LEAST NORM SOLUTION OF NON-MONOTONE LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. We show that a least 2-norm solution of a general linear complementarity problem (LCP) can be obtained by solving a sequence of perturbed quadratic programs. The norm of the solutions of the perturbed problems approaches monotonically from below the norm of a least 2-norm solution of the LCP, as the perturbation parameter approaches zero. For sufficiently large value of the perturbation, the quadratic program is strongly convex and easily solved by Lemke's method. This yields a guaranteed lower bound to the norm of a least 2-norm solution. For some LCP's, even non-monotone ones, the perturbed quadratic program may give a solution to the LCP for a finite value of the perturbation parameter. For monotone LCP's the perturbed quadratic program yields the least 2-norm solution for a finite value of the perturbation parameter if and only if the least 2-norm solution of the linearized LCP equals that of the original LCP.

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We consider the fundamental linear complementarity problem [4]

$$Mx + q \geq 0,\ x \geq 0,\ x(Mx+q) = 0 \tag{1}$$

where M is an arbitrary $n \times n$ real matrix and q is in the real n-space R^n. Since M is arbitrary it may not be copositive plus [4] nor positive semidefinite and hence the affine function $Mx+q$ need not be monotone. Consequently, fundamental algorithms for solving (1) such as Lemke's algorithm [4] fail in general and the problem is known to be NP-complete [3]. Obviously there is no simple way of solving the problem then. When the problem is solvable we propose the solution of a sequence of decreasingly perturbed quadratic programs (see (3) below) the solution of which have an accumulation point which is a least 2-norm solution of (l). When the perturbation parameter ε exceeds the absolute value of the most negative eigenvalue of $(M+M^T)$, the perturbed quadratic program (2) becomes strongly convex and the norm of its unique solution gives a lower bound to the norm of a least 2-norm solution of (1). If this strongly convex quadratic gives a complementary solution, that is $x(Mx+q) = 0$, then it is a least 2-norm solution to the non-monotonic linear complementarity problem (see Example 2 and Corollary 3 below). However in general the perturbed quadratic programs are nonconvex as the perturbation parameter ε tends to zero. When M is positive semidefinite the perturbed quadratic programs are always strongly convex. For such a case the least 2-norm solution of (1) is obtained by solving (2) for a finite ε if and only if the linearization of the linear complementarity problem (1) around any solution point of (1) has the same least 2-norm solution as the original problem (1) (see Theorem 5 below). Our first result (Theorem 1) gives useful monotonic properties of solutions of the perturbed quadratic programs (3) for a non-monotonic linear complementarity problem. Throughout $\|x\|$ will denote the 2-norm $(xx)^{1/2}$.

We begin our analysis by considering the following perturbation of (1) for $\varepsilon > 0$

$$\min\ x(Mx+q) + \frac{\varepsilon}{2}xx \quad \text{subject to} \quad Mx+q \geq 0,\ x \geq 0 \tag{2}$$

When $\varepsilon = 0$ and the minimum of (2) is zero, we obtain an exact solution to (1). For a sequence of decreasing positive numbers $\{\varepsilon_i\}$ converging to zero, the subproblems

$$\text{(3)} \qquad \min\ x(Mx+q) + \frac{\varepsilon_i}{2}\,xx \quad \text{s.t.} \quad Mx+q \geq 0,\ x \geq 0$$

can be considered subproblems of an exterior penalty function method for the following problem of determining the least 2-norm solution of (1)

$$\text{(4)} \qquad \min\ \frac{1}{2}\,xx \quad \text{s.t.} \quad Mx+q \geq 0,\ x \geq 0,\ x(Mx+q) \leq 0$$

By using standard results of exterior penalty functions such as given in [7] we can state the following important properties of solutions x^i of (3).

1. Theorem Let $\{\varepsilon_i\} \downarrow 0$, let $\{x^i\}$ be a corresponding sequence of solutions of (3) and let $\bar{x}$ be a solution of (1) with least 2-norm. Then

(a) $0 \leq x^{i+1}(Mx^{i+1}+q) \leq x^i(Mx^i+q)$

(b) $\|x^{i+1}\| \geq \|x^i\|$

(c) $\|x^i\| \leq \left(\|\bar{x}\|^2 - \frac{2}{\varepsilon_i}\,x^i(Mx^i+q)\right)^{1/2} \leq \|\bar{x}\|$

(d) If $x^i(Mx^i+q) = 0$ then x^i is a least 2-norm solution of (1).

