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An Approach to Robust Stability of Matrix Polytopes Through Copositive Homogeneous Polynomials

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***Abstract*—This technical note proposes an approach to guaranteeing stability in a polytope of matrices. Previous results in the literature have made the obvious connection between various robust stability problems and test for positivity of a multivariable polynomial. This note extends such results to demonstrate that all matrices within a polytope are stable if and only if an associated *homogeneous* polynomial is strictly "copositive." The additional structure obtained by exploiting homogeneity of this multivariable polynomial leads to several computationally tractable sufficiency tests for establishing either robust stability or instability of a polytope of matrices.**

I. Introduction

The study of families of linear time-invariant systems obtained when the system model contains uncertain parameters has given rise to a considerable literature under the title of robust stability. Research in this area has generally sought computationally efficient schemes for guaranteeing stability (or "desirable" locations) for roots of characteristic polynomials or eigenvalues of matrices when either vary over a family. A naive approach to a robust stability test would simply invoke repeated root or eigenvalue computations over a "sufficiently fine" grid of points within the family. However, the cost of such a computation grows exponentially with problem size. A practical scheme must have a cost of computation that grows much more slowly with problem size. For polynomial problems, [1] showed that for a certain structure of polynomial family (interval polynomials), left half plane root locations for the whole family can be decided by root computations on a fixed number of polynomials, independent of the number of uncertain parameters. When the uncertain polynomial possesses a polytope structure, [2] showed that stability of the family of polynomials is guaranteed by stability of all exposed edges.[1] While such results have proven valuable for certain types of system models, variation in physical parameters often does not yield a model whose characteristic polynomial has coefficients varying within a polytope. To allow treatment of a wider class of system models, one may assume that variations in physical parameters yield a state-space normal form in which the family of system matrices forms a polytope. This model structure will be the focus of this note.

For $n \times n$ matrix families, [3] established that the conjecture arising from extension of the results of [2] to a polytope of matrices is false, i.e., checking the exposed edges of a polytope of matrices is not sufficient to guarantee the stability of the entire family. Instead, [4] demonstrates that stability of all faces with dimension lower than $(2n-4)$ is not in general sufficient to guarantee robust stability of a matrix polytope, while stability of all $(2n-4)$-dimensional faces does guarantee robust stability. Restricting the structure to interval matrices (i.e., a "rectangular" polytope) reduces the dimension of the faces to be tested to n [5]. Clearly, none of these results decreases the dimensionality of the stability tests enough to make them easily computed in reasonably sized systems.

In this technical note, we expose an additional structure in the study of robust stability for a polytope of matrices that may serve as a step towards reducing computational complexity. Many results in the literature have shown equivalence of various robust stability problems and positivity of a multivariable polynomial, see, for example, [6]. We show that a necessary and sufficient condition for a matrix polytope to be robustly stable is that an associated *homogeneous* polynomial be strictly "copositive." This additional structure (in particular, homogeneity of the associated polynomial) provides several useful sufficient conditions for testing a polytope of matrices to be either robustly stable or unstable.

II. Preliminaries

A. The Robust Stability Problem

Consider a polytope of matrices represented as

$$\mathcal{A} = \text{convex hull of } \{A_1, A_2, \cdots, A_l\} = \left\{ A(\lambda) = \sum_{k=1}^{l} \lambda_k A_k : \lambda \in \Gamma \right\} \quad (1)$$

where

$$A_k \in \boldsymbol{R}^{n \times n}, \quad k = 1, 2, \cdots, l; \quad \text{and}$$

[1] Note, however, that the number of exposed edges may grow exponentially with the number of uncertain parameters.

