From Special Relativity to Gyrogroups

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ABSTRACT

We will start with special relativity and derive the velocity addition formula and the Lorentz Transformation in one spatial dimension, and we show that velocity addition yields a group law. We will then move into higher spatial dimensions and prove that relativistic velocity addition, although not a group operation, yields a gyrogroup.

Special Relativity

Definition 1. An *inertial observer* is an observer with no external forces acting on the observer.

Definition 2. An *inertial frame of reference* is a frame of reference of an inertial observer where isolated bodies are seen to move in straight lines at constant velocities.

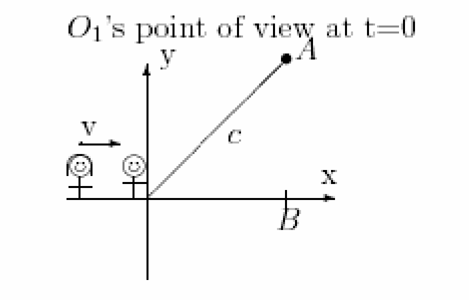
Definition 3. $\mathbb{R}^n$ is called *space-time* where $n-1$ is the dimension of space. An n-tuple in space-time is called an *event*}. We assume the n^{th} coordinate of an event represents time.

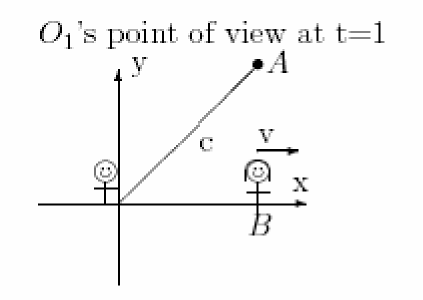
To derive special relativity, we will use the following assumptions:

1. Newton's laws are the same for all inertial observers.
2. All inertial observers see light move at speed *c*.

First, let's assume that each inertial observer measures time and distance with the same units and that they pass each other when $t=0$. We will usually denote distance as *d* and time as *t*. We will also assume they agree what a straight line is, as well as what horizontal and vertical lines are at the moment they pass each other.

For our diagrams, let's assume O_1 is bald and O_2 has hair. Now, let's assume O_1 and O_2 meet at $t=0$, the path traveled is the x-axis, O_1 sees O_2 move at velocity *v*, and O_2 sees O_1 move at velocity -*v*. Let's also assume that there is a rock at point *A*, which lies directly above point *B* and a distance *c* from the origin. Suppose now that at $t=0$ a flashlight is shone towards *A* from the origin. Then the light hits the rock at $t=1$, and at $t=1$ O_2 is at *B*.





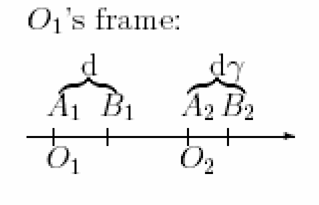
Now for $O_2's$ point of view. O_2 thinks light is moving straight up. So, how far is it from *A* to *B*? Using the Pythagorean Theorem gives $\sqrt{c^2-v^2}=c\sqrt{1-v^2/c^2}$.

Let *d* be the distance from O_2 to B at $t=0$, according to O_2, and let τ be the time when *B* reaches O_2, according to O_2. So, since $d=r\tau$, $d=\frac{c\sqrt{1-v^2/c^2}}{c}=\sqrt{1-v^2/c^2}$. This means $d=v\sqrt{1-v^2/c^2}$. So to an observer in motion, the distance changes.

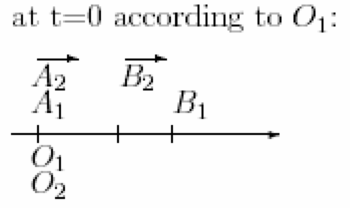
Definition 4.

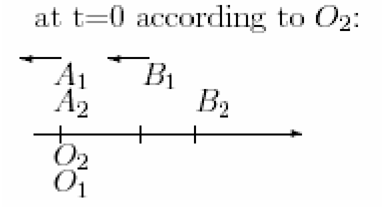
$$\gamma_v=\gamma=\sqrt{1-\frac{v^2}{c^2}}$$

In $\mathbb{R}$, given two inertial observers where one is moving at velocity v with respect to the other, γ is the factor by which time and space are contracted.



