



Constructing 4-Dimensional Tops

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Introduction

String Theory predicts that the universe has several extra dimensions, which have the structure of *Calabi-Yau varieties*; the universes defined by these varieties are conjectured to occur in physically indistinguishable pairs. The mathematical field of *mirror symmetry* seeks to understand the geometric correspondences between paired Calabi-Yau varieties.

The polar duality transformation takes a polytope with integer lattice points to its polar dual.

Let Δ be a lattice polytope which contains $\vec{0}$. The polar polytope Δ° is the polytope given by:

$$\{(m_1, \dots, m_k) : (n_1, \dots, n_k) \cdot (m_1, \dots, m_k) \geq -1 \\ \forall (n_1, \dots, n_k) \in \Delta\}$$

A lattice polytope is defined to be *reflexive* if its polar dual is also a lattice polytope.

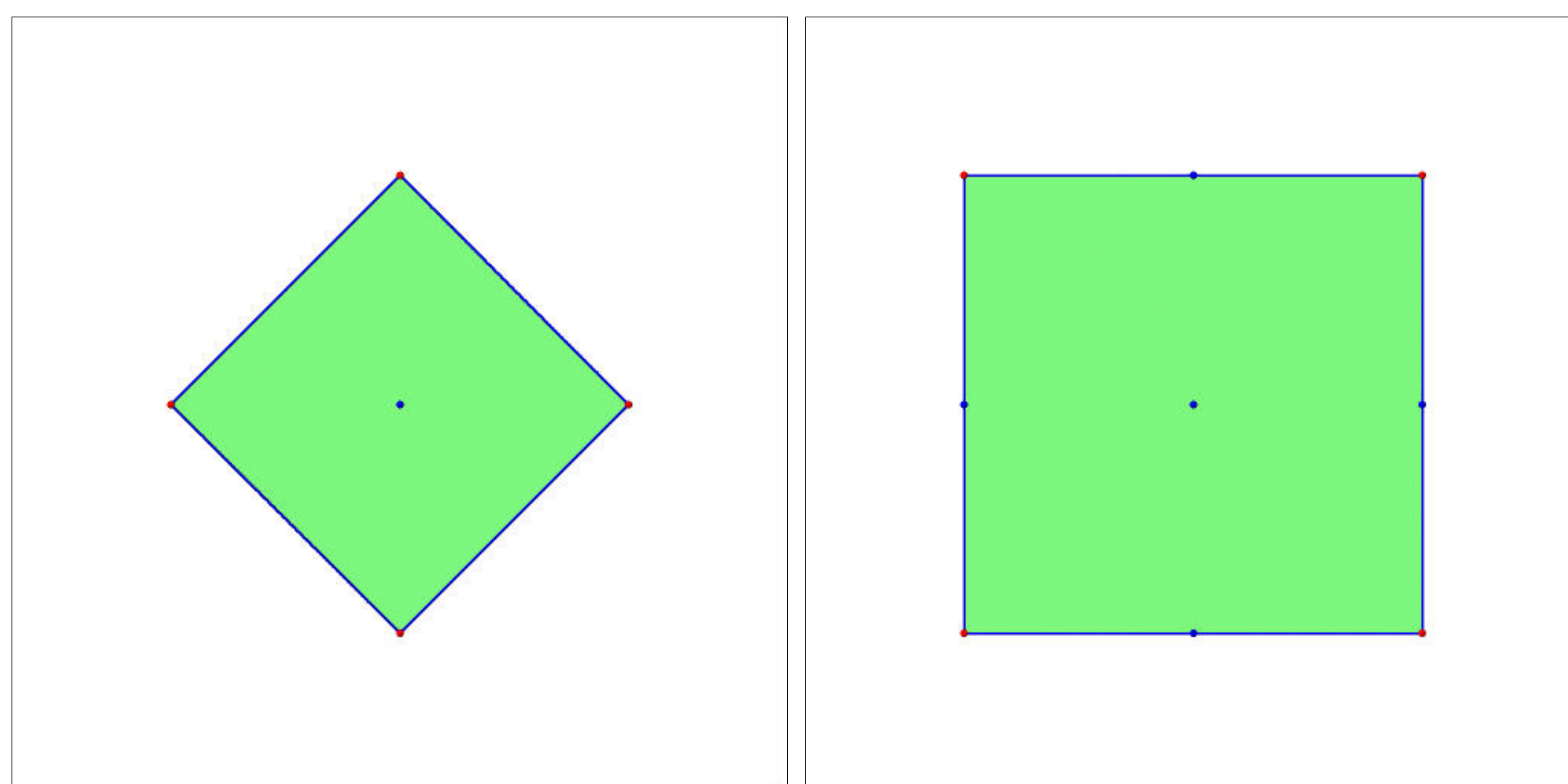


Figure 1: 2D Octahedron (left) and its dual

Reflexive polytopes can be used to describe Calabi-Yau hypersurfaces, so by studying reflexive polytopes we may gain important insight into the nature of hidden dimensions in space. Reflexive polytopes have been classified in 3D and 4D, with 4,319 and 473,800,776 classes of equivalent polytopes respectively. A top generalizes the idea of slicing a reflexive polytope. A top is a lattice polytope Δ such that one of its defining inequalities is of the form

$$(n_1, \dots, n_k) \cdot (0, \dots, 0, 1) \geq 0$$

and the rest are of the form

$$(n_1, \dots, n_k) \cdot (m_{j1}, \dots, m_{jk}) \geq -1,$$

where (m_{j1}, \dots, m_{jk}) is a point in the lattice.

The points of a top with the last coordinate of 0 are a reflexive polytope that is one dimension less than the top.

The polar duality transformation can also be applied to tops and the structure changes to that of a dual top, with one facet infinitely extended.