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$$\Gamma = \left\{ \lambda : \lambda \in \boldsymbol{R}^l,\ \lambda_k \geq 0,\ k = 1, 2, \cdots, l,\ \sum_{k=1}^{l} \lambda_k = 1 \right\}.$$

The polytope matrix family $\mathscr{A}$ is said to be *robustly stable* if $A(\lambda)$ is stable (i.e., has all eigenvalues in the open left half plane) for all $\lambda \in \Gamma$; if there exists any $\hat{\lambda} \in \Gamma$ such that $A(\hat{\lambda})$ is unstable, the family $\mathscr{A}$ is said to be unstable. Suppose that $\mathscr{A}$ has at least one stable matrix in the family $A(\lambda^s)$, where $\lambda^s \in \Gamma$. Given one stable matrix in the polytope, an obvious necessary and sufficient condition for $\mathscr{A}$ to be robustly stable is that $A(\lambda)$ has no pure imaginary eigenvalues for any $\lambda \in \Gamma$. Without loss of generality, we assume in the sequel that $\mathscr{A}$ always has at least one stable matrix in the family.

B. The Robust Nonsingularity Problem and Transformation Methods

A number of authors (see, for example, [11]) have observed that a useful step in robust stability analysis for a family of matrices is to transform the original *stability* problem into an equivalent *nonsingularity* problem. For a polytope matrix family, one creates an associated polytope of transformed matrices with the property that the original polytope is robustly stable if and only if the polytope of transformed matrices is "robustly nonsingular." One of the most commonly used transformation methods employs Kronecker sum operations ($\oplus$) [7], [8].

Let $\boldsymbol{A} = [a_{ij}] \in \boldsymbol{R}^{m \times n}$, $\boldsymbol{B} = [b_{ij}] \in \boldsymbol{R}^{p \times q}$. The Kronecker product of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$, is defined as

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B} \\ \cdots & & \cdots \\ a_{n1}\boldsymbol{B} & \cdots & a_{nn}\boldsymbol{B} \end{bmatrix} \in \boldsymbol{R}^{mp \times nq}.$$

If $m = n$ and $p = q$, the Kronecker sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A} \oplus \boldsymbol{B}$, is defined as

$$\boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{A} \otimes \boldsymbol{I}_p + \boldsymbol{I}_m \otimes \boldsymbol{B} \in \boldsymbol{R}^{mp \times mp}. \tag{2}$$

Let $\mu_i(\boldsymbol{A})$, $i = 1, 2, \cdots, m$, denote the (not necessarily distinct) eigenvalues of $\boldsymbol{A}$. The spectrum set of the transformed matrix, denoted by $sp(\boldsymbol{A} \oplus \boldsymbol{B})$, is then given by

$$sp(\boldsymbol{A} \oplus \boldsymbol{B}) = \{\mu_i(\boldsymbol{A}) + \mu_j(\boldsymbol{B}),\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, p\}. \tag{3}$$

Let $A(\lambda) \in \mathscr{A}$ and construct the Kronecker sum of $A(\lambda)$ with itself. Then

$$\tilde{A}(\lambda) := A(\lambda) \oplus A(\lambda) = \sum_{k=1}^{l} \lambda_k \tilde{A}_k$$

where $\tilde{A}_k := A_k \oplus A_k$, $k = 1, 2, \cdots, l$. Define $\tilde{\mathscr{A}} = \{\tilde{A}(\lambda): \lambda \in \Gamma\}$. The set $\tilde{\mathscr{A}}$ is a polytope of sparse matrices with dimension $n^2 \times n^2$. It follows from (3) that $A(\lambda)$ has no imaginary eigenvalues if and only if $\tilde{A}(\lambda)$ remains nonsingular. Therefore, $\mathscr{A}$ is robustly stable if and only if the new polytope matrix family $\tilde{\mathscr{A}}$ is robustly nonsingular. There exist several other transformation methods that also allow examination of the robust stability problem through a robust nonsingularity problem. These include power transformations (Schläflian forms) [9]–[11], and bialternate product operations [8], [11].

C. Determinants for a Matrix Polytope

From the discussions above, the original robust stability problem for a polytope of matrices may be treated as a robust nonsingularity problem for an associated polytope of transformed matrices.

