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New Tools for Robustness Analysis *

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Summary of Plenary

My objective in this lecture is to provide a personal perspective on a new line of research—a line of research which many of us believe will have a profound impact on the analysis and design of robust control systems. The motivation for this work undoubtedly comes from papers in the Russian literature by Kharitonov (1978a),(1978b). Since the time of these seminal publications, more than a hundred papers have appeared (most in the last four years) providing extensions, generalizations, new proofs and applications of Kharitonov's Theorem. Given this explosive rate of publication, much "smoke" has emerged. The researcher who is unfamiliar with this area faces an enormous pile of papers and may not know which ones to read first. To this end, I will discuss only a selected subset of the published literature which I feel provides the uninitiated reader with a coherent path through the rapidly expanding jungle of results. I apologize in advance to those authors whose work I do not mention. Perhaps any controversy resulting from my citation of references will spark discussion motivating further work.

Why is Kharitonov's Theorem Important?

In this part of the lecture, I will provide some motivation for the type of robustness problems under consideration. Various examples of specific physical systems will be used to focus attention on the issue of robust performance with respect to structured perturbations. The typical scenario which I have in mind occurs in analyzing the behavior of the closed loop poles when various physical parameters are known only within given bounds. For example, in the well known track-guided bus problem considered by Ackermann and Turk (1982), the coefficient of friction and the mass of the bus can be viewed as perturbation parameters. These parameters depend on whether the road is slippery or dry and the extent to which the bus is loaded with passengers. A robust controller is one which leads to satisfaction of some given set of specifications for all admissible values of the perturbation parameters.

Given this framework, the following question is natural to ask: Does all this "Kharitonov business" lead to solutions of new and important problems or can the problems under consideration be solved using existing results? After all, over the last two decades, we have seen many new advances in robustness theory; e.g., quantitative feedback theory in the spirit of Horowitz (1963), singular value theory in the spirit of Doyle and Stein (1981), H^∞ theory as in Zames and Francis (1983).

One objective of this talk is to describe important robustness problems which
(a) do not easily fit into existing frameworks;
(b) become easy to solve using results emanating from Kharitonov's work.
Furthermore, from an engineering point of view, I feel that the new results are quite easy to use and the input requirements for a computer implementation are also readily comprehensible. Typically, the user specifies a nominal system (say using transfer functions) indicating which parameters are subject to perturbation. In addition, the user specifies upper and lower bounds on these perturbation parameters.

Granted, some of the robustness problems to be discussed can be solved using more than one technique. With this in mind, I was quite careful in choosing the title for this lecture; i.e., my plan is to expose new tools for robustness analysis without excessive concern about the existence of competing tools. At this point in time, I see nothing wrong with carrying around a toolbox which may be a little heavier than needed to get the job done. I will avoid any judgements as to which of two possible tools is better for any given robustness problem.

One final motivational comment about the new tools to be discussed: When many of us first became aware of Kharitonov's Theorem, I think there was a widespread belief that its range of applicability would turn out to be rather narrow. A most elegant result stood before but it only applied for left half plane stability problems with rather restrictive assumptions about the independence of perturbations entering into the characteristic polynomial. More recent work, however, convinces us that our intial belief was erroneous.

It turns out that the type of thinking involved in the simplest analysis can readily be adapted to deal with a wide variety of other robustness problems—multidimensional system stability (see, for example, Bose (1988)), and stability in the presence of both structured perturbations and unmodelled dynamics (see Barmish and Khargonekar (1988)), stability in the presence of structured perturbations and time delays (see Mori and Kokame (1987)) and Root Locus in the presence of structured perturbations (see Barmish and Tempo (1988)). Another interesting paper, although not directly emanating from the Kharitonov's analysis, has much of the same flavor of other work in the area. Namely, Bailey, Panzer and Gu (1988) consider transfer functions whose coefficients obtain perturbation parameters which are known only within given bounds. A novel technique is then given for finding the so-called frequency tem-

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plates in the log gain versus phase plane. That is, at each frequency ω, the template is a "cloud" on the Nichol's chart reflecting the fact that the perturbation parameters are known only within given bounds. Note that the availability of these clouds is important in the robustness theory of Horowitz (1963). The recent innovations above have added more impetus to research in this area.

