College Board says AP Calculus AB is meant to match the level of a five-credit college course. MATH 231 is a four-credit course and teaches similar material. Because AP Calculus AB follows a national standard and prepares students for a final exam, AP Calculus AB includes more structure and details.

The way these courses are taught is different. AP Calculus AB is usually taught over a full school year in high school. Teachers often have training in how to support student learning in the summer. This gives students more time and contact hours with the teacher. MATH 231 is taught in a 15-week college semester. It moves faster and expects students to work more on their own. Instructors are often graduate students, and they give less day-to-day guidance.

There are also differences in calculator use and testing. The AP Calculus AB exam includes parts where calculators are allowed. This lets students use technology to help solve problems. In MATH 231, calculators are not allowed on quizzes or exams. Students are expected to do all work by hand and rely on their algebra skills.

After reviewing the AP Calculus AB curriculum and MATH 231 at UWM, I noticed some differences in the types of questions used. The questions in MATH 231 are usually more direct and focus on solving problems with less context. In comparison, AP Calculus AB questions often require more thinking and include more steps or real-life situations. The design of AP questions seems to push students to think more deeply and understand the concepts better. Both styles are helpful in different ways, and the difference seems to be more about the goals of the courses than which one is harder. To answer the question I posed at the beginning, AP Calculus does adequately prepare students for college courses beyond Calculus I.

Chapter 5: Reflection

When we encourage students to take College Calculus I or higher while still in high school, we are often accelerating their academic timeline without fully considering their readiness. I have seen many students believe they are ready to take several AP math courses at once, even when teachers advise against it. This is often due to outside pressures, not a deep understanding of the subject. Schools sometimes push students into these advanced tracks because of enrollment rules. Classes might not run without enough students. This can lead to decisions based on logistics instead of what is best for the student.

When I teach AP Calculus AB, I notice how structured and intense the course is. The College Board outlines every unit in detail, and training for the course focuses heavily on understanding the rubrics. The free response questions are long and can take a whole class to complete. Even with extra support time, many students struggle to keep up. Many students take AP Calculus in high school, but a significant number do not pass the AP exam. This raises concerns about whether the course was a good use of their time and effort. Not only do these students miss out on earning college credit, but they may also develop misunderstandings or bad habits that can make college mathematics even harder. The course requires daily practice and effort, and students who are used to getting good grades often find AP Calculus AB challenging.

It is important to provide students with strong content, but it is equally important to teach them how to learn independently, think clearly about ideas, and persist through challenges. We should not assume that moving faster means doing better. Instead, our teaching should focus on helping students build a deep understanding of mathematics, rather than pushing them to earn good grades or impressive titles. I also hope this research encourages teachers to adopt more flexible approaches to instruction. With so much content packed into the course, we need to ask whether students truly need to master every detail now, or if they would benefit more from spending time exploring the main ideas of calculus in a thoughtful and meaningful way.

One way to improve AP Calculus AB instruction is to place more emphasis on conceptual understanding. Students should not only know how to solve problems but also understand why certain methods work. Teachers can support this by using multiple representations such as graphs, tables, and written explanations, in addition to symbolic work. For example, when teaching the concept of a derivative, it is important to help students understand what the derivative represents and how it describes the behavior of a function.

Another way to improve instruction is to slow down and focus more deeply on the core ideas. The course moves quickly, and teachers may feel pressure to cover every topic before the exam. However, students often learn more when they have time to ask questions, reflect on their understanding, and revisit key concepts. Deep learning takes time, and students are more likely to retain and connect ideas when instruction is not rushed.

Encouraging students to explain their thinking can also strengthen learning. Through written responses, classroom discussions, or small group conversations, students benefit from expressing their ideas and hearing how others think. These opportunities help students develop mathematical reasoning and prepare them for the type of work expected on the free response section of the exam.

A supportive classroom environment that values effort, growth, and curiosity can help students navigate the demands of the course while staying engaged in learning. By focusing on these priorities, we can create a more balanced and meaningful experience for students in AP Calculus AB.