Lemma 1: Suppose $\tilde{\mathscr{A}}$ is a polytope of matrices and $\tilde{A}(\lambda^s) \in \tilde{\mathscr{A}}$ is nonsingular. Then $\tilde{\mathscr{A}}$ is robustly nonsingular if and only if

$$\gamma \det \tilde{A}(\lambda) > 0 \qquad \text{for all } \lambda \in \tilde{\Gamma} \tag{4}$$

where

$$\gamma = \operatorname{sign}\left(\det \tilde{A}(\lambda^s)\right)$$

and

$$\tilde{\Gamma} = \{\lambda: \lambda \in \boldsymbol{R}^l,\ \lambda_k \geq 0,\ k = 1, 2, \cdots, l, \text{ and } \lambda \neq 0\}.$$

Proof of Lemma 1: For any $\lambda \in \Gamma$, $\tilde{A}(\lambda)$ is nonsingular if and only if $\tilde{A}(\alpha\lambda)$ remains nonsingular for any $\alpha > 0$, and $\alpha\lambda$ remains an element of $\tilde{\Gamma}$. With this simple observation, the remainder of the proof follows from the discussion in Section II-B. ■

Let N denote the dimension of $\tilde{A}(\lambda)$. By definition

$$\det \tilde{A}(\lambda) = \sum_{(j_1, \cdots, j_N)} s(j_1, \cdots, j_N) \tilde{a}_{1j_1}(\lambda) \tilde{a}_{2j_2}(\lambda) \cdots \tilde{a}_{Nj_N}(\lambda) \tag{5}$$

where

$$s(j_1, \cdots, j_N) := \operatorname{sign} \prod_{1 \leq p < q \leq N} (j_q - j_p)$$

and

$$\tilde{a}_{ij}(\lambda) = \sum_{k=1}^{l} \lambda_k \tilde{a}_{ij}^k, \qquad i, j = 1, 2, \cdots, N$$

is the (i, j)th element of $\tilde{A}(\lambda)$. Note that the summation in (5) is taken over all permutations of N integers, denoted by $(j_1, \cdots, j_N)$. Each $\tilde{a}_{ij}(\lambda)$ is a linear combination of λ_k's, so the right-hand side of (5) expands to an Nth-order homogeneous polynomial

$$\det \tilde{A}(\lambda) = \sum_{i_1 + \cdots + i_l = N} a_{i_1 i_2 \cdots i_l} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_l^{i_l}. \tag{6a}$$

The coefficients $a_{i_1 i_2 \cdots i_l}$ in (6a) have the form

$$a_{i_1 i_2 \cdots i_l} = \sum_{c(i_1 i_2 \cdots i_l)} \det \tilde{A}_{c(i_1 i_2 \cdots i_l)} \tag{6b}$$

where $\tilde{A}_{c(i_1 i_2 \cdots i_l)}$ is a square matrix constructed by selecting i_1 columns from $\tilde{A}_1$, i_2 columns from $\tilde{A}_2$, and so on. The position of each column in $\tilde{A}_{c(i_1 i_2 \cdots i_l)}$ is the same as that in the original matrix. The subscript $c(i_1, i_2, \cdots, i_l)$ denotes one such column selection, and the summation is taken over all such possible selections. This approach does therefore exhibit a cost of computation with potentially combinatorial growth. As a simple example, suppose $l = 2$, $\tilde{A}_1 = [a_1^1 a_2^1]$ and $\tilde{A}_2 = [a_1^2 a_2^2] \in \boldsymbol{R}^{2 \times 2}$. Then we have

$$\det \tilde{A}(\lambda) = a_{20}\lambda_1^2 + a_{11}\lambda_1\lambda_2 + a_{02}\lambda_2^2$$

where

$$a_{20} = \det \tilde{A}_1,\ a_{11} = \det\left[a_1^1 a_2^2\right] + \det\left[a_1^2 a_2^1\right], \text{ and } a_{02} = \det \tilde{A}_2.$$

D. Copositive Homogeneous Polynomials

Suppose $p(\lambda)$ is a homogeneous multivariable polynomial of order N and dimension (number of variables) l, i.e.,

$$p(\lambda) = p(\lambda_1, \lambda_2, \cdots, \lambda_l) = \sum_{i_1 + \cdots + i_l = N} a_{i_1 i_2 \cdots i_l} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_l^{i_l}.$$