Some Historical Notes and Early Work

I have often wondered how a gem such as Kharitonov's Theorem remained unheralded for more than four years. The result first came to my attention at a 1982 robustness workshop in Switzerland organized by Jurgen Ackermann. At that time, I remember sitting next to Manfred Morari listening to Andrej Olbrot present a paper on delay systems which exploited Kharitonov's Theorem. Apparently, Olbrot was aware of the importance of this result at least one year earlier. In a 1981 letter from Olbrot to Ackermann (following a workshop in Bielefeld), Kharitonov's Theorem was stated precisely and Olbrot recognized that this result had possible applications to "insensitive stabilization." The next time Kharitonov's name surfaced was in my paper (see Barmish (1983)) exposing his theorem. In that work, I demonstrated that the theorem makes it easy to compute the "maximal stability box" about a stable polynomial [1]. This generalized the result of Guiver and Bose (1983) for polynomials up to fourth order.

Kharitonov's original proof is rather cryptic and difficult to understand. Upon first reading his paper, it was obvious that the result was important but the proof provided little insight as to what was really going on. There was a clear need for demystification. I imagine that others had a similar experience because a number of papers were subsequently written providing new proofs. In my opinion, the best of these is the one by Minnichelli, Anagnost and Desoer (1988). Their proof is so simple that it can be quickly taught in an undergraduate course. In contrast, other proofs in the literature require some preliminaries about the so-called interlacing property for zeros of a polynomial; see the Hermite-Biehler Theorem in Gantmacher (1959). These interlacing ideas are not really complicated but on the other hand, they are not needed.

I believe that it is fair to say that one of the breakthrough ideas for the Minnichelli, Anagnost and Desoer proof is motivated by the work of Dasgupta (1988). In his paper, Dasgupta provides a nice geometric interpretation of the behavior of an interval polynomial as a function of frequency. Namely, at each fixed frequency, the so-called *value set* is a rectangle in the complex plane. This value set concept will be discussed in the sequel.

Statement of Kharitonov's Theorem and its Limitations

For the sake of completeness, I now paraphrase Kharitonov's Theorem: Indeed, we use $q_0, q_1, \ldots, q_n$ to represent perturbation parameters with known apriori bounds

$$q_i^- \leq q_i \leq q_i^+$$

[1]More recently, an even simpler description of the maximal stability box has been given in Fu and Barmish (1988)—a one shot type formula.

for $i = 0, 1, \ldots, n$. Next, we consider an it interval polynomial family $\mathcal{P}$ described by

$$p(s, q) = q_0 + q_1 s + q_2 s^2 + \cdots + q_n s^n,$$

where the q_i take values in the allowed ranges $[q_i^-, q_i^+]$ indicated above. The problem addressed by Kharitonov is whether all polynomials in $\mathcal{P}$ are strictly stable (all roots in the strict left half plane). His theorem provides a simple and elegant solution to this problem. Namely, one first forms the 4 polynomials

$$K_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + q_6^+ s^6 + \cdots;$$

$$K_2(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + q_6^- s^6 + \cdots;$$

$$K_3(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + q_6^- s^6 + \cdots;$$

$$K_4(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + q_6^+ s^6 + \cdots$$

Then, it follows that every polynomial in $\mathcal{P}$ is strictly stable if and only if the four $K_i(s)$ are strictly stable. This says that a solution to the problem is obtained by simply applying Routh's stability test four times—once for each $K_i(s)$. It is also interesting to note that for low order polynomials, fewer than four $K_i(s)$ are needed; see Anderson, Jury and Mansour (1987).

From a control engineering point of view, an important limitation of Kharitonov's Theorem is that independent coefficient perturbations are assumed in the coefficients of the characteristic polynomial. Said another way, no q_i enters into more than one coefficient. For the more general case when a perturbation parameter *does* enter into more than one coefficient, the theorem can often be used after "overbounding." This usually amounts to treating dependent perturbations as if they are independent and leads to conservative conclusions about stability. This limitation motivated a number of researchers to consider more general versions of the robust stability problem to allow for the fact that coefficient perturbations are dependent. The first level of generalization is discussed in the next section.

Another important limitation of Kharitonov's Theorem is that it does not apply to discrete-time systems. For discrete-time systems, it is known that the obvious conjecture does not hold. That is, stability of the four Kharitonov polynomials is not sufficient for stability of the entire family. In fact, for an n-th order interval polynomial (say monic), even stability of all 2^n extreme polynomials is not sufficient for stability of the entire family. These facts are established by counterexamples in the literature; e.g., see Bose and Zeheb (1986), Hollot and Bartlett (1986) and Cieslik (1987). Note, however, that for discrete-time interval polynomials of degree 5 or less, Kraus, Anderson and Mansour (1987) obtain a 2^n type result with the added condition that a finite set of "supplementary polynomials" are stable. In fact, their result also applies in the more general polytope setting in the section to follow.