To generalize, we need four physical markers, two in each observer's frame. Let's imagine markers A_1 and A_2 at the position of O_1 and O_2 respectively. We will use B_1 and B_2 to represent markers a distance *d* from A_1 and A_2 respectively. More precisely, according to O_1, A_1 and B_1 are a distance *d* apart, and they are stationary. According to O_2, A_2 and B_2 are a distance *d* apart and are stationary.





Let E_1 be the event A_1 and A_2 meet. Let E_2 be the event B_1 and B_2 meet. According to O_1, E_1 happens before E_2. But, according to O_2, E_2 happens before E_1. *It's backwards.*

Let the coordinates of E_i with respect to O_j be $E_{i,j}$. So,

- $E_{1,1} = (0,0)$
- $E_{1,2} = (0,0)$
- $E_{2,1} = \left(d, \dfrac{d - d_\gamma}{v} \right)$

- $E_{2,2} = \left(d, \frac{d_{\gamma} - d}{v} \right)$

We're assuming for now that space is 1-dimensional, so an event is a point in $\mathbb{R}^2$. The translation of the event coordinates from O_1's frame to O_2's frame is a function $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2$.

Now, let's place our markers B_1 and B_2 strategically to determine how the coordinates of a general event change from one reference frame to another. So, given $(x,t) \in \mathbb{R}$ that is the coordinates of our event E_2 according to O_1, we want to find the coordinates of E_2 according to O_2. So, let $d_1 = x$ be the distance from A_1 to B_1 according to O_1. Let d_2 denote the distance from A_2 to B_2 according to O_2. Now, we know the time coordinate must be the distance traveled divided by the velocity, and we know the distance between A_2 and B_2 is $d_2\gamma$ according to O_1. Then $t = \frac{d_1 - d_2\gamma}{v}$ so $d_2 = \frac{x - vt}{\gamma}$. We also know the distance from A_1 to B_1 according to O_2 is $d_1\gamma$. We can use these distances to find the coordinates of our event according to O_2, and we can see

$$\begin{aligned} E_{2,2} &= \left(d_2, \frac{\gamma d_1 - d_2}{v} \right) \\ &= \left(\frac{x - vt}{\gamma}, \frac{\gamma x - \left(\frac{x - vt}{\gamma} \right)}{v} \right) \end{aligned}$$

$$\left(\frac{x-vt}{\gamma},\frac{\gamma x-\left(\frac{x-vt}{\gamma}\right)}{v}\right)=\left(\frac{x-vt}{\gamma},\frac{\gamma^2 x-x+vt}{v\gamma}\right)$$

$$=\left(\frac{x-vt}{\gamma},\frac{(1-v^2/c^2)x-x+vt}{v\gamma}\right)$$

$$=\left(\frac{x-vt}{\gamma},\frac{-xv^2/c^2+vt}{v\gamma}\right)$$

$$=\left(\frac{x-vt}{\gamma},\frac{-xv/c^2+t}{\gamma}\right)$$

$$=\left(\frac{x-vt}{\gamma},\frac{t-xv/c^2}{\gamma}\right)$$

Definition 5. The *Lorentz Transformation,* $\mathcal{L}_v:\mathbb{R}^2\to\mathbb{R}^2$ is given by

$$(x,t)\mapsto\left(\frac{x-vt}{\gamma},\frac{t-xv/c^2}{\gamma}\right)$$

Theorem 1. $\mathcal{L}_v^{-1}=\mathcal{L}_{-v}$ where $v\in\mathbb{R}$ and $-c<v<c$.