The homogeneous polynomial $p(\lambda)$ is said to be strictly copositive if $p(\lambda) > 0$ for all $\lambda \in \tilde{\Gamma}$. This terminology is adopted from that of copositive matrices [13], where a matrix $\boldsymbol{Q}$ is said to be strictly copositive if $\lambda^T \boldsymbol{Q} \lambda > 0$ for all $\lambda \in \tilde{\Gamma}$. Note that the quadratic form $\lambda^T \boldsymbol{Q} \lambda$ is an order-two homogeneous polynomial. Condition (4) of Lemma 1 can then be restated as a requirement that $p(\lambda) = \gamma \det \tilde{A}(\lambda)$ be strictly copositive.

We say $p(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k})$ is a k-dimensional $(1 \leq k \leq l)$ sub-homogeneous polynomial of $p(\lambda)$ if $p(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k})$ is obtained by setting to zero the remaining l-k components of λ in $p(\lambda)$. The following lemma summarizes some observations on copositive polynomials.

Lemma 2: Suppose $p(\lambda)$ is a homogeneous polynomial of order N. Then:

i) $p(\lambda)$ is strictly copositive if and only if all its subhomogeneous polynomials are strictly copositive;

ii) if $p(\lambda)$ is strictly copositive, then the coefficients of λ_k^N, $k = 1,\cdots,l$, are greater than zero, and the sum of all coefficients in any subhomogeneous polynomial of $p(\lambda)$ is greater than zero;

iii) if the coefficients of λ_k^N's are positive and the rest of the coefficients are nonnegative, then $p(\lambda)$ is strictly copositive.

In Lemma 2, the first property provides an equivalent condition for $p(\lambda)$ being strictly copositive, but does not reduce the computational cost of the test. In general, tests for positivity of a multivariable polynomial have computational cost that grows exponentially with problem size [6]. As a step towards providing tests of lower computational cost, ii) lists two more easily tested necessary conditions for a homogeneous polynomial to be strictly copositive. Failure to satisfy these conditions will yield a sufficient condition for instability of an associated polytope of matrices. Property iii) states a simple sufficient condition for a homogeneous polynomial to be strictly copositive, and will yield a test for stability.

III. Main Results

The first theorem is an immediate consequence of Lemma 1 and the equivalence of the stability problem and the robust nonsingularity problem.

Theorem 1: A polytope of matrices $\mathscr{A}$ is robustly stable if and only if $\gamma \det \tilde{A}(\lambda)$ strictly copositive, where $\tilde{A}(\lambda)$ is the transformed matrix function associated with $A(\lambda) \in \mathscr{A}$.

The next theorem follows from Theorem 1 and properties ii) and iii) of Lemma 2.

Theorem 2: Suppose $\mathscr{A}$ is a polytope family of matrices. Then:

i) $\mathscr{A}$ is robustly stable if $\gamma \det \tilde{A}_i > 0$, $i = 1, 2, \cdots, N$, and the remaining coefficients of $\gamma \det \tilde{A}(\lambda)$ are nonnegative;

iii) $\mathscr{A}$ is unstable if there exits at least one subhomogeneous polynomial of $\gamma \det \tilde{A}(\lambda)$ such that the sum of its coefficients is less than or equal to zero.

In Theorem 1, a necessary and sufficient condition for $\mathscr{A}$ being robustly stable is given in terms of strict copositivity of $\gamma \det \tilde{A}(\lambda)$. The closed form expression for $\det \tilde{A}(\lambda)$ shown in (6) facilitates tests for $\gamma \det \tilde{A}(\lambda)$ copositive. Hence, Theorem 1 provides an alternative approach to test if a polytope matrix family is robustly stable. In Theorem 2, sufficient conditions for testing either stability or instability of a polytope of matrices are provided. Symbolic algebra software packages can assist in obtaining the desired expressions (*Mathematica* [15] is employed for the examples in this note).