Despite the lack of parallelism between continuous-time and discrete-time results, there are still at least three methods available for stability analysis of discrete-time interval polynomials: First, one can test for stability using the edge theory described in the section to follow. The second possibility is to replace the given "box" of polynomials by a certain type of diamond approximation involving pairs of coefficients; see Kraus, Mansour and Anderson (1988). A problem, however, is that the computational demands associated with both of these approaches becomes excessive as the degree of the polynomial increases. The recent work of Barmish (1988) alleviates this computational problem. To this end, a "scalar testing" function plays an important role; see next section for further discussion.

In concluding this section, we consider the following question: Are there other important regions besides the left half plane for which Kharitonov's Theorem holds? In Petersen (1987), an interesting class of such regions is described. These regions include many that one encounters when specifications are given on the damping ratio for the dominant poles, degree of stability, etc. Note, however, that the unit circle is not one of Petersen's regions.

Polytope Stability Problems and the Edge Theorem

To begin this section, recall that one of the main limitations associated with Kharitonov's Theorem is the assumption of independent perturbations entering into each polynomial coefficient. Before Bartlett, Hollot and Lin (1987) provided the Edge Theorem, I think that there was a widespread belief that Kharitonov's result, although beautiful, was rather restrictive and that attempts at significant generalizations would turn out to be futile. I believe that the main contribution of Bartlett, Hollot and Lin was to dramatically change this opinion. They considered the case when perturbations in the polynomial's coefficients were allowed to be linearly dependent. More precisely, consider a polynomial of the form

$$p(s,q) = s^n + \sum_{i=0}^{n-1} a_i(q)s^i$$

where the coefficients $a_i(q)$ depend affine linearly on underlying physical paramters $q_1, q_2, \ldots, q_\ell$. When each of these parameters are known only within given bounds $[q_i^-, q_i^+]$, the resulting set of polynomials turns out to be a polytope. That is, this set is the convex hull of the 2^ℓ polynomials obtained by setting q to an extreme point.

In their important paper, Bartlett, Hollot and Lin proved that a necessary and sufficient condition for strict stability of the entire family is strict stability of the exposed edges. Since exposed edges are only one dimensional, a fantastic reduction in computational complexity results. In fact, a large number of computationally tractable tests are available for checking the stability of an edge; e.g., brute force gridding, root locus generation, the eigenvalue test of Bialas (1985). Another important point to note about the Edge Theorem is that it not only applies to continuous-time systems but to discrete-time systems as well. The authors accomplish this feat by formulating the stability problem in terms of a simply connected region $\mathcal{D}$ within which the roots of the perturbed polynomial are required to lie. By considering different choices of $\mathcal{D}$, one can study a wide range of pole assignment problems. In its full generality, the Edge Theorem indicates that $\mathcal{D}$-stability of the polynomials associated with the exposed edges of the polytope is both necessary and sufficient for $\mathcal{D}$-stability of the entire polytope of polynomials.

There is, however, one "hooker" in the Edge Theorem. As pointed out in Barmish (1988), there is a combinatorial explosion in the number of exposed edges as a function of the number of perturbation parameters ℓ; an ℓ-dimensional box for the q_i parameters has $\ell 2^{\ell-1}$ exposed edges! This raises the following question: Can robust stability criteria be provided which avoids the "two at a time" combinatorics of edges? The paper by Saridereli and Kern (1987) was the first to answer this question in the affirmative[2]. In essence, their robust stability test amounts to solving a linear programming problem at each frequency ω within some bounded range. I believe considerable motivation for this approach is derived from the earlier paper by Fam and Meditch (1978).

[2] I don't believe that Saridereli and Kern were aware of the Edge Theorem at the time they wrote their paper.

In the paper by Barmish (1988), a much simpler robust stability test is given. In that paper, I argue that linear programming is entirely unnecessary and one can avoid the combinatorics of edges by working with a 2-dimensional polygon in the complex plane. This polygon is the generalization of the "Kharitonov rectangle" representing the value set of the polynomial at a fixed frequency. The net result is a stability test which involves plotting the graph of a scalar function of a scalar variable over a bounded interval. Positivity of this scalar "testing function" is both necessary and sufficient for robust stability.

More General Coefficient Dependencies on Perturbations

From a control systems point of view, the Edge Theorem is not a cureall because it is unduly restrictive to assume that the polynomial's coefficients $a_i(q)$ depend on the perturbation parameters in an affine linear manner. Therefore, this part of the lecture will concentrate on the existing machinery which is available for more complicated coefficient structures.

