UNIQUENESS OF SOLUTION IN LINEAR PROGRAMMING

by

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Abstract

A number of characterizations are given which are both necessary and sufficient for the uniqueness of a solution to a linear programming problem.

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The main purpose of this work is to give some conditions which are both necessary and sufficient that a solution to a linear programming be unique. Dantzig [1] gives a sufficient condition which is clearly not necessary for uniqueness. Dantzig's uniqueness condition for a linear program in canonical form is that the optimal reduced costs are positive. Fiacco and McCormick [2] give a second order sufficiency condition for a nonlinear programming problem to have an isolated local minimum. This condition can, as they point out, be specialized to give a sufficient uniqueness condition for a linear programming problem. It turns out that this condition is also necessary for uniqueness in linear programming as we show below, Theorem 2 (iv). Another interesting fact about this specific characterization of uniqueness is that it implies the equivalent of the LI (linear independence) condition [4] or more generally Robinson's regularity condition [5,6,7] for the dual linear program constraints, which in turn ensures their stability in the sense of Robinson [5,6,7].

Our principal results are contained in two theorems. The first one, Theorem 1, is a particularly simple characterization of uniqueness which states that a linear programming solution is unique if and only if it remains a solution to each linear program obtained by an arbitrary but sufficiently small perturbation of its cost row. Theorem 2 gives a number of other characterizations of uniqueness which are equivalent to the condition of Theorem 1. Among these characterizations, (v) and (x) are probably the easiest to verify computationally as indicated in Remark 2 following Theorem 2. Uniqueness of the optimal dual variables can be characterized in a similar way to that of the optimal primal variables. We state such characterizations in Corollaries 1 and 2.

We shall consider the linear programming problem of finding an $\bar{x}$ in the n-dimensional real Euclidean space R^n that will

$$\begin{aligned} \text{Minimize} \quad & p^T x \\ \text{subject to} \quad & Ax = b \\ & Cx \geqq d \end{aligned} \tag{1}$$

where p, b and d are given vectors in R^n, R^m and R^k respectively, and A and C are given $m \times n$ and $k \times n$ matrices respectively. Associated with (1) is the dual linear programming problem [1,3] of finding $\bar{u}$ in R^m and $\bar{v}$ in R^k that will

$$\begin{aligned} \text{Maximize} \quad & b^T u + d^T v \\ \text{subject to} \quad & A^T u + C^T v = p \\ & v \geqq 0 \end{aligned} \tag{2}$$

The choice of the form of the linear program (1) is rather arbitrary although it somewhat simplifies the statement of the results of Theorem 2. Note that by appropriately partitioning the constituents p,A,b,C,d of problem (1), a very general linear programming problem can be obtained, that is one with equalities and inequalities which involve both unrestricted and nonnegative variables. For forms other than (1), Theorem 1 would not change in essence, that is we would still consider arbitrary but small perturbations of the cost row. However, the results of Theorem 2 would have to be modified appropriately. Our first principal result is the following.

Theorem 1 A solution $\bar{x}$ of the linear program (1) is unique if and only if it remains a solution to all linear programs obtained from (1) by arbitrary but sufficiently small perturbation of its cost vector p, or equivalently for each q in R^n there exists a real positive number ε such that $\bar{x}$ remains a solution of the perturbed linear program

$$\begin{aligned} \text{Minimize} \quad & (p+\varepsilon q)^T x \\ \text{subject to} \quad & Ax = b \\ & Cx \geqq d \end{aligned} \tag{3}$$

Proof $\left\langle \bar{x} \text{ is a unique solution of (1)} \right\rangle$

$$\Longleftrightarrow \left\langle \begin{array}{l} \forall q \in R^n, \nexists\, x,u,v,\lambda \in R^{n+m+k+1}: \\ \begin{array}{rrrl} & Ax & -b\lambda & = 0 \\ & Cx & -d\lambda & \geqq 0 \\ -A^Tu - C^Tv & & +p\lambda & = 0 \\ v & & & \geqq 0 \\ b^Tu + d^Tv - p^Tx & & & \geqq 0 \\ & -q^Tx & +q^T\bar{x}\lambda & > 0 \\ & & \lambda & > 0 \end{array} \end{array} \right\rangle$$

(The forward implication holds because otherwise $\lambda^{-1}x$ would be another solution of (1). The backward implication becomes evident if we note that $q^T(x-\bar{x}) \geqq 0$ for all q in R^n is equivalent to $x-\bar{x} = 0$.)