Next, we consider the simple symmetric matrix problem, where well-known results [16], [17] show that a polytope of symmetric matrices is robustly stable if and only if the extreme matrices are stable. The following theorem provides an alternate viewpoint on this standard result.

Theorem 3: Suppose $\mathscr{A}$ is a polytope of symmetric matrices; i.e., $\mathscr{A} =$ convex hull $\{A_1, A_2, \cdots, A_l\}$ where A_1, A_2, $\cdots$, and A_l are symmetric. Then the following statements are equivalent: i) $\mathscr{A}$ is robustly stable; ii) the extreme matrices A_i, $i = 1, 2, \cdots, l$, are negative definite; and iii) all the coefficients in $\gamma \det A(\lambda)$ are greater than zero.

Proof of Theorem 3:

i) → ii) Obvious, given that the extreme matrices are members of the family.

iii) → i) Since $A(\lambda)$ is symmetric, and hence has purely real eigenvalues, it follows by applying Theorem 2 i) to $\gamma \det A(\lambda)$.

ii) → iii) Proceed by induction. W.L.O.G., we assume that all A_i's are positive definite and show that all the coefficients in $\det A(\lambda)$ have a unique positive sign.

Step 1: For $l = 1$, $\det A(\lambda) = a_1 \lambda_1^N$, and $a_1 = \det A_1 > 0$ since A_1 is positive definite.

Step 2: Suppose that the proposition holds for $l = k$, i.e., all the coefficients in $\det A(\lambda)$ are greater than zero when $l = k$. We must induce that the proposition also holds for $l = k + 1$. To this end, denote $\lambda_{(k)} = [\lambda_1, \lambda_2, \cdots, \lambda_k]^T$ and $A(\lambda_{(k)}) = \sum_{j=1}^{k} \lambda_j A_j$. Then

$$\begin{aligned}\det A(\lambda_{(k+1)}) &= \det\{A(\lambda_{(k)}) + \lambda_{k+1} A_{k+1}\}.\\ &= \det Q^T \det\{B(\lambda_{(k)}) + \lambda_{k+1} I\} \det Q \quad (7)\end{aligned}$$

where Q is a positive definite matrix with the property that $A_{k+l} = Q^T Q$ [18] and

$$B(\lambda_{(k)}) = Q^{-T} A(\lambda_{(k)}) Q^{-1} = \sum_{i=1}^{k} \lambda_i B_i, \; B_i = Q^{-T} A_i Q^{-1}, \quad i = 1, \cdots, k. \quad (8)$$

The matrices B_i in (8) are positive definite since A_i are positive definite. Hence, $B(\lambda_{(k)})$ as well as its principal submatrices are k-dimensional polytopes of positive definite matrices. By the assumption made for $l = k$, all the coefficients in $\det B(\lambda_{(k)})$ and in all principal minors of $B(\lambda_{(k)})$ are greater than zero. Meanwhile, expressing $\det\{B(\lambda_{(k)}) + \lambda_{k+1} I\}$ as a polynomial in λ_{k+1}, it follows that [19, p. 126]

$$\det\{B(\lambda_{(k)}) + \lambda_{k+1} I\} = \lambda_{k+1}^N + \Delta_1(\lambda_{(k)}) \lambda_{k+1}^{N-1} + \cdots + \Delta_{N-1}(\lambda_{(k)}) \lambda_{k+1} + \Delta_N(\lambda_{(k)})$$

where

$$\Delta_j(\lambda_{(k)}) = \text{sum of all principal minors of order } j \text{ from } B(\lambda_{(k)}).$$

This implies that the coefficients in $\det\{B(\lambda_{(k)}) + \lambda_{k+l} I\}$ are greater than zero. The proposition then holds for $l = k + 1$ since in (8), $\det Q^T = \det Q = \sqrt{\det A_{k+l}} > 0$ are positive real constants. ∎

Next we consider the polytope matrix family of convex combinations of two matrices. In the following theorem, a result similar to that of [12] is obtained.