Proof. Applying $\mathcal{L}_v$ gives

$$(x,t)\mapsto\left(\frac{x-vt}{\gamma},\frac{t-xv/c^2}{\gamma}\right)$$

We now apply $\mathcal{L}_{-v}$, which gives

$$\left(\frac{1}{\gamma}\left(\frac{x-vt}{\gamma}\right)+\frac{v}{\gamma}\left(\frac{t-xv/c^2}{\gamma}\right),\frac{1}{\gamma}\left(\frac{t-xv/c^2}{\gamma}+\left(\frac{x-vt}{\gamma}\right)\frac{v}{c^2}\right)\right)=\left(\frac{x-xv^2/c^2}{\gamma^2},\frac{t-tv^2/c^2}{\gamma^2}\right)$$

$$=\left(\frac{x\gamma^2}{\gamma^2},\frac{t\gamma^2}{\gamma^2}\right)$$

$$=(x,t)$$

Since $\mathcal{L}_v$ is a linear function and has a left inverse $\mathcal{L}_{-v}$, the right inverse must also be $\mathcal{L}_{-v}$. □

Definition 6. The *worldline* of an object or person is $L = \{x(t), y(t), z(t) \mid t \in \mathbb{R}\}$ where $\big(x(t), y(t), z(t)\big)$ is the parameterization of the path of the object or person with respect to time *t*.

In our notation, when O_i and O_j are inertial observers, $L_{i,j}$ will represent the worldline of O_i according to O_j.

Now suppose we have 3 inertial observers, O_1, O_2, and O_3. Suppose O_2 is moving at velocity ***v*** with respect to O_1 and that O_3 is moving at velocity ***w*** with respect to O_2 in the same direction. How fast is O_3 moving with respect to O_1? (Note that we are taught at an early age the answer is ***v+w***, but we will see this is not true.)

Assume all three observers met at *t=0*. The worldline of O_3 according to O_2 is $L_{3,2} = \{(wt, t) \mid t \in \mathbb{R}\}$. Applying the Lorentz Transformation $\mathcal{L}_{-v}$ to a general point on $L_{3,2}$ will give a general point on the worldline $L_{3,1}$.

$$\begin{aligned}\mathcal{L}_{-v}(wt,t) &= \left(\frac{wt+vt}{\gamma_{-v}}, \frac{t-wt(-v)/c^2}{\gamma_{-v}}\right)\\ &= \left(\frac{t(w+v)}{\gamma_{-v}}, \frac{t(1+wv/c^2)}{\gamma_{-v}}\right)\end{aligned}$$

So $L_{3,1} = \left\{\left(\frac{t(w+v)}{\gamma_{-v}}, \frac{t(1+wv/c^2)}{\gamma_{-v}}\right) \mid t \in \mathbb{R}\right\}$.

Since *d=rt*, the velocity of O_3 with respect to O_1 is

$$\begin{aligned}r &= \frac{d}{t}\\ &= \frac{\dfrac{t(w+v)}{\gamma_{-v}}}{\dfrac{t(1+wv/c^2)}{\gamma_{-v}}}\\ &= \frac{t(w+v)}{t(1+wv/c^2)}\\ &= \frac{w+v}{1+wv/c^2}\end{aligned}$$

Definition 7. Let $G = \{v \in \mathbb{R} \mid -c < v < c\}$ and define $\oplus : G \times G \to \mathbb{R}$,

$$v \oplus w = \frac{w+v}{1+\frac{wv}{c^2}}$$

Theorem 2. G is closed under $\oplus$.

Proof. Fix $w \in G$. Then,

$$\frac{\partial}{\partial v}\frac{w+v}{1+\frac{wv}{c^2}} = \frac{1+\frac{wv}{c^2}-(w+v)\frac{w}{c^2}}{\left(1+\frac{wv}{c^2}\right)^2}$$

$$= \frac{1+\frac{wv}{c^2}-\frac{w^2}{c^2}-\frac{wv}{c^2}}{\left(1+\frac{wv}{c^2}\right)^2}$$

$$= \frac{1+-\frac{w^2}{c^2}}{\left(1+\frac{wv}{c^2}\right)^2}$$

For any $w, v \in G$, $1+\frac{wv}{c^2} > 0$, and $1+-\frac{w^2}{c^2} > 0$, so $\frac{\partial}{\partial v}\frac{w+v}{1+\frac{wv}{c^2}} > 0$.