$$\Longleftrightarrow \left\langle \begin{array}{l} \forall q \in R^n,\ \exists x,y,u,v,\beta,\varepsilon,\gamma \in R^{n+k+m+k+3}: \\ -Ax + b\beta = 0 \\ -Cx + y + d\beta = 0 \\ A^Tu + C^Tv - p\beta - q\varepsilon = 0 \\ -b^Tu - d^Tv + p^Tx + q^T\bar{x}\varepsilon + \gamma = 0 \\ v,y,\beta,\varepsilon,\gamma \geqq 0,\ \varepsilon + \gamma > 0 \end{array} \right\rangle \text{(By Motzkin's theorem [3])} \qquad (4)$$

$$\Longleftrightarrow \left\langle \begin{array}{l} \forall q \in R^n,\ \exists x,u,v,\beta,\varepsilon \in R^{n+m+k+2}: \\ Ax = \beta b,\ Cx \geqq \beta d,\ p^Tx = \beta p^T\bar{x},\ \beta \geqq 0 \\ A^Tu + C^Tv = \beta p + \varepsilon q,\ b^Tu + d^Tv \geqq (\beta p+\varepsilon q)^T\bar{x},\ v \geqq 0,\ \varepsilon > 0 \end{array} \right\rangle$$

(The backward implication is evident once we substitute $p^Tx = \beta p^T\bar{x}$ in $b^Tu + d^Tv \geqq (\beta p+\varepsilon q)^T\bar{x}$. The forward implication follows from the following considerations. Let (4) hold. The case of $\varepsilon = 0$ is excluded because that would contradict the existence of an optimal solution $\bar{x}$ to (1). So $\varepsilon > 0$, $\gamma \geqq 0$, and it only remains to show that $p^Tx = \beta p^T\bar{x}$. From $A\bar{x} = b$, $C\bar{x} \geqq d$, $v \geqq 0$ we have that $u^TA\bar{x} = b^Tu$, $v^TC\bar{x} \geqq d^Tv$ and consequently

$$p^Tx + \varepsilon q^T\bar{x} \leqq b^Tu + d^Tv \leqq \bar{x}^T(A^Tu+C^Tv) = \beta p^T\bar{x} + \varepsilon q^T\bar{x}$$

Hence $p^Tx \leqq \beta p^T\bar{x}$. But $\beta\bar{x}$ is a solution of the linear program: Min $\{p^Tx: Ax = \beta b,\ Cx \geqq \beta d\}$, for $\beta \geqq 0$, and x is feasible for this linear program, so $p^Tx = \beta p^T\bar{x}$.)

$$\Longleftrightarrow \quad \left\langle \begin{array}{l} \forall q \in R^n,\ \exists u,v,\beta,\varepsilon \in R^{m+k+2}: \\ A^T u + C^T v = \beta p + \varepsilon q,\ b^T u + d^T v = (\beta p + \varepsilon q)^T \bar{x} \\ v \geqq 0,\ \beta \geqq 0,\ \varepsilon > 0 \end{array} \right\rangle \qquad (5)$$

(Because $x = \beta\bar{x}$ satisfies the conditions $Ax = \beta b$, $Cx \geqq \beta d$, $p^T x = \beta p^T \bar{x}$ and $(\beta p+\varepsilon q)^T \bar{x} = u^T A\bar{x} + v^T C\bar{x} \geqq b^T u + d^T v$.)

$$\Longleftrightarrow \quad \left\langle \begin{array}{l} \forall q \in R^n,\ \exists u,v,\varepsilon \in R^{m+k+1}: \\ A^T u + C^T v = p + \varepsilon q,\ b^T u + d^T v = (p + \varepsilon q)^T \bar{x} \\ v \geqq 0,\ \varepsilon > 0 \end{array} \right\rangle \qquad (6)$$

(The backward implication is evident if we set $\beta = 1$. The forward implication is again evident for the case of $\beta > 0$ upon normalizing the relations of (5) with respect to β. Suppose now that for some q the relations of (5) hold with $\beta = 0$. Then $\bar{x}$ is a solution of the linear program: Min $\{\varepsilon q^T x: Ax = b,\ Cx \geqq d\}$. But since $\bar{x}$ is also a solution of the linear program (1), it follows that $\bar{x}$ is a solution of the linear program: Min $\{(p+\varepsilon q)^T x: Ax = b,\ Cx \geqq d\}$. Conditions (6) are the necessary optimality conditions for $\bar{x}$ to solve this last linear program.)