Theorem 4: Suppose $\mathscr{A} =$ convex hull $\{A_1, A_2\}$, where A_1 and A_2 are $n \times n$ square matrices, and $\tilde{\mathscr{A}}$ is the matrix family of convex combinations of the transformed matrices $\tilde{A}_1$ and $\tilde{A}_2$. Let

$$p(\lambda_1, \lambda_2) = \gamma \det \tilde{A}(\lambda) = \sum_{i=0}^{N} \tilde{a}_i \lambda_1^i \lambda_2^{N-i}$$

where $\tilde{a}_i = \gamma \sum_{c(i, N-i)} \det \tilde{A}_{c(i, N-i)}$ and N is the dimension of $\tilde{\mathscr{A}}$. Then the polytope matrix $\mathscr{A}$ is robustly stable if and only if $\tilde{a}_N = \gamma \det \tilde{A}_1 > 0$ ($\tilde{a}_0 = \gamma \det \tilde{A}_2 > 0$) and the polynomial $p(\lambda, 1)$ ($p(1, \lambda)$) has no roots on the closed positive half real axis.

Proof of Theorem 4: Note that $p(\lambda_1, \lambda_2)$ is strictly copositive if and only if: i) for $\lambda_2 = 0$

$$p(\lambda_1, 0) = \gamma \det(\lambda_1 \tilde{A}_1) = (\gamma \det \tilde{A}_1) \lambda_1^N > 0$$

for all $\lambda_1 > 0$, and ii) for any fixed $\lambda_2 > 0$, the polynomial $p(\lambda_1, \lambda_2) > 0$ for all $\lambda_1 \geq 0$. It is easily seen that the condition i) holds if and only if $\tilde{a}_N = \gamma \det \tilde{A}_1 > 0$. For the condition ii) to hold, we divide $p(\lambda_1, \lambda_2)$ by λ_2^N and define

$$p(\lambda, 1) := \frac{1}{\lambda_2^N} p(\lambda_1, \lambda_2) = p\left(\frac{\lambda_1}{\lambda_2}, 1\right) = \sum_{i=0}^{N} \tilde{a}_i \lambda^i \quad (9)$$

where $\lambda = (\lambda_1/\lambda_2)$. For any fixed $\lambda_2 > 0$, $p(\lambda_1, \lambda_2) > 0$ for all $\lambda_1 \geq 0$ if and only if $p(\lambda, 1) > 0$ for all $\lambda \geq 0$ if and only if $p(\lambda, 1)$ has no nonnegative real roots. ∎

IV. Examples

The following example was originally studied in [4].

Example 1 (Sufficient Test for Instability): Let

$$A = \text{convex hull } (A_1, A_2, A_3, A_4)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

and $A(\lambda)$ the associated matrix function of λ. The extreme matrices all have the same set of eigenvalues at $\{-0.3194 + 1.6332i, -0.3194 - 1.6332i, -0.3611\}$. Define

$$\tilde{A}(\lambda) := A(\lambda) \oplus A(\lambda).$$

Expanding the determinant $\gamma \det(\tilde{A}(\lambda))$ yields a four variable polynomial; straightforward (with the assistance of symbolic manipulation software) calculations verify that the sum of its coefficients is zero, implying the polytope is unstable. An alternate approach explored in [4] verifies that the polytope does contain a single matrix with an eigenvalue at the origin.

Example 2 (Sufficient Test for Stability): This example tests the stability of an interval matrix A_I which was first considered by Bialas [20], and later discussed by Heinen [21], Yedavalli [22], and Juang, Kuo, and Hsu [23]. The lower and upper bounds of the elements of the interval matrix which passed the stability tests in these previous works are listed as follows:

Bialas [20]:

$$A_I = \begin{bmatrix} [-5, -3] & [1, 2] \\ [4, 5] & [-6, -4] \end{bmatrix},$$

Heinen [21]:

$$A_I = \begin{bmatrix} [-\infty, -3] & [-2, 2] \\ [-5, 5] & [-\infty, -4] \end{bmatrix},$$