$$\lim_{v\to c}\frac{w+v}{1+\frac{wv}{c^2}} = \frac{w+c}{1+\frac{wc}{c^2}}$$

$$= \frac{w+c}{1+w/c}$$

$$= \frac{c(w+c)}{w+c}$$

$$= c$$

$$\begin{aligned}\lim_{v\to -c}\frac{w+v}{1+\frac{wv}{c^2}} &= \frac{w-c}{1+\frac{w(-c)}{c^2}}\\ &= \frac{w-c}{1-w/c}\\ &= \frac{-c(c-w)}{c-w}\\ &= -c\end{aligned}$$

Since $\frac{w+v}{1+\frac{wv}{c^2}}$ is increasing as a function of *v*, and the limits are *c* and -*c*, $v\oplus w\in G$. □

As an illustration of time-dilation, let's take the above worldline of $L_{3,2}=\{(wt,t)\,|\,t\in\mathbb{R}\}$ and apply the Lorentz Transformation $\mathcal{L}_w$ to find O_3's worldline according to O_3. Because of the way we set up our Lorentz Transformations, we expect the x-coordinate in his worldline should be 0.

$$\begin{aligned}\mathcal{L}_w(wt,t) &= \left(\frac{wt-wt}{\gamma_w},\frac{t-wtw/c^2}{\gamma_w}\right)\\ &= \left(0,\frac{t-tw^2/c^2}{\gamma_w}\right)\\ &= \left(0,\frac{t\left(1-w^2/c^2\right)}{\gamma_w}\right)\\ &= \left(0,\frac{t\gamma_w^2}{\gamma_w}\right)\\ &= (0,t\gamma_w)\end{aligned}$$

So, $\mathcal{L}_w(L_{3,2}) = \{(0, t\gamma_w) \mid t \in \mathbb{R}\} = \{(0,t) \mid t \in \mathbb{R}\}$. We can see that O_3's x-coordinate is indeed 0 which means that he does not see himself moving. Notice also that his time coordinate was transformed to $t\gamma_w$. The worldline of O_3 according to O_2 showed O_3's time coordinate as *t*. So, each inertial observer will record a different time coordinate for the same event. In particular, when a Lorentz Transformation is applied to an event from the worldline of an inertial observer, the time coordinates will be off by a factor of γ.

Groups

The following definition is from [1].

Definition 8. Let *G* be a nonempty set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a,b) of elements of *G* an element in *G* denoted by *ab*. We say *G* is a *group* under this operation if the following three properties are satisfied.

1. *Associativity*. The operation is associative; that is, $\forall a,b,c \in G$, $(ab)c = a(bc)$.
2. *Identity*. There is an element *e* (called the *identity*) in *G* such that $\forall a \in G$, $ae = ea = a$.
3. *Inverses*. For each element *a* in *G*, there is an element *b* in *G* (called an *inverse* of *a*) such that $ab = ba = e$.

Theorem 3. Velocity addition from Definition 7 forms a group.

Proof.
(Associative law)

$$
\begin{aligned}
(v\oplus w)\oplus z &= \left(\frac{w+v}{1+\frac{wv}{c^2}}\right)\oplus z \\
&= \frac{z+\frac{w+v}{1+\frac{wv}{c^2}}}{1+\frac{z}{c^2}\left(\frac{w+v}{1+\frac{wv}{c^2}}\right)} \\
&= \frac{\frac{z\left(1+vw/c^2\right)+w+v}{1+wv/c^2}}{\frac{z(w+v)+c^2\left(1+vw/c^2\right)}{c^2\left(1+vw/c^2\right)}} \\
&= \frac{z\left(1+vw/c^2\right)+w+v}{\left(z(w+v)+c^2\left(1+vw/c^2\right)\right)/c^2} \\
&= \frac{z+zwv/c^2+w+v}{\left(zw+zv+c^2+vw\right)/c^2} \\
&= \frac{v\left(1+zw/c^2\right)+z+w}{\left(v(z+w)+c^2+zw\right)/c^2}
\end{aligned}
$$

$$\frac{v\left(1+zw/c^2\right)+z+w}{\left(v\left(z+w\right)+c^2+zw\right)/c^2}=\frac{\dfrac{v\left(1+zw/c^2\right)}{1+zw/c^2}+\dfrac{z+w}{1+zw/c^2}}{\dfrac{v+w}{c^2\left(1+zw/c^2\right)}+\dfrac{c^2+zw}{c^2\left(1+zw/c^2\right)}}$$