In my opinion, the most significant work to date (for more general coefficient dependencies on perturbations) can be attributed to a revival of two rather old ideas:

First Idea: Given a polynomial (say monic to keep the discussion simple) $p(s,q)$ with continuous dependence of the coefficients $a_i(q)$ on the perturbation vector q, suppose that a closed and bounded set $Q \subset \mathbf{R}^\ell$ is specified for admissible values of q and assume that $p(s,q^*)$ is strictly stable for at least one $q^* \in Q$. Then strict stability is guaranteed for all $q \in Q$ if and only if, for each frequency $\omega \in \mathbf{R}$, the following "zero exclusion condition" is satisfied:

$$0 \notin p(j\omega, Q) \doteq \{p(j\omega, q) : q \in Q\}.$$

Notice that $p(j\omega, Q)$ is the value set which we discussed in the earlier exposition. The definition above indicates that this set is in the complex plane and obtained by holding $s = j\omega$ fixed and evaluating the polynomial for all possible $q \in Q$. Hence, when Q is a box and the polynomial coefficients $a_i(q)$ depend on q in a "nice" way, the value set is easy to construct and we have a computationally tractable stability test which readily lends itself to graphic display. That is, we display the motion of the value set as a function of the frequency ω and look to see whether the origin, $s = 0$, enters this set. This rather old Nyquist-like idea is discussed in the textbook of Zadeh and Desoer (1963) and was well known to researchers at Bell Labs in the early fifties.[3] If one digs even deeper into the literature, we find that this idea was used in a robustness context more than 20 years earlier; see Frazer and Duncan (1929)[4].

In view of the discussion above, the robust stability problem (for more complicated coefficient dependencies) boils down to the following question: Are there important classes of coefficient dependencies for which the value set is easy to generate? Recalling the earlier discussion, we already know that the value set is a rectangle for an interval polynomial and a polygon for

[3] This fact was communicated to me in a personal correspondence by Professor Charles Desoer.

[4] I was not aware of this reference until quite recently when Jurgen Ackermann brought it to my attention.

a polytope of polynomials. The question is: Can we generate the value set for other significant perturbation classes?

One naive approach to value set generation involves gridding the bounding set Q and simply evaluating the polynomial $p(s, q)$ at each grid point. The obvious flaw in this approach, however, is that the number of grid points explodes with the dimension ℓ of q. Furthermore, this gridding has to be carried out at each frequency. Hence, this idea is only practical for a small number of parameters. In the second idea below, we see that there is an important class of coefficient dependencies for which a convex hull approximation to the value set is readily available.

Second Idea: When Q is a box and the polynomial coefficients $a_i(q)$ are multilinear in q, it follows that the convex hull of the value set is a polygon—generated by applying $p(j\omega, \cdot)$ to the vertices of the box. This fact is a consequence of the so-called Mapping Theorem; e.g., see Zadeh and Desoer (1963). Hence, for this class of multilinear coefficient perturbations, For the robust stability problem with multilinear dependence of this result (in combination with the first idea above) proves to be quite useful. Namely, one generates a convex hull approximation to the value set at each frequency and then the zero exclusion condition becomes easy to check. The price paid for this computational tractability, however, is conservatism in the analysis. That is, zero exclusion from the convex hull of the value set is sufficient but not necessary for stability.

I personally credit the revival of the two ideas above to Saeki (1986). The conservatism in Saeki's analysis paved the way for an interesting paper by de Gaston and Safonov (1986). These authors also deal with multilinear $a_i(q)$ and in essence, the objective in their paper is to eliminate conservatism in the robustness analysis by refining the convex hull approximation to the value set. To this end, the authors develop a "domain splitting" algorithm which involves iteratively slicing the Q box; for an extension of this technique to the more general case of polynomic dependence of coefficients on perturbations, see Sideris and de Gaston (1986). Perhaps the main drawback of this "iterative approach" is that computation times can become excessive for more than a few perturbation parameters. In this regard, the domain splitting algorithm must be executed at every frequency. In practice, this means that one must execute the algorithm a large number of times over some "closely spaced" frequency grid.

I believe that the issues of computational tractability associated with the de Gaston-Safonov approach provided the motivation for the the paper by Sideris and Pena (1988). With the goal of making computational requirements realistic, Sideris and Pena develop a new version of the domain splitting algorithm which avoids the frequency sweep. They indicate that computational times are quite reasonable for a large test bed of problems—many randomly generated.

In my opinion, the emergence of the Sideris-Pena algorithm brings us to an important juncture in the study of robust stability. On one hand, there are a number of researchers who continue to work on the case when the coefficients depend multilinearly and polynomially on the perturbations. On the other hand, we have available the iterative algorithm of Sideris and Pena for precisely this class of problems. Hence, my question: What is driving these researchers if the problem is supposedly solved? I will provide two plausible explanations.