$$\Longleftrightarrow \quad \left\langle \forall q \in R^n,\ \exists \varepsilon \in R,\ \varepsilon > 0 \text{ such that } \bar{x} \text{ solves (3)} \right\rangle$$

(Because $\bar{x}$ is feasible for problem (3), the pair (u,v) is feasible for the dual of problem (3) and $b^T u + d^T v = (p+\varepsilon q)^T \bar{x}$.) □

By similar arguments we can characterize the uniqueness of the dual optimal variables as follows.

<u>Corollary 1</u> (Uniqueness of optimal dual variables) The optimal dual solution $(\bar{u},\bar{v})$ of (2) associated with a primal optimal solution $\bar{x}$ of (1) is unique if and only if it remains an optimal dual solution to all linear programs obtained from (1) by arbitrary but sufficiently small perturbation of the right hand side vector $\begin{pmatrix} b \\ d \end{pmatrix}$, or equivalently for each $g,h \in R^{m+k}$ there exists a real number $\varepsilon > 0$ such that $(\bar{u},\bar{v})$ remains dual optimal for the perturbed linear program

$$\begin{aligned} &\text{Minimize} && p^T x \\ &\text{subject to} && Ax = b + \varepsilon g \\ & && Cx \geqq d + \varepsilon h \end{aligned} \tag{7}$$

We proceed now to our second principal result which gives other equivalent characterizations of uniqueness of a linear programming solution. For this purpose we introduce some notation. Let $\bar{x}$ be a solution of (1) and let $(\bar{u},\bar{v})$ be any solution of (2). Let C_i denote the ith row of C. Define

$$J = \{i \mid C_i\bar{x}=d_i\}$$
$$K = \{i \mid \bar{v}_i>0\} = \{i \mid C_i\bar{x}=d_i,\ \bar{v}_i>0\}$$
$$L = \{i \mid C_i\bar{x}=d_i,\ \bar{v}_i=0\}$$

Note that $J = K \cup L$, and that any of these three sets may be empty.

Let C_J, C_k and C_L be matrices whose rows are C_i, $i \in J$, $i \in K$ and $i \in L$ respectively. For x in R^n we shall use in addition to the notation $x \geqq 0$, which denotes $x_i \geqq 0$, $i = 1,\ldots, n$, the notation $x \geq 0$ and $x > 0$ which denote respectively $0 \neq x \geqq 0$ and $x_i > 0$, $i = 1,\ldots, n$. To simplify notation in the sequel we shall not explicitly state the dimensionality of some vectors, it being obvious from the context. The vector e will denote a vector of ones of appropriate dimension.

<u>Theorem 2</u> Let $\bar{x}$ be a solution of the linear program (1). The following statements are equivalent:

(i) $\bar{x}$ is unique

(ii) For each q in R^n there exists a positive real number ε such that $\bar{x}$ solves (3).

(iii) There exists no x satisfying

$$Ax = 0,\ C_J x \geqq 0,\ p^T x \leqq 0,\ x \neq 0$$

(iv) There exists no x satisfying

$$Ax = 0,\ C_K x = 0,\ C_L x \geqq 0,\ x \neq 0$$

(v) The rows of $[A^T\ C_K^T\ C_L^T]$ are linearly independent and there is no x satisfying

$$Ax = 0,\ C_K x = 0,\ C_L x \geq 0$$

(vi) For each a, c, h, the set $\{x | Ax=a,\ C_K x=c,\ C_L x \geqq h\}$ is empty or bounded.

(vii) For each q in R^n there exists a positive number ε such that the system

$$A^T u + C_J^T v_J = p + \varepsilon q,\ v_J \geqq 0$$

has a solution (u, v_J).

(viii) For each s in R^n the system

$$A^T u + C_K^T v_K + C_L^T v_L = s, \; v_L \geqq 0$$

has a solution (u, v_K, v_L).

(ix) For each s in R^n the system

$$A^T u + C_K^T v_K + C_L^T v_L = s, \; v_L > 0$$

has a solution (u, v_K, v_L).

(x) The rows of $[A^T \; C_K^T \; C_L^T]$ are linearly independent and the system

$$A^T u + C_K^T v_K + C_L^T v_L = 0, \; v_L > 0$$

has a solution (u, v_K, v_L).