(e) $\lim\limits_{i\to\infty} x^i(Mx^i+q) = 0$

(f) $\lim\limits_{i\to\infty} \|x_i\| = \|\bar{x}\|$

(g) Each accumulation point $\hat{x}$ of the bounded sequence $\{x^i\}$ is a least 2-norm solution of (1). If the least 2-norm solution $\bar{x}$ of (1) is unique, then $\{x^i\}$ converges to $\bar{x}$ and

$$\lim_{i\to\infty} \frac{x^i(Mx^i+q)}{\varepsilon_i} = 0$$

Proof Consider the exterior penalty problem

$$\min_{x\varepsilon X} P(x, \frac{1}{\varepsilon}) := \min_{x\varepsilon X} \frac{1}{2}\,xx + \frac{1}{\varepsilon}\,x(Mx+q)$$

associated with (4) where

$$X := \{x | Mx + q \geq 0,\ x \geq 0\}$$

Note that the penalty term $x(Mx+q)$ is nonnegative on X and is zero on the feasible region of the least 2-norm problem (4). The assertions of this theorem are then direct consequences of results of exterior penalty functions, for example as given in [7]. Thus (a) and (b) above are a consequence of Proposition 2.1 [7]. Parts (c) and (d) are a consequence of Proposition 2.2 [7] and its proof which give $P(x^i, \frac{1}{\varepsilon^i}) \leq \frac{1}{2}\,\bar{x}\bar{x}$. Parts (e) and (f) are a consequence of Theorems 2.5 and 2.8 [7]. Part (g) is a consequence of Theorem 2.8 [7] and the fact that a bounded sequence in R^n all of whose accumulation points are the same, converges [2]. ∎

When M is positive semidefinite the subproblems (3) are strongly convex quadratic programs and are easily solvable by standard methods such as Lemke's or other methods [4]. The subproblems (3) are different from those of the Tihonov regularization proposed in [11] in which the following subproblems are solved for $\{\varepsilon_i\} \downarrow 0$

$$(M + \varepsilon_i I)x + q \geq 0,\ x \geq 0,\ x\big((M + \varepsilon_i I)x + q\big) = 0 \tag{5}$$

The approach proposed here is probably more robust because our results as stated in Theorem 1 do not require any monotonicity as is the case in [11]. Furthermore our approach may give a solution to a non-monotone linear complementarity problem for a finite nonzero ε as demonstrated by the following example which was generated such that $M + M^T$ is not positive semidefinite and such that $\tilde{x} = (0\ \ 1\ \ 1)$ is a solution of (1).

2. Example

$$M = \begin{pmatrix} -5 & 1 & 1 \\ 3 & 0 & 2 \\ -9 & 1 & -7 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix}$$

Eigenvalues $(M + M^T) = (-21.3426,\ -4.0640,\ 1.4066)$. An initial value of ε was taken as $\varepsilon_0 = 22$ in order to make problem (3) strongly convex. This gave

$$x(\varepsilon_0) = (0.125 \quad 0.8125 \quad 0.8125)$$

$$Mx(\varepsilon_0) + q = (0 \quad 0 \quad 0)$$

$$\|x(\varepsilon_0)\| = 1.1558 < \sqrt{2} = \|\tilde{x}\|$$

Hence we have obtained an exact solution $x(\varepsilon_0)$ to the non-monotone (and hence nonconvex) complementarity problem (1) by solving a simple strongly convex quadratic program! Moreover, by Theorem 1(d), $x(\varepsilon_0)$ is a least 2-norm solution of the linear complementarity problem (1). If we take $\varepsilon = 0$, Lemke's method fails to solve the resulting nonconvex quadratic program (2) for this example. Also Lemke's method applied directly to the non-monotone complementarity problem (1) for this example also fails. It is worthwhile to formalize part of these results as follows.

3. Corollary For $\varepsilon_0 > \frac{|\lambda|-\lambda}{2}$, where λ is the smallest eigenvalue of $M + M^T$, the quadratic program (2) is strongly convex and has a unique solution $x(\varepsilon_0)$. If (1) is solvable then $\|x(\varepsilon_0)\| \leq \|\bar{x}\|$ where $\bar{x}$ is a least 2-norm solution of (1). If $x(\varepsilon_0)(Mx(\varepsilon_0) + q) = 0$, then $x(\varepsilon_0)$ is a least 2-norm solution of (1).

When M is positive semidefinite and $X \neq \phi$, the convex program (2) has a unique solution $x(\varepsilon)$ for each $\varepsilon > 0$ and by Theorem 1 $\lim_{\varepsilon \to 0} x(\varepsilon) = \bar{x}$, where $\bar{x}$ the unique least 2-norm solution of (1). For many randomly generated test cases for which M is positive semidefinite it turned out that $\bar{x} = x(\varepsilon)$ for $0 < \varepsilon \leq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. One would have expected that this is also true in general for all positive semidefinite M. Unfortunately, this is not the case as evidenced by the following simple counterexample provided by the author's colleague, Professor T.-H. Shiau [10].