My first explanation is that given all the excitement resulting from Kharitonov and its aftermath, researchers refuse to believe that one cannot find a stability test which is more elegant than an iterative algorithm—the ultimate is a simply stated analytical criterion in the spirit of Kharitonov's Theorem or perhaps the Edge Theorem. In addition to this issue of elegance and aesthetics, I think that many researchers believe that a more analytical test would be helpful as far as design is concerned; i.e., the iterative algorithm obfuscates the effect of the controller on stability.

My second possible explanation: It is my perception that some researchers are concerned that under intense scrutiny, the domain splitting algorithm may not prove to be computationally tractable for more than a few parameters. Such concerns could easily be dispelled if a code were widely available so that an array of benchmark problems could be tried. The issue here is not whether the code is "optimal." The question is whether the algorithm works on reasonable problems involving more than a few parameters. If the algorithm works well for (say 80% of) some "reasonable collection" of benchmark problems, then I believe that this would force researchers to re-evaluate the directions in which they are working. I am aware of several research efforts aimed at developing "alternative" iterative algorithms. Personally, if the Sideris-Pena algorithm truly works, then what's the main point of these other research efforts?

Dressing up Polynomial Results in Control Clothing

Many people working in this area take it for granted that the results they develop at the level of polynomials have obvious control theoretic interpretations. To some degree, however, what is being overlooked is that the polynomials generated for a closed loop system often have special structure which can be exploited to simplify the bookkeeping involved in the robustness test. In this regard, I feel that the papers by Biernacki, Hwang and Bhattacharyya (1987), Keel Bhattacaryya and Howze (1988) and the book by Bhattacharyya (1987) are of considerable interest. The reader should be forewarned, however, that in many cases, these authors consider a framework is somewhat different from the one considered here—the perturbation parameters are often restrained to a sphere rather than a box. In my opinion, researchers should not make a "big deal" about this distinction because in practice, there is usually a degree of arbitrariness in shaping the bounding set for the perturbations. If the spherical formulation ultimately leads to a richer theory, then so be it.

To conclude this section, I feel that it is important to mention two driving forces behind the recent developments for spherically bounded uncertainties: In my opinion, the paper by Soh, Berger and Dabke (1985) provided critical motivation for Bhattacharyya's work. In turn, the paper by Fam and Meditch (1978) made Soh, Berger and Dabke's work possible.

Conclusion: A Brief Mention of Closely Related Research

Over the last decade, there have also been a number of advances on "matrix versions" of the polynomial problems discussed thus far. First, I mention the so-called *interval matrix stability problem*: Given a matrix A whose entries a_{ij} are known only within bounds $[a_{ij}^-, a_{ij}^+]$, determine if strict stability is guaranteed over the entire range of parameter variations. There are two key points to note about this problem: First, the interval matrix stability problem is a special case of the polynomial problem with multilinear $a_i(q)$. That is, $det\ (sI - A)$ has coefficients which depend multilinearly on the a_{ij}. It may turn out, however, that the interval matrix problem is easier to solve than the more general polynomial problem because of the special structure of $det\ (sI - A)$. Second, in my opinion, results to date on the interval matrix stability problem are not strong. With computational tractability as a prerequisite, only restrictive sufficient conditions are available. To conclude the discussion of interval matrices, I note that the obvious conjectures motivated by polynomial theory are false; see Karl, Greschak and Verghese (1984), Barmish and Hollot (1984) and Barmish, Fu and Saleh (1988).

The second matrix problem which I mention is an ℓ^2 version of the matrix stability problem: Suppose that A is a strictly stable matrix and consider the standard norm

$$||A|| \doteq \lambda_{max}[A^T A]^{1/2}.$$

Then the problem is to find the largest number $\beta > 0$ such that

$$A + \Delta A$$

is strictly stable for all ΔA satisfying

$$||\Delta A|| < \beta.$$

For the case when the perturbation matrix ΔA is allowed to be complex, a simple solution of the problem is in provided in Qin and Davison (1986), Hinrichsen and Pritchard (1986) and Hewer and Martin (1987). Namely, the largest β is given by

$$\beta_{max} = \frac{1}{||(sI - A)^{-1}||_\infty}$$

where

$$||(sI - A)^{-1}|| = sup\{||(j\omega - A)^{-1}|| : \omega \in \mathbf{R}\}$$

is the H^∞ norm. Finally, I note that the references cited above also include extensions of this result to allow for some structuring of the perturbations. From a control systems point of view, an important open problem is to find β_{max} with the added restriction that ΔA is real. Hinrichsen and Pritchard (1988) continue work on this important problem.

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