Yedavalli [22]:

$$A_I = \begin{bmatrix} [-7, -3] & [0, 2] \\ [3, 5] & [-8, -4] \end{bmatrix},$$

Juang *et al.* [23]:

$$A_I = \begin{bmatrix} [-5, -2.9] & [1, 2.05] \\ [4, 5.05] & [-6, -3.9] \end{bmatrix}.$$

Again, using Kronecker sum operation, define $\tilde{A}_I := A_I \oplus A_I$. The transformed interval matrix family involves $2^4 = 16$ 4×4 extreme matrices. The result from a simple *Mathematica* program shows that $\gamma \det \tilde{A}_I$ has all positive coefficients for

$$A'_I = \begin{bmatrix} [-100, -2.85] & [-2.15, 2.15] \\ [-5.1, 5.1] & [-100, -3.85] \end{bmatrix}.$$

Hence, A'_I passes the sufficient condition for stability offered by Theorem 2 i). Further calculations confirm that the lower bound of -100 on the (1, 1) and (2, 2) elements can be extended down to $-\infty$ if desired.

V. Conclusions

This technical note has defined copositivity for a homogeneous polynomial, and showed that all matrices within a matrix polytope are stable if and only if an associated homogeneous polynomial $\gamma \det \tilde{A}(\lambda)$ is strictly copositive. It also provides practically computable necessary and sufficient tests for robust stability of two simple classes of matrix polytopes in terms of strictly copositive homogeneous polynomials. However, obtaining an effective necessary and sufficient criterion for testing if a general homogeneous polynomial is strictly copositive is difficult. As an alternative, this note has offered sufficient tests both for robust stability and for instability of a matrix polytope. While experience of the authors with randomly generated polytopes indicates that some cases will be left undecided using these sufficiency tests, these methods can reliably classify many matrix polytopes as robustly stable or unstable at relatively low computational cost.

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On Obtaining a Realization of a Polynomial Matrix Description of a System

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Abstract—**This note refers to the computation of a state-space realization of a linear dynamical system, starting from a polynomial matrix description of it. Simple procedures are given for obtaining its dynamic matrix (possibly in some canonical or block-companion form) through unimodular transformations, starting from a square diagonal polynomial matrix, obtained as an intermediate step of the overall realization algorithm.**

I. Introduction

For processes which can be modeled by linear differential (or difference) equations with constant coefficients, Rosenbrock [1] introduced the polynomial matrix description in the form of the following pair of vector equations:

$$T(s)\xi = U(s)u \tag{1.1a}$$

$$y = V(s)\xi + W(s)u \tag{1.1b}$$

where $T(s)$, $U(s)$, $V(s)$, and $W(s)$ are polynomial matrices in the indeterminate s. He showed that under the polynomial transformations on (1.1) that he called strict system equivalence, if $T(s)$ is square, det $T(s) \neq 0$ [and has a degree equal to the dimension of $T(s)$] and the input-output transfer matrix corresponding to (1.1) is proper, then it is possible to obtain a description of the same process in state-space form, i.e., of the type

$$\Delta x(t) = Ax(t) + Bu(t) \tag{1.2a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.2b}$$

where Δ means either differentiation or one-step forward shift operator. Since then, several authors studied the polynomial matrix description (1.1) and the procedures for the computation of a state-space realization (1.2) strict system equivalent to (1.1) (see, e.g., [2]-[8]).

In the original procedure proposed by Rosenbrock [1], one of the steps consists of finding two square polynomial unimodular matrices $\tilde{L}(s)$ and $\tilde{R}(s)$, both of dimension n, for a given $n \times n$ diagonal polynomial matrix $\hat{T}(s)$ in Smith form, with the degree of det $\hat{T}(s)$ equal to n (obtained from $T(s)$ through elementary operations represented by some other unimodular transformations), such that:

$$\tilde{L}(s)\hat{T}(s)\tilde{R}(s) = sI_n - A \tag{1.3}$$

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for some constant $n \times n$ square matrix A, with I_n being the $n \times n$ identity matrix. Notice that, in order to complete the computation of a representation (1.2) strict system equivalent to (1.1), the explicit knowledge of $\tilde{L}(s)$ and $\tilde{R}(s)$ is needed, that of matrix A being not sufficient.