Many students take AP Calculus in high school but do not always benefit from it in college, especially if they perform poorly on the AP exam. According to research by Bressoud (2015), about one third of AP Calculus students score below a 3. These students tend to perform no better in college calculus than peers who never took calculus in high school. In 2024, only 64.4 percent of AP Calculus AB students scored a 3 or higher, meaning more than one third may not be gaining a meaningful advantage from the course (College Board, 2024).

This is further supported by outcome data comparing students' performance in college calculus:

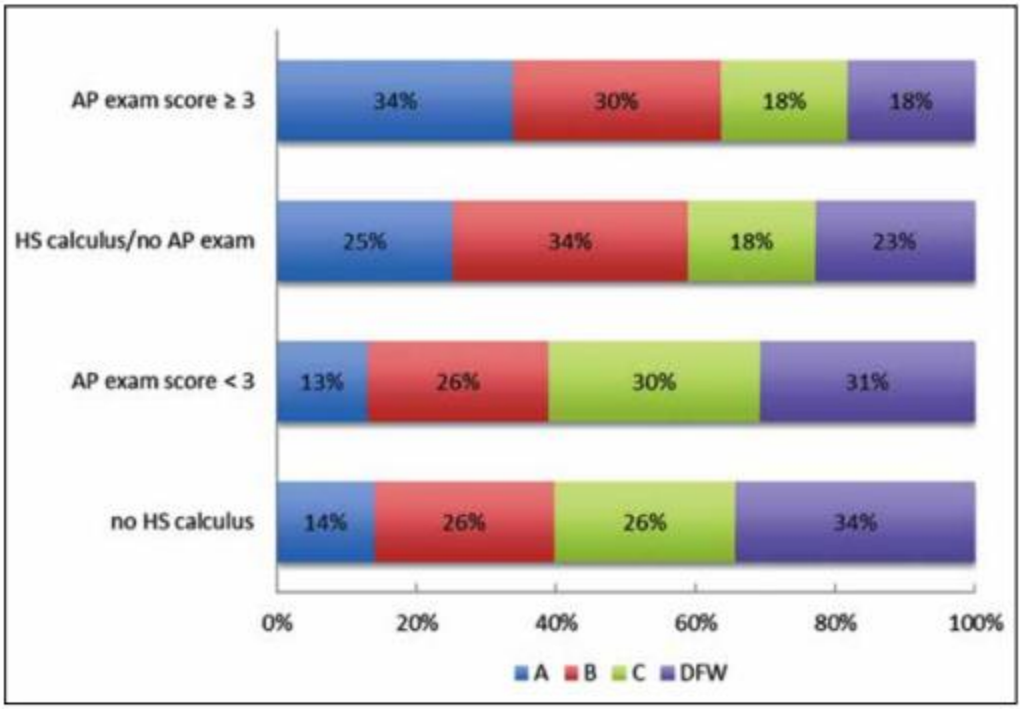


Table 5- Final grades in Calculus 1 are compared by mathematics experience in high school.

(Figure 7. Final grades in Calculus 1 are compared by mathematics experience in high school. Retrieved from https://www.maa.org/sites/default/files/pdf/cspcc/BressoudInsights.pdf)

The difference in failure rates between the two groups — students who did not take calculus in high school and those who took AP Calculus but did not take the AP exam — is only three percentage points. This small change suggests that simply enrolling in AP Calculus, without demonstrating mastery through the exam, does not significantly improve students' performance in college Calculus I. For some students, taking the course under these circumstances may even represent a missed opportunity to develop stronger foundational skills in other areas of mathematics, such as geometry, trigonometry, or statistics. These statistics are drawn from the MAA National Study of College Calculus, which examined student outcomes across a broad range of institutions and highlighted the limited benefit for students who took high school calculus but did not pass the AP exam (Bressoud, 2015).

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