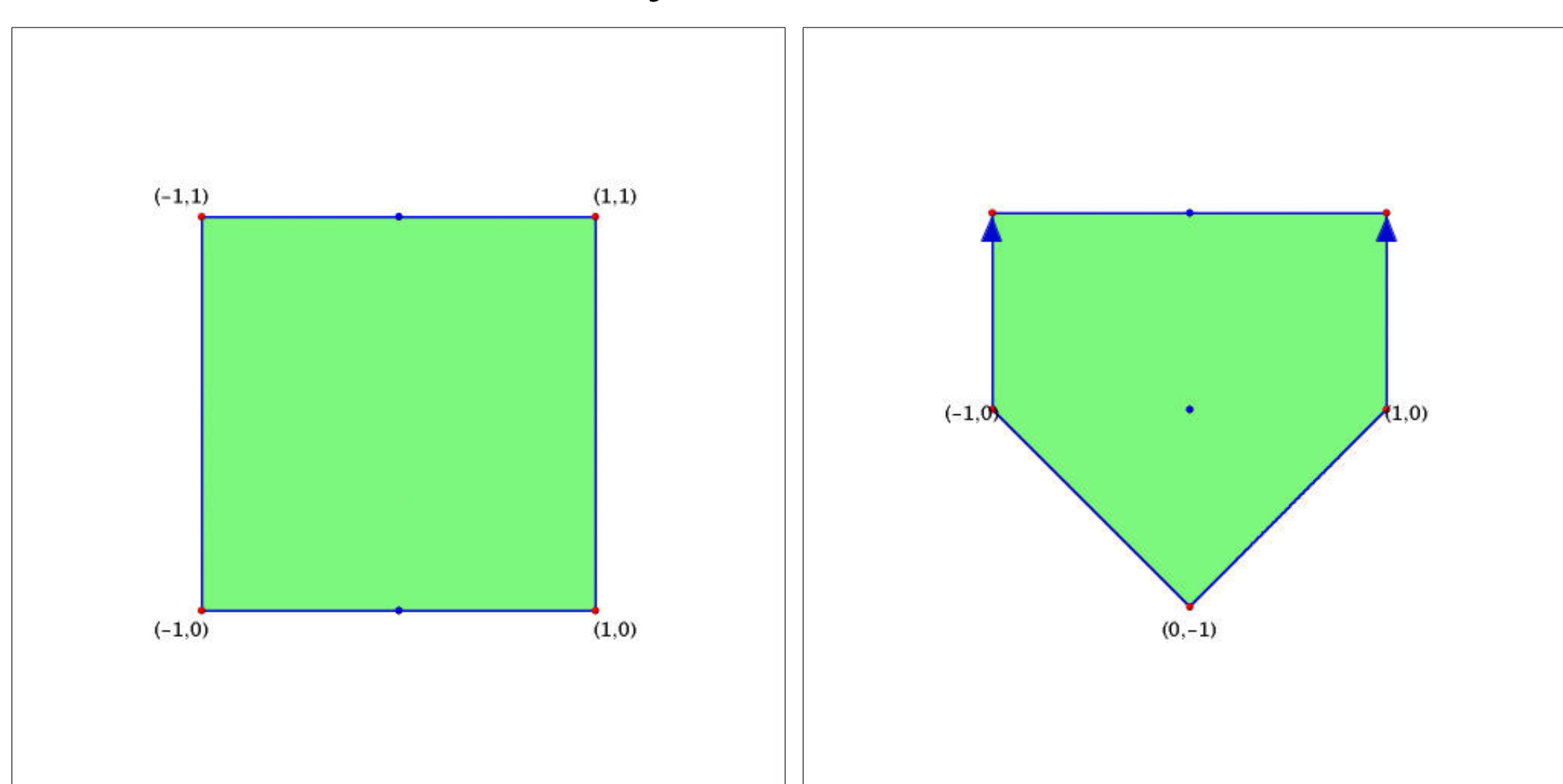


Figure 2: 2D top (left) and associated dual top

We use triangulations of 3-dimensional reflexive polytopes to construct new examples of tops.

From Reflexive Polytopes to Dual Tops

Bouchard and Skarke classified families of 3D tops by relating them to 2D reflexive polytopes. To generate these tops, a reflexive 2D polytope was chosen as a base and points were defined beneath it to form a dual top. We require that the dual tops be convex, so we must constrain the points beneath the polytope accordingly. By generating dual tops in this fashion, we can extend into higher dimensions. The classification of reflexive polytopes yields a finite number of equivalence classes. In contrast, the classification of tops yields infinite families of equivalence classes.

Constructing 4-Dimensional Tops

- Choose reflexive polytope base for dual top
- Fix a lattice triangulation of this polytope
- Construct a 4-dimensional top with triangulated boundary. We choose the fourth coordinate for each lattice point of base while satisfying the convexity condition (Four of these are fixed up to overall change of coordinates)

Our construction shows that we can use lattice triangulations to describe all tops with a given reflexive polytope as its base.

