CONSTRUCTING MODULI SPACES OF LOW DIMENSIONAL A_∞-ALGEBRAS BY EXTENSIONS

Josh Frinak & Austen Ott
Faculty Mentor: Michael Penkava
University of Wisconsin-Eau Claire

1. WHAT IS AN A_∞ ALGEBRA?

If $\alpha \in \mathrm{Hom}(V^k, V)$, then it determines a coderivation of $T(V)$ by

$$\alpha(v_1 \cdots v_n) = \sum_{i=0}^{n-k} (-1)^{(v_1+\cdots+v_k)\alpha} v_1 \cdots v_i \cdot \alpha(v_{i+1} \cdots v_{i+k}) \cdot v_{i+k+1} \cdots v_n.$$

Using the formula above we are able to identify the Lie algebra $\mathrm{Coder}(T(V))$ of coderivations with

$$C(V) = \mathrm{Hom}(T(V), V) = \bigoplus_{k=0}^{\infty} \mathrm{Hom}(V^k, V) = \bigoplus_{k=0}^{\infty} C^k(V).$$

As a consequence, any coderivation δ has a decomposition as a power series $\delta = \sum_{k=0}^{\infty} \delta_k$. An odd coderivation $d = d_1 + d_2 + \cdots$ such that $[d, d] = 0$ is called an A_∞ algebra structure on V.

Define the coboundary operator on an A_∞ structure d by $D(\varphi) = [d, \varphi]$ for any $\varphi \in C(V)$. Then $D^2 = 0$, so we can define the cohomology $H(d)$ by

$$H(D) = \ker(D)/\operatorname{Im}(D).$$

When d consists of a single term, we can refine this cohomology to define $H^n(d)$ for all natural numbers n.

2. EXTENSIONS OF A VECTOR SPACE

What is an Extension?

An extension of an algebra W by an algebra M is represented by a *short exact sequence*

$$0 \to M \to V \to W \to 0.$$

where V is the vector space $V = M \oplus W$, equipped with some algebra structure.

In the language of algebra, we have M is an ideal in V, and $W = V/M$ is the quotient algebra. Why we are interested in this construction is that we want to determine the moduli space of all algebras on V by looking at the moduli spaces of algebras on smaller dimensional spaces.

If V has a proper nontrivial ideal M, then we can use the idea of extensions to express V as an extension of the algebra $W = V/M$ by M. Thus, we also have to understand the case when V has no proper nontrivial ideals. In this case we say V is simple.

When V is $\mathbb{Z}_2$-graded we require that ideals be graded, meaning it is a graded subspace of a vector space V.

3. THE MODULI SPACE OF A_∞ ALGEBRAS

An invertible even linear map $\lambda : V \to V$ extends in a natural way to a coalgebra automorphism of $T(V)$. Moreover, if $\alpha_k \in C^k(V)$ for $k > 1$, then $\exp(\alpha)$ is always defined. An arbitrary coalgebra morphism g can be written in the form $g = \lambda \exp(\alpha_2) \exp(\alpha_3) \cdots$, where $\lambda \in \mathbb{GL}(V)$ and $\alpha_k \in C^k(V)$. Then

$$g^* = (\prod_{k=\infty}^{2} \exp(-\operatorname{ad}_{\alpha_k}))\lambda^*.$$

The important fact about the above formula is that it is computable!

We say that d and d' are *equivalent* A_∞ algebra structures if there is a coalgebra automorphism g of the tensor coalgebra such that $g^*(d') = d$, and write $d' \sim d$.

Theorem 1 *Suppose that*

$$d = d_k + d_{k+1} + \cdots$$

$$d' = d'_l + d'_{l+1} + \cdots .$$

Then $k = l$ and there is a linear automorphism λ of $T(V)$ such that $d' = \lambda^*(d)$.

Because of this theorem, we know the first step in classifying the A_∞ algebras is to classify all nonequivalent A_∞ algebras consisting of a single term $d_k \in C^k$.

4. THREE TYPES OF DEFORMATION PROBLEMS

We give three types of deformation problems which have the same formal solution.

1. Formal deformations of an algebra. Let $\delta_i = t^i \psi_i$, where $\psi_i \in Hom(T(V), V)$.
2. Extension of a degree n codifferential to a multiterm A_∞ algebra. Let $\delta_i \in \mathrm{Hom}(T^i(V), V)$ for $i \geq n$.
3. Extending a degree n codifferential $\delta \in \mathrm{Hom}(T^n(W), W)$ by a degree n codifferential $\mu \in \mathrm{Hom}(T^n(M), W)$ to a degree n codifferential on $V = M \oplus W$. Let $\delta_0 = \mu$, $\delta_{n-1} = \lambda^{n-1} + \delta$ and $\delta_i = \lambda^i$ otherwise, where

$$\lambda^i \in \mathrm{Hom}(T^{n-i,i}, M),$$

$$T^{0,0} = \mathbb{K}$$
$$T^{k,l} = M \otimes T^{k-1,l} \oplus W \otimes T^{k,l-1}, \quad k, l \geq 0.$$

Then if $d = \delta_0 + \delta + 1 + \cdots$, $[d, d] = 0$ is equivalent to the sequence of equations $\xi_i = 0$, $i = 0, \ldots,$ where

$$\xi_n = [\delta_n, \delta_0] + [\delta_{n-1}, \delta_1] + \cdots + [\delta_0, \delta_n].$$

The third problem is the one we are interested in solving, and we note that it only has $n + 1$ nonzero terms, $\delta_0, \ldots \delta_n$.