Proof (i) $\Longleftrightarrow$ (ii): This is Theorem 1.

(i) $\Longleftrightarrow$ (vii): From the proof of Theorem 1 we have that (i) is equivalent to (6) which in turn is equivalent to this: For each q in R^n there exists a positive number ε such that

$$A^T u + C^T v = p + \varepsilon q, \; v^T(C\bar{x} - d) = 0, \; v \geqq 0$$

has a solution $(u, v) \in R^{m+k}$. This is equivalent to (vii) because it follows from $C\bar{x} - d \geqq 0$ and $v \geqq 0$ that $v_i = 0$ for $i \notin J$, $i \in \{1, 2, \ldots, k\}$.

(vii) $\Rightarrow$ (iv): Let $x \neq 0$ satisfy $Ax = 0$, $C_K x = 0$, $C_L x \geqq 0$ and we shall exhibit a contradition. Let $q = -x$ in (vii) and let $(\bar{u}, \bar{v})$ be any solution to (2). Then (vii) and $p = A^T\bar{u} + C^T\bar{v} = A^T\bar{u} + C_J^T\bar{v}_J$ give

$$A^T u + C_J^T v_J = A^T\bar{u} + C_J^T\bar{v}_J - \varepsilon x, \; v_J \geqq 0$$

Premultiplication by x^T and rearranging gives the contradition

$$0 > -\varepsilon x^T x = (u-\bar{u})^T Ax + (v_J-\bar{v}_J)^T C_J x$$

$$= (u-\bar{u})^T Ax + (v_K-\bar{v}_K)^T C_K x + v_L^T C_L x \geqq 0$$

(iv) $\Rightarrow$ (i): We shall assume that (i) does not hold and shall exhibit a contradition. Let $\hat{x} \neq \bar{x}$ be another solution of (1). Because the solution set of (1) is convex it follows that for $0 < \lambda \leqq 1$, $(1-\lambda)\bar{x} + \lambda\hat{x}$ also solves (1). Hence for $0 < \lambda \leqq 1$

$$p^T\bar{x} = p^T((1-\lambda)\bar{x}+\lambda\hat{x}),\ A((1-\lambda)\bar{x}+\lambda\hat{x}) = b = A\bar{x},$$

$$C_J((1-\lambda)\bar{x}+\lambda\hat{x}) \geqq d_J = C_J\bar{x}$$

Consequently

$$p^T(\hat{x}-\bar{x}) = 0,\ A(\hat{x}-\bar{x}) = 0,\ C_J(\hat{x}-\bar{x}) \geqq 0,\ \hat{x} - \bar{x} \neq 0$$

If $C_K(\hat{x}-\bar{x}) = 0$, or if K is empty we have a contradition to (iv). Suppose now that K is nonempty, that $C_i(\hat{x}-\bar{x}) > 0$ for at least one i in K and that $(\bar{u},\bar{v})$ is a solution of (2). Then we have the contradition

$$0 = p^T(\hat{x}-\bar{x}) = (\bar{u}^T A+\bar{v}^T C)(\hat{x}-\bar{x}) = \bar{v}_K^T C_K(\hat{x}-\bar{x}) > 0.$$

(iv) $\Rightarrow$ (v): We shall prove the contrapositive implication. If the rows of $[A^T\ C_K^T\ C_L^T]$ are linearly dependent, then there exists an $x \neq 0$ such that $Ax = 0$, $C_K x = 0$, $C_L x = 0$ which is a negation of (iv). If there exists an x satisfying $Ax = 0$, $C_K x = 0$, $C_L x \geq 0$, then $x \neq 0$, and again we have a negation of (iv).

(v) $\Leftrightarrow$ (x): This follows from Tucker's theorem of the alternative [3].

(x) $\Rightarrow$ (ix): Let (x) be satisfied and let s be any point in R^n. Since the rows of $[A^T\ C_K^T\ C_L^T]$ are linearly independent, there exist $u(s)$, $v_K(s)$, $v_L(s)$ such that

$$A^T u(s) + C_K^T v_K(s) + C_L^T v_L(s) = s$$

By using the u, v_K, v_L of (x) we have that for a sufficiently large positive number $\lambda(s)$

$$v_L(s) + \lambda(s) v_L > 0$$

and

$$A^T(u(s)+\lambda(s)u) + C_K^T(v_K(s)+\lambda(s)v_K) + C_L^T(v_L(s)+\lambda(s)v_L) = s$$

Hence (ix) is satisfied.