Octahedron

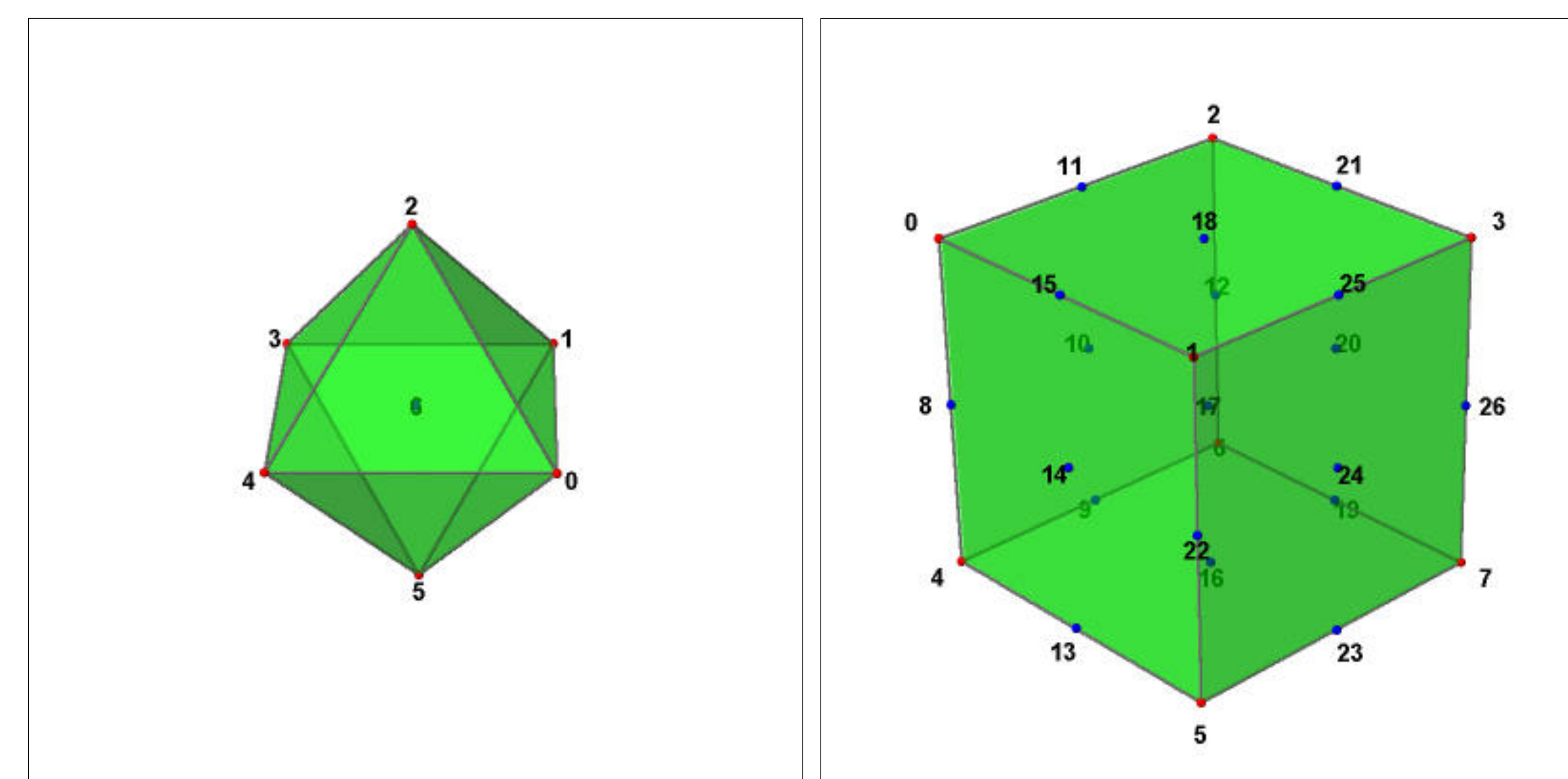


Figure 3: Octahedron (left) and dual cube

We labeled the summit vertices of our octahedron as follows: 0) (0, 1, 0, -1); 1) (1,0,0,t); 2) (0,0,1,s); 3) (0,-1,0,0); 4) (-1,0,0,-1); 5) (0,0,-1,-1); 6) not a vertex

There are two ways to triangulate (up to symmetry): origin is a vertex of all simplices, and 4 simplices where origin lies on edge. Ryan Davis (UW-EC, 2012) and Adam Gewiss (UW-EC, 2013) showed that by using one tetrahedron for every face we are given a 3-parameter infinite number of families of tops with all lattice points having last coordinate of 0 or 1. We used the second triangulation to describe a new family with 2 parameters. It is important to check the convexity for the dual top. We use determinants of the matrices that are made of the points from the dual top.

Now we must go from dual top to top. We use the definition of polar duality to find the vertices of the top corresponding to each simplex in our triangulation. A

complete description of the dual will also require consideration of the vertical sidewalls. Because we created a dual top in this fashion, the vertical sidewalls correspond to the facets of the cube base (polar dual of octahedron). Then the dual of the constructed 4D dual top has vertices given by:

$$\begin{pmatrix} -1 - 2t, 1, -1 - 2s, 2 \\ -1, 1, -1 - 2s, 2 \\ -1 - 2t, 1, -1, 2 \\ -1, 1, -1, 2 \end{pmatrix}$$