$$=\frac{v+\dfrac{z+w}{1+zw/c^2}}{1+\dfrac{v}{c^2}\left(\dfrac{z+w}{1+zw/c^2}\right)}$$

$$=v\oplus\left(\frac{z+w}{1+zw/c^2}\right)$$

$$=v\oplus\left(w\oplus z\right)$$

(Identity Element)

$$v\oplus 0=\frac{0+v}{1+\dfrac{0v}{c^2}}$$

$$=\frac{v}{1}$$

$$=v$$

$$0\oplus v=\frac{v+0}{1+\dfrac{v0}{c^2}}$$

$$=\frac{v}{1}$$

$$=v$$

(Inverses)

$$v\oplus(-v)=\frac{-v+v}{1+v(-v)/c^2}$$

$$=\frac{0}{1-v^2/c^2}$$

$$=0$$

$$\begin{aligned} -v \oplus v &= \frac{v+(-v)}{1+(-v)v/c^2} \\ &= \frac{0}{1-v^2/c^2} \\ &= 0 \end{aligned}$$

Velocity addition (in $\mathbb{R}$) is associative, has an identity (namely 0), and each element *v* has an inverse (namely -*v*) so it does form a group.

We note that $\mathcal{L}_v \mathcal{L}_w = \mathcal{L}_{v\oplus w}$ and we know that the composition of functions is associative, so this provides a simpler way to show associativity of velocity addition. □

Definition 9. The *Special Orthogonal Group* (the set of all n-dimensional rotation matrices) is

$$SO_n(\mathbb{R}) = \left\{ R \in M_{n\times n}(\mathbb{R}) \mid R^T R = I_n, \det(R) = 1 \right\}$$

Theorem 4. (SO_n ,matrix multiplication) forms a group.

Proof. Let $A, B, C \in SO_n$.

(Closure)

$$\begin{aligned} (AB)^T (AB) &= B^T A^T AB \\ &= B^T IB \\ &= I \end{aligned}$$

(Associativity)

It is a well known fact that matrix multiplication is associative.

(Identity)

Using the identity matrix I_n , $I_n^T I_n = I_n I_n = I_n$, and $\det(I_n) = 1$ so $I_n \in SO_n$.

$$AI_n = I_n A = A$$

(Inverse)

$\det(A) = 1$ so A^{-1} exists.

$$\begin{aligned} A^T A &= I_n \\ (A^T)^{-1} A^T A &= (A^T)^{-1} I_n \\ A &= (A^T)^{-1} I_n \\ AA^{-1} &= (A^T)^{-1} I_n A^{-1} \\ I_n &= (A^T)^{-1} I_n A^{-1} \\ I_n &= (A^T)^{-1} A^{-1} \\ I_n &= (A^{-1})^T A^{-1} \end{aligned}$$

Additionally, $A^{-1} = A^T$. We also know $\det(A^{-1}) = \dfrac{1}{\det A} = 1$. So, $A^{-1} \in SO_n$. ▯

Definition 10. The *Lorentz Group* in $n-1$ spatial dimensions, where $n \geq 2$, is

$$L_n = \left\{ A \in M_{n\times n}(\mathbb{R}) \mid A^T V A = V, \det(A) = 1, A_{n,n} > 0 \right\}$$

where *V* is a diagonal matrix with diagonal entries of 1 except $V_{n,n} = -c^2$.

Theorem 5. (L_n, matrix multiplication) forms a group.

Proof. Let $A, B, C \in L = L_n$

(Closure)

$$\begin{aligned} (AB)^T V (AB) &= B^T A^T V A B \\ &= B^T V B \\ &= V \end{aligned}$$

Also, $\det(AB) = \det A \cdot \det B = 1 \cdot 1 = 1$. Since $B \in L$ we know that the last column must satisfy the equation

$$b_{1,n}^2 + \ldots + b_{n-1,n}^2 - c^2 b_{n,n}^2 = -c^2 \qquad (1)$$

To see if the last entry of *AB* is positive, it will help to know an equation that the last row of *A* satisfies. We know

$$\begin{aligned} A^T V A &= V \\ A^T V A V^{-1} &= I \\ V A V^{-1} &= A^{-T} \\ V A V^{-1} A^T &= I \\ A V^{-1} A^T &= V^{-1} \end{aligned}$$