(ix) $\Rightarrow$ (viii): Obvious.

(viii) $\Rightarrow$ (iv): If (iv) does not hold then for some $x \neq 0$, $Ax = 0$, $C_K x = 0$ and $C_L x \geqq 0$. By picking s in (viii) equal to $-x$ and premultiplying the equality of (viii) by x^T we get the contradiction

$$0 > -x^T x = x^T A^T u + x^T C_K^T v_K + x^T C_L^T v_L \geqq 0$$

(viii) $\Leftrightarrow$ (vi):

(viii) $\Leftrightarrow$ $\{u, v_K, v_L \mid A^T u + C_K^T v_K + C_L^T v_L = s, v_L \geqq 0\} \neq \emptyset$ for each $s \in R^n$

$$\Leftrightarrow \max_{u, v_K, v_L} \{a^T u + c^T v_K + h^T v_L \mid A^T u + C_K^T v_K + C_L^T v_L = s, v_L \geqq 0\}$$

has a solution for each $s \in R^n$ and for each a, c, h for which the set $\{x | Ax=a, C_K x=c, C_L x \geqq h\} \neq \emptyset$. (By linear programming duality [1,3])

$\iff$ $\underset{x}{\text{Min}} \{s^T x | Ax=a, C_K x=c, C_L x \geqq h\}$ has a solution for each $s \in R^n$ and for each a, c, h for which the set $\{x | Ax=a, C_K x=c, C_L x \geqq h\} \neq \emptyset$. (By linear programming duality [1,3])

$\iff$ $\{x | Ax=a, C_K x=c, C_L x \geqq h\}$ is bounded if it is nonempty

$\iff$ (vi).

(iii) $\iff$ (vii): Condition (iii) is equivalent to this: For each s in R^n there exists no x satisfying

$$Ax = 0, \; C_J x \geqq 0, \; -p^T x \geqq 0, \; -s^T x > 0$$

which in turn is equivalent, by Motzkin's Theorem [3], to the existence of u, v_J, ξ, η satisfying

$$A^T u + C_J^T v_J - p\xi - s\eta = 0, \;\; v_J, \xi, \eta \geqq 0, \; \eta > 0$$

Normalization with respect to η and letting $s = q + p$ gives that for all q in R^n there exists u, v_J, ξ satisfying

$$A^T u + C_J^T v_J = p(1+\xi) + q, \;\; v_J, \xi \geqq 0$$

Normalization with respect to $1 + \xi$ and defining $\varepsilon = \frac{1}{1+\xi} > 0$, gives (vii). □

Remark 1 Just as conditions (v), (vi), (viii), (ix) and (x) are all derivable from condition (iv), which utilizes the index sets K and L, similar conditions can also be derived from (iii) which utilizes the index set J only. Because of this similarity and to avoid repitition of the obvious we refrain from listing or deriving these conditions.

Remark 2 The easiest way to verify computationally the uniqueness of the linear programming solution $\bar{x}$ is probably to paraphrase conditions (v) or (x) of Theorem 2 above as another linear program as follows:

(v') The rows of $[A^T\ C_K^T\ C_L^T]$ are linearly independent and the linear program

$$\text{Maximize } \{e^T C_L x \mid Ax=0,\ C_K x=0,\ C_L x \geqq 0\}$$

has a zero maximum.

(x') The rows of $[A^T\ C_K^T\ C_L^T]$ are linearly independent and the linear program

$$\text{Maximize } \{\delta \mid A^T u + C_K^T v_K + C_L^T v_L = 0,\ v_L \geqq e\delta\}$$

is unbounded above.

We state now a similar uniqueness result for the dual problem (2) but omit its proof.

Corollary 2 Let $(\bar{u},\bar{v})$ be a solution of the dual problem (2). The following statements are equivalent:

(i) $(\bar{u},\bar{v})$ is unique.

(ii) For each $g,h\in R^{m+k}$ there exists a positive real number ε such that $(\bar{u},\bar{v})$ remains dual optimal for (7).

(iii) There exists no (u,v) satisfying

$$A^T u + C^T v = 0,\ b^T u + d^T v \geqq 0,\ v_{\tilde{K}} \geqq 0,\ (u,v) \neq 0$$

where $\tilde{K}$ is the complement of K in $\{1,2,\ldots,k\}$.