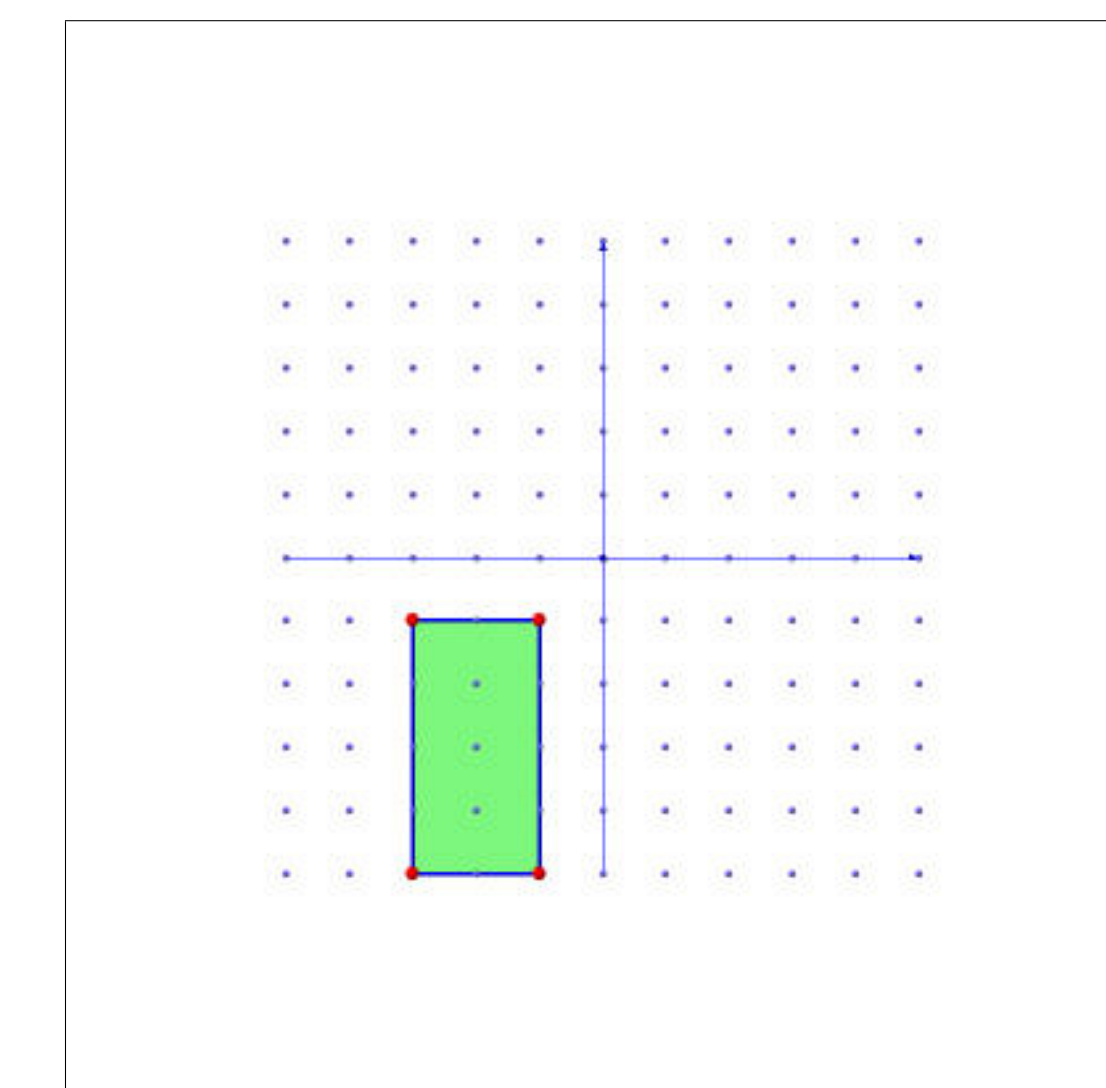


Figure 4: Graph of vertices when t=1, s=2

As you can see, the final coordinate is a consistent 2 for each vertex. This gives us a quadrilateral (2-face), thus a new infinite family of tops. This is a complete classification of 4-dimensional tops with a 3-dimensional cube as its base.