5. A DESCENDING SEQUENCE OF COHOMOLOGY

The set of equations $\xi_k = 0$ give a sequence of cohomology operators D_k, each defined on the previous cohomology space H_{k-1}. Define

$$D_0(\varphi) = [\delta_0, \varphi], \quad H_0 = \ker D_0/\operatorname{Im} D_0.$$

Since $D_0^2 = 0$, H_0 is well defined, and since

$$D_0([\varphi, \psi]) = [D_0\varphi, \psi] + (-1)^{\varphi}[\varphi, D_0\psi],$$

H_0 is a graded Lie algebra. We proved the following theorem:

Theorem 2 *An element φ gives rise to a cohomology class $[\varphi]_n \in H_n$ precisely when there are elements $\varphi_1, \ldots, ph_n$, such that the sequence of equations*

$$[\delta_0, \varphi] = 0, \quad [\delta_1, \varphi] + [\delta_0, \varphi_1] = 0, \quad \ldots [\delta_n, \varphi] + [\delta_{n-1}, \varphi_1] \cdots + [\delta_0, \varphi_n] = 0.$$

Then the map $D_{n+1} : H_n \to H_n$, given by $D_{n+1}([\varphi]_n) = [\delta_{n+1}, \varphi] + \cdots [\delta_1, \varphi_n]]_n$ is well defined, satisfies $D_{n+1}^2 = 0$, and the cohomology $H_{n+1} = \ker D_{n+1}/\operatorname{Im} D_{n+1}$ has the structure of a graded Lie algebra. Moreover $[\delta_1]_n$ is well defined for all n.

The theorem above plays a role in the calculation of extensions of A_∞ algebras. In the next slide, we give an example of this construction.

6. A_∞ ALGEBRAS OF DEGREE 3 ON A $1|1$ DIMENSIONAL SPACE

Codifferential	H^0	H^1	H^2	H^3
$d_1 = \psi_1^{211} + \psi_1^{222} + \psi_2^{212} + \psi_2^{221}$	$0\|0$	$0\|1$	$0\|0$	$0\|0$
$d_2 = -\psi_1^{112} + \psi_1^{222} + \psi_2^{122} + \psi_2^{212}$	$0\|0$	$0\|1$	$0\|0$	$0\|0$
$d_3 = -\psi_1^{121} + \psi_1^{211} + \psi_2^{111} + \psi_2^{221}$	$0\|1$	$0\|1$	$0\|1$	$0\|0$
$d_4 = -\psi_1^{112} + \psi_1^{121} + \psi_2^{111} + \psi_2^{122}$	$0\|1$	$0\|1$	$0\|1$	$0\|0$
$d_5 = \psi_2^{212}$	$0\|0$	$1\|0$	$1\|0$	$1\|0$
$d_6 = \psi_1^{121}$	$0\|0$	$1\|0$	$0\|1$	$1\|0$
$d_7 = \psi_1^{211} + \psi_2^{2121} + \psi_2^{221}$	$0\|0$	$1\|1$	$0\|0$	$1\|1$
$d_8 = -\psi_1^{112} + \psi_2^{122} + \psi_2^{212}$	$0\|0$	$1\|1$	$0\|0$	$1\|1$
$d_9 = -\psi_1^{121} + \psi_1^{211} + \psi_2^{221}$	$0\|0$	$1\|1$	$1\|1$	$1\|1$
$d_{10} = -\psi_1^{112} + \psi_1^{121} + \psi_2^{122}$	$0\|0$	$1\|1$	$1\|1$	$1\|1$
$d_{11} = \psi_2^{111}$	$0\|1$	$1\|1$	$2\|1$	$2\|2$
$d_{12} = \psi_1^{222}$	$0\|1$	$1\|1$	$0\|2$	$2\|2$

The codifferentials d_5, d_6 d_{11} and d_{12} arise as extensions. We have jump deformations $d_7 \rightsquigarrow d_1$, $d_8 \rightsquigarrow d_2$, $d_{12} \rightsquigarrow d_1, d_2$, $d_9 \rightsquigarrow d_3$, $d_{10} \rightsquigarrow d_4$, and $d_{11} \rightsquigarrow d_3, d_4$. Note that unlike the associative case, some of the simple algebras (those which are not extensions) have nontrivial deformations!

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