(iv) There exists no (u,v_K,v_L) satisfying

$$A^T u + C_K^T v_K + C_L^T v_L = 0,\ v_L \geqq 0,\ (u,v_K,v_L) \neq 0$$

(v) The rows of $\begin{pmatrix} A \\ C_K \end{pmatrix}$ are linearly independent and there is no (u,v_K,v_L) satisfying

$$A^T u + C_K^T v_K + C_L^T v_L = 0,\ v_L \geq 0$$

(vi) For each $q\in R^n$ the set

$$\{u,v_K,v_L \mid A^T u + C_K^T v_K + C_L^T v_L = q,\ v_L \geqq 0\}$$

is empty or bounded.

(vii) For each $g,h\in R^{m+k}$ there exists a positive number ε such that the system

$$Ax = b + \varepsilon g,\ C_K x = d_K + \varepsilon h_K,\ C_{\tilde{K}} x \geqq d_{\tilde{K}} + \varepsilon h_{\tilde{K}}$$

has a solution x, where $\tilde{K}$ is the complement of K in $\{1,2,\ldots k\}$.

(viii) For each r,s,t the system

$$Ax = r,\ C_K x = s,\ C_L x \geqq t$$

has a solution x.

(ix) For each r,s the system

$$Ax = r,\ C_K x = s,\ C_L x > 0$$

has a solution x.

(x) The rows of $\begin{pmatrix} A \\ C_K \end{pmatrix}$ are linearly independent and the system

$$Ax = 0,\ C_K x = 0,\ C_L x > 0$$

has a solution x.

Remark 3 It is easy to verify that any of the conditions of Theorem 1 imply regularity of the constraints of the dual problem (2) in the sense of Robinson [5,6] and that any of the conditions of Corollary 2 imply regularity of the constraints of the primal problem (1). It thus follows that if any of the conditions of Theorem 2 holds and that if any of the conditions of Corollary 2 holds then both problems (1) and (2) are stable in the sense of Robinson [7, Theorem 1] for sufficiently small perturbation of p,A,b,C,d.

Remark 4 When Theorem 1 holds then for each q in R^n, $\bar{x}$ solves the linear program (3) for both ε set to zero and to some positive value $\bar{\varepsilon}$, say, which depends on q. It is easy to show that for a given q, $\bar{x}$ also solves (3) for all values of ε in the closed interval $[0,\bar{\varepsilon}]$. For if we let X denote the feasible region of (3) then for ε in $[0,\bar{\varepsilon}]$, a solution to the linear program (3) exists because for all x in X the objective function of (3) is bounded below as follows

$$\begin{aligned}(p+\varepsilon q)^T x &= (1-\varepsilon/\bar{\varepsilon})\ p^T x + (\varepsilon/\bar{\varepsilon})(p+\bar{\varepsilon} q)^T x \\ &\geqq (1-\varepsilon/\bar{\varepsilon})\ p^T \bar{x} + (\varepsilon/\bar{\varepsilon})(p+\bar{\varepsilon} q)^T \bar{x} \\ &= (p+\varepsilon q)^T \bar{x}\end{aligned}$$

Consequently

$$\min_{x\in X}\ (p+\varepsilon q)^T x \geqq (p+\varepsilon q)^T \bar{x}$$

But because $\bar{x}$ is in X

$$(p+\varepsilon q^T)\ \bar{x} \geqq \min_{x\in X}\ (p+\varepsilon q)^T x$$

These last inequalities give the desired result that

$$(p+\varepsilon q^T)\ \bar{x} = \min_{x\in X}\ (p+\varepsilon q)^T x \qquad \text{for all } \varepsilon\in[0,\bar{\varepsilon}]$$

By taking $q = \nabla f(\bar{x})$ where f is any numerical function on R^n which is differentiable at $\bar{x}$ it follows from the above that if $\bar{x}$ is a unique solution of (1) then there exists a positive $\bar{\varepsilon}$ such that for all ε in $[0,\bar{\varepsilon}]$, $\bar{x}$ satisfies the Karush-Kuhn-Tucker conditions [3] for: $\min_{x\in X} p^T x + \varepsilon f(x)$. If in addition f is pseudoconvex or convex at $\bar{x}$ (or on R^n) then $\bar{x}$ solves: $\min_{x\in X} p^T x + \varepsilon f(x)$ for all ε in $[0,\bar{\varepsilon}]$.

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