So we know the last row of *A* must satisfy

$$a_{n,1}^2 + ... + a_{n,n-1}^2 - \frac{1}{c^2} a_{n,n}^2 = -\frac{1}{c^2} \qquad (2)$$

Rewriting these equations gives us $b_{1,n}^2 + ... + b_{n-1,n}^2 = c^2(b_{n,n}^2 - 1)$ and $a_{n,1}^2 + ... + a_{n,n-1}^2 = \frac{1}{c^2}(a_{n,n}^2 - 1)$. Let $\vec{a}\,' = < a_{n,1}, ..., a_{n,n-1} >$ and $\vec{b}\,' = < b_{1,n}, ..., b_{n-1,n} >$. Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\vec{a}\,'\cdot\vec{b}\,'| &\le |\vec{a}\,'||\vec{b}\,'| \\ &= \sqrt{1/c^2(a_{n,n}^2 - 1)}\sqrt{c^2(b_{n,n}^2 - 1)} \\ &= \sqrt{(a_{n,n}^2 - 1)(b_{n,n}^2 - 1)} \\ &\le \sqrt{a_{n,n}^2 b_{n,n}^2} \\ &= a_{n,n} b_{n,n} \end{aligned}$$

So, $(AB)_{n,n} > 0$ which means $AB \in L$.

(Associativity)

It is a well known fact that matrix multiplication is associative.

(Identity)

Using the matrix identity *I*, $I^T VI = IVI = V$, and $\det(I) = 1$ so $I \in L$.

$$AI = IA = A$$

(Inverse)

$\det(A) = 1$ so A^{-1} exists. Let $B = A^{-1}$.

$$\begin{aligned} A^T VA &= V \\ (A^T)^{-1} A^T VA &= (A^T)^{-1} V \\ VA &= (A^T)^{-1} V \\ VAA^{-1} &= (A^T)^{-1} VA^{-1} \\ V &= (A^T)^{-1} VA^{-1} \\ V &= (A^{-1})^T VA^{-1} \\ V &= B^T VB \end{aligned}$$

Using equation (2), we know that the last row in *A* must satisfy $a_{n,1}^2 + ... + a_{n,n-1}^2 - \frac{1}{c^2} a_{n,n}^2 = -\frac{1}{c^2}$. We also know from (2) that the last column of A^{-1} must satisfy $b_{1,n}^2+...+b_{n-1,n}^2-c^2b_{n,n}^2=-c^2$. So, let $\vec{a}\,'=<a_{n,1},...,a_{n,n-1}>$ and $\vec{b}\,'=<b_{1,n},...,b_{n-1,n}>$. Then we know

$$\begin{aligned}|\vec{a}\,'\cdot\vec{b}\,'|&\leq|\vec{a}\,'||\vec{b}\,'|\\&=\sqrt{1/c^2(a_{n,n}^2-1)}\sqrt{c^2(b_{n,n}^2-1)}\\&=\sqrt{(a_{n,n}^2-1)(b_{n,n}^2-1)}\\&\leq\sqrt{a_{n,n}^2b_{n,n}^2}\\&=a_{n,n}|b_{n,n}|\end{aligned}$$

Now, we know that $a_{n,1}b_{1,n}+...+a_{n,n-1}b_{n-1,1}+a_{n,n}b_{n,n}=1$. Knowing that $a_{n,1}b_{1,n}+...+a_{n,n-1}b_{n-1,1}\leq a_{n,n}|b_{n,n}|$ and that $a_{n,n}>0$ we see that $b_{n,n}>0$ because if $b_{n,n}\leq 0$ then

$a_{n,1}b_{1,n}+...+a_{n,n-1}b_{n-1,1}+a_{n,n}b_{n,n}$ would be at most 0. Also, we know that $\det(A^{-1})=\dfrac{1}{\det(A)}=1$, so $A^{-1}\in L$. □

Gyrogroups

To see where the first examples of gyrogroups came from, let's start by choosing $\vec{v}$ to be a velocity vector in $\mathbb{R}^3$ such that $|\vec{v}|<c$. Then we wish to define a function $\mathcal{L}_{\vec{v}}:\mathbb{R}^4\to\mathbb{R}^4$.