4. Example

$$M = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The unique solution of (1) is $\bar{x} = (0 \;\; 1)$. For $\varepsilon > 0$, the unique solution $x(\varepsilon)$ to (2) is given by

$$x_1(\varepsilon) = \frac{\varepsilon}{2(3+\varepsilon)}, \quad x_2(\varepsilon) = \frac{6+\varepsilon}{2(3+\varepsilon)}$$

We note that $\lim_{\varepsilon \to 0} x(\varepsilon) = \bar{x}$, but that $x(\varepsilon) \neq \bar{x}$ for $\varepsilon > 0$.

We can give a necessary and sufficient condition under which problem (2) gives a least 2-norm solution of the linear complementarity problem (1) for a finite value ε when M is positive semidefinite.

5. Theorem Statement (a) below implies statement (b):

(a) For all $\varepsilon \in (0, \bar{\varepsilon}]$ for some $\bar{\varepsilon} > 0$, the quadratic program (2) has fixed solution $\bar{x}$ independent of ε.

(b) $\bar{x}$ is a least 2-norm solution of the linear complementarity (1).

If in addition M is positive semidefinite and $\tilde{x}$ is any solution of the linear complementarity problem (1) then statement (a) and the following are equivalent:

(c) The least 2-norm solution $\bar{x}$ of the linear complementarity problem (1) is also the least 2-norm solution of the linearized complementarity problem around $\tilde{x}$:

$$Mx + q \geq 0,\ x \geq 0,\ \left(\tilde{x}(M + M^T) + q\right)(x - \tilde{x}) = 0 \tag{6}$$

Proof (a) $\Rightarrow$ (b): By Theorem 1(g).

(c) $\Rightarrow$ (a): By [1] or [9] the solution set of (1) is polyhedral and the linearization (6) around any solution $\tilde{x}$ of (1) is equivalent to a linearization around $\bar{x}$, the least 2-norm solution $\bar{x}$ of (1). Since by the minimum principle [5, Theorem 9.3.3]

$$\left(\bar{x}(M + M^T) + q\right)(x - \bar{x}) \geq 0 \ \text{ for all } \ x \ \text{ in } \ X$$

the linearization (6) is equivalent to the linear program

$$\min_{x \varepsilon X} \left(\bar{x}(M + M^T) + q\right)(x - \bar{x}) = 0 \tag{7}$$

It follows by [6, Theorem 2.1a(i)] or [8, Theorem 1], that there exists an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, the solution of

$$\min_{x \varepsilon X} \left(\bar{x}(M + M^T) + q\right)(x - \bar{x}) + \frac{\varepsilon}{2} xx \tag{8}$$

is the least 2-norm solution of (7) and hence that of (6), and by assumption is also the least 2-norm solution of (1), $\bar{x}$. Hence $\bar{x}$ and some $\bar{u}(\varepsilon) \in R^n$ satisfy the Karush-Kuhn-Tucker conditions of (8), for $\varepsilon \in (0, \bar{\varepsilon}]$. But these are precisely the Karush-Kuhn-Tucker conditions for (2), and since (2) has a unique solution for each $\varepsilon > 0$, it follows that $\bar{x}$ solves (2) for $\varepsilon \in (0, \bar{\varepsilon}]$.

(a) $\Rightarrow$ (c): Since (a) $\Rightarrow$ (b), we know that $\bar{x}$ is the least 2-norm solution of the linear complementarity problem (1). By (a), $\bar{x}$ is a solution of (2) for $\varepsilon \in (0, \bar{\varepsilon}]$, hence $\bar{x}$ and some $\bar{u}(\varepsilon) \in R^n$ satisfy its Karush-Kuhn-Tucker conditions, which as noted above are the Karush-Kuhn-Tucker conditions for $\bar{x}$ to solve (8) for $\varepsilon \in (0, \bar{\varepsilon}]$. Hence $\bar{x}$ solves (8) for $\varepsilon \in (0, \bar{\varepsilon}]$ and hence by [6, Theorem 2.1a(ii)], $\bar{x}$ is a least 2-norm solution of (7) or equivalently of (6). ∎

We note that as asserted by Theorem 5, Example 2 violates the linearization condition (c) because there is no $\bar{\varepsilon}$ such that for $\varepsilon \in (0, \bar{\varepsilon}]$, $\bar{x}$ solves the quadratic program (2).

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