Simplex

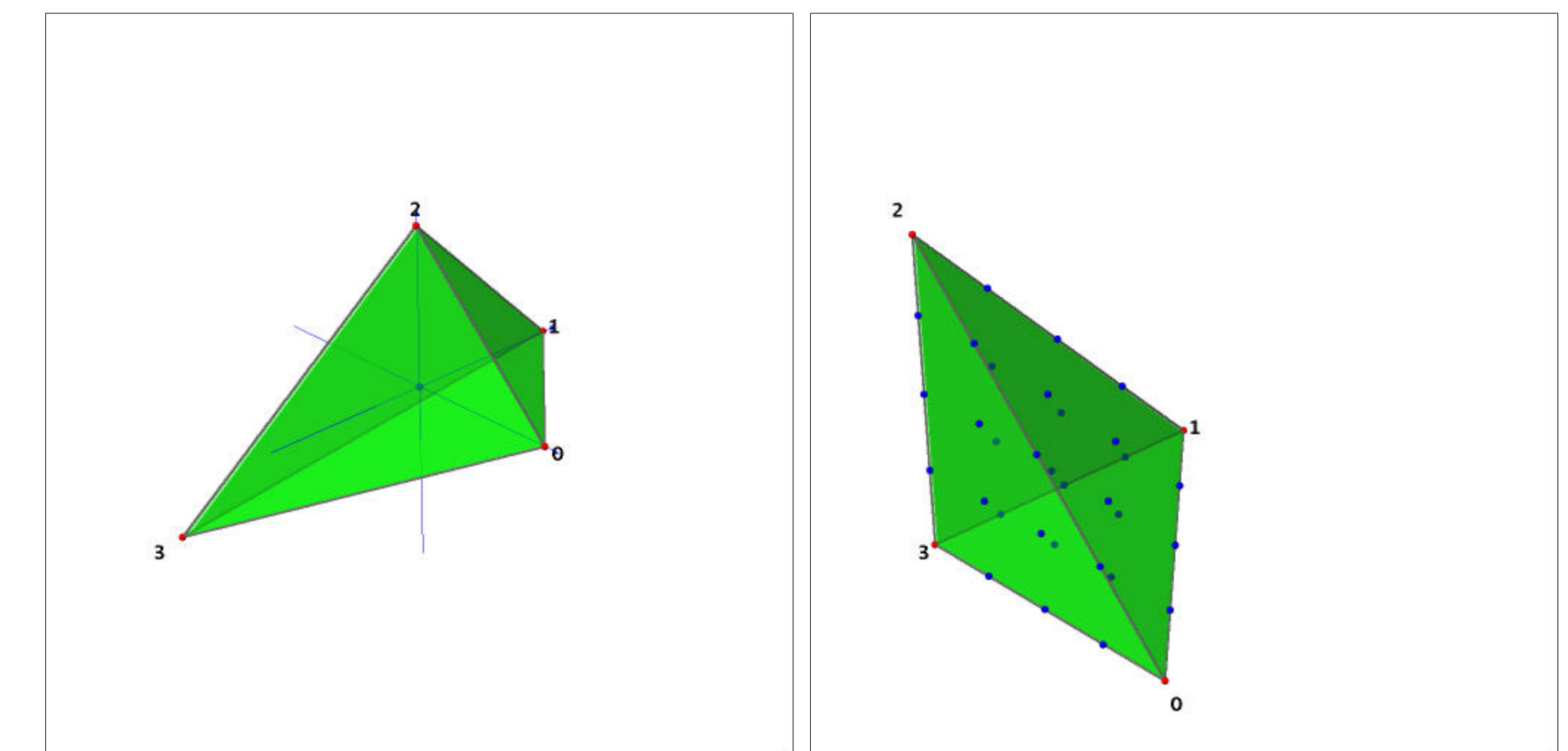


Figure 5: Simplex (left) and dual simplex

We labeled the vertices of our simplex as follows: 0) (1,0,0,0); 1) (0,1,0,0); 2) (0,0,1,0); 3) (-1,-1,-1,-1) Again, there are two ways to triangulate this simplex. To stay consistent we did not include the origin in our triangulation, therefore our calculations were different from Adam and Ryan's. After checking convexity for the dual simplex we extended from dual top to top. We used the definition of polar duality to find the vertices of the top corresponding to each simplex in our triangulation. Since we created a dual this way our calculations were similar to that of the octahedron, but we found our vertices for our new top to be different than the octahedron's top. Our dual of the constructed 4D dual top has the summit vertex of: $(-1, -1, -1, 4)$.

As you can see, this simplex gives us a last coordinate of a consistent 4. Therefore, we have a new infinite family of tops that extend all the way to 4, in 4-dimensional space.

Our constructions show that we can use lattice triangulations to describe all tops with a given reflexive polytope as its base.