In our 1-dimensional case, all motion was along was the x-axis. So, in $\mathbb{R}^3$ if the motion is along the x-axis, the $\hat{i}$ component is the one that changes, while the $\hat{j}$ and $\hat{k}$ components do not change. Therefore, we have the following special cases where $v=|\vec{v}|$ and $\gamma=\gamma_v$:

- $\mathcal{L}_{v\hat{i}}:(x,y,z,t)\mapsto\left(\dfrac{x-vt}{\gamma},y,z,\dfrac{t-xv/c^2}{\gamma}\right)$
- $\mathcal{L}_{v\hat{j}}:(x,y,z,t)\mapsto\left(x,\dfrac{y-vt}{\gamma},z,\dfrac{t-yv/c^2}{\gamma}\right)$
- $\mathcal{L}_{v\hat{k}}:(x,y,z,t)\mapsto\left(x,y,\dfrac{z-vt}{\gamma},\dfrac{t-zv/c^2}{\gamma}\right)$

What if our velocity vector $\vec{v}$ points in a more general direction? Well, we know that physics does not care which way we orientate our coordinate axes. So, we could imagine that the direction of $\vec{v}$ is the direction of our positive x-axis. Well, then we have a formula for that Lorentz Transformation. It is our special case transformation in the $\hat{i}$ direction. Now that we understand the physics, we need to look at it mathematically. We can rotate any vector in $\mathbb{R}^3$ to point in any direction we wish, so, let's rotate $\vec{v}$ to point in the $\hat{i}$ direction. This rotation is equivalent to doing a change of basis of one coordinate system with respect to another. Notice that our Lorentz Transformation is a function from $\mathbb{R}^4 \to \mathbb{R}^4$, but our rotation is a function from $\mathbb{R}^3 \to \mathbb{R}^3$. We can compensate for this by having our rotation map $\mathbb{R}^4 \to \mathbb{R}^4$; we will just leave the fourth parameter, time, untouched. We will call this a *spatial rotation*.

Definition 11. Let $R_{\vec{v}} \in SO_4(\mathbb{R})$ be a spatial rotation satisfying $R_{\vec{v}}(\hat{i}) = \frac{\vec{v}}{|\vec{v}|}$. Then, the general Lorentz Transformation is a composition,

$$\mathcal{L}_{\vec{v}} = R_{\vec{v}} \mathcal{L}_{|\vec{v}|\hat{i}} R_{\vec{v}}^{-1}$$

Where $\mathcal{L}_{|\vec{v}|\hat{i}}$ is our special case Lorentz Transformation in the $\hat{i}$ direction.

Notice that even though $R_{\vec{v}}$ is not uniquely determined by $\vec{v}$, nevertheless $\mathcal{L}_{\vec{v}}$ is well defined.

As an example of our Lorentz Transformations, let's take a vector $\vec{v} = < 0, v, 0 >$ and apply our general Lorentz Transformation from definition 11. First we need a rotation that will take $\hat{i}$ to the direction of $\vec{v}$. We find that one such rotation is

$$R_{\vec{v}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $R_{\vec{v}} \in SO_4(\mathbb{R})$, we know that $R_{\vec{v}}^T = R_{\vec{v}}^{-1}$. Now,

$$\mathcal{L}_{\vec v} = R_{\vec v}\mathcal{L}_{|\vec v|\hat i}R_{\vec v}^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma} & 0 & 0 & \frac{-v}{\gamma} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v}{\gamma c^2} & 0 & 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\gamma} & 0 & 0 & \frac{-v}{\gamma} \\ 0 & 0 & 1 & 0 \\ \frac{-v}{\gamma c^2} & 0 & 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma} & 0 & \frac{-v}{\gamma} \\ 0 & 0 & 1 & 0 \\ \frac{-v}{\gamma c^2} & 0 & 0 & \frac{1}{\gamma} \end{pmatrix}$$

Now, looking at our special case $\mathcal{L}_{v\hat j}$, we see that our vector is transformed the same as when we use our general Lorentz Transformation formula.