In this note, some very simple procedures are presented for the computation of a pair of unimodular matrices $\tilde{L}(s)$ and $\tilde{R}(s)$ such that (1.3) holds, possibly with A in some canonical form. In these procedures the computation of the Smith form of $T(s)$ is not required, since they are effective even when $\hat{T}(s)$, although diagonal, is not in Smith form, thus allowing us to shorten the overall realization algorithms deriving a state-space description of the form (1.2) from the original description of the form (1.1). A couple of these procedures allows us to obtain the matrix A in (1.3) in some block-diagonal forms, having one diagonal block for each nonunit diagonal element of $\hat{T}(s)$; the diagonal blocks can be obtained either in companion form, or in an upper triangular form, having the elements on the superdiagonal equal to one, and the elements above the superdiagonal equal to zero. The other procedures, which do not obtain A in these canonical forms, are faster, thus providing a further computational advantage.

II. Notations and Preliminaries

Henceforth, any m-dimensional square polynomial matrix $Z(s) = [z_{ij}(s)]$ will be assumed to have either real or complex coefficients in each entry, unless otherwise specified; its ith row (column) will be denoted by $[Z(s)]^i([Z(s)]_i)$; in addition, the degree of det $Z(s)$ will be called the *order* of $Z(s)$ and denoted by ord$(Z(s))$. Moreover, denoting with $d(p(s))$ the degree of any polynomial $p(s)$, matrix $Z(s)$ will be said to be in *special diagonal form* if:

a) $Z(s)$ is diagonal and nonsingular, and its diagonal elements $z_{ii}(s)$, $i = 1, \cdots, m$, are monic;

b) $d(z_{ii}(s)) \leq d(z_{i+1,i+1}(s))$, $i = 1, \cdots, m-1$;

obviously, matrix $Z(s)$ is in *Smith form* if, in addition to a) and b):

c) $z_{ii}(s)$ is a divisor of $z_{i+1,i+1}(s)$, $i = 1, \cdots, m-1$.

Let $Z(s)$ be the Smith form of a given m-dimensional square nonsingular polynomial matrix $H(s)$, and $\lambda_{ij} \in \mathbb{C}$ be a root with multiplicity n_{ij} of the nonunit entry $z_{ii}(s)$; then, with the usual terminology, $\gamma_{ij}(s) := (s - \lambda_{ij})^{n_{ij}}$ will be called an *elementary divisor* of $H(s)$; it will be also called an elementary divisor of $z_{ii}(s)$.

If $Z(s)$ is diagonal and nonsingular, but not in special diagonal form, it can be put in special diagonal form by reordering and constant scaling its diagonal elements. It is well known that, if $Z(s)$ is not diagonal, it can be put in Smith form by a sequence of row and column elementary transformations represented by unimodular matrices (see, e.g., [9, ch. 6]). It is stressed that the same algorithm allows $Z(s)$ to be put in special diagonal form, if it is nonsingular and all the steps specifically needed for obtaining and/or checking c) (which are at least $m - 1$), are deleted.

If $Z(s)$ is in special diagonal form, then $\nu(Z(s))$ will denote the number of its entries equal to 1.

Remark 1: For an m-dimensional square polynomial matrix $Z(s)$ in special diagonal form, let its order be equal to its dimension, i.e., ord $(Z(s)) = m$. Then, the following claims are stressed:

d) if $\nu(Z(s)) = 0$, then $z_{ii}(s) = (s - \lambda_i)$, for some λ_i, $i = 1, \cdots, m$;

e) if $z_{mm}(s) = (s - \lambda_m)$, then $z_{ii}(s) = (s - \lambda_i)$, for some λ_i, $i = 1, \cdots, m-1$;