Theorem 6. If $A \in L_3$, then $A = R_1\mathcal{L}_{v\hat i}R_2$ for some spatial rotations $R_1, R_2 \in SO_3(\mathbb{R})$ and some $v \in \mathbb{R}$.

Proof. Given $A \in L_3$, let's left multiply *A* by $R_{\vec v}$ where $\vec v$ is the first two entries of the third column of *A* and $R_{\vec v}$ rotates $\hat i$ to the direction of $\vec v$. We know that $R_{\vec v}$ must have its last row and column all zeros except the bottom right corner must have a 1 to preserve time. This gives

$$\begin{aligned} A'' &= R_{\vec{v}} A \\ &= \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \\ &= \begin{pmatrix} ? & ? & ? \\ ? & ? & 0 \\ ? & ? & ? \end{pmatrix} \end{aligned}$$

Now, in our new matrix, let's call $\vec{w}$ the first two entries of the third row of *A*. We can now right multiply A'' by the transpose of the rotation that takes $\vec{w}$ to the $\hat{i}$ direction. Because we still need to preserve time, this rotation will leave our last column untouched.

$$\begin{aligned} A' &= A'' R_{\vec{w}} \\ &= \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} ? & ? & ? \\ ? & ? & 0 \\ ? & 0 & ? \end{pmatrix} \end{aligned}$$

Let's start by naming our unknowns. This gives

$$A' = \begin{pmatrix} a & b & d \\ e & f & 0 \\ g & 0 & h \end{pmatrix}$$

Since $A, R_{\vec{v}}, R_{\vec{w}} \in L$, we know $A' \in L$, so $A'^{T} V A' = V$ where *V* is as in Definition 10. So,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c^2 \end{pmatrix} = \begin{pmatrix} a & e & g \\ b & f & 0 \\ d & 0 & h \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -c^2 \end{pmatrix} \begin{pmatrix} a & b & d \\ e & f & 0 \\ g & 0 & h \end{pmatrix}$$

$$= \begin{pmatrix} a & e & -gc^2 \\ b & f & 0 \\ d & 0 & -hc^2 \end{pmatrix} \begin{pmatrix} a & b & d \\ e & f & 0 \\ g & 0 & h \end{pmatrix}$$

$$= \begin{pmatrix} a^2+e^2-g^2c^2 & ab+ef & ad-ghc^2 \\ ab+ef & b^2+f^2 & bd \\ ad-ghc^2 & bd & d^2-h^2c^2 \end{pmatrix}$$

We also know the determinant of A' must be one. This gives the following equations.

$$a^2+e^2-g^2c^2=1 \tag{3}$$

$$ab+ef=0 \tag{4}$$

$$ad-ghc^2=0 \tag{5}$$

$$b^2+f^2=1 \tag{6}$$

$$bd=0 \tag{7}$$

$$d^2-h^2c^2=-c^2 \tag{8}$$

$$afh-beh-dfg=1 \tag{9}$$

By **Error! Reference source not found.** we know either $b=0$ or $d=0$. We will start with the case $b=0$. We know by **Error! Reference source not found.** that $f=\pm 1$ so $e=0$ by **Error! Reference source not found.**. Now, if $h=0$, then by **Error! Reference source not found.** $d^2=-c^2$ so $d\in\mathbb{C}-\mathbb{R}$, but $d\in\mathbb{R}$ so $h\neq 0$. Let $v=\frac{-d}{h}$ and let $\gamma=\frac{1}{h}$. Then $d=\frac{-v}{\gamma}$ and $h=\frac{1}{\gamma}$. By **Error! Reference source not found.** we know

$$\begin{aligned}\frac{v^2}{\gamma^2}-\frac{c^2}{\gamma^2}&=-c^2\\ v^2-c^2&=-c^2\gamma^2\\ \gamma^2&=1-\frac{v^2}{c^2}\\ \gamma&=\pm\sqrt{1-\frac{v^2}{c^2}}\end{aligned}$$

Since $h\geq 0$ by definition, $\gamma=\sqrt{1-\frac{v^2}{c^2}}$.

Solving **Error! Reference source not found.** and **Error! Reference source not found.** simultaneously gives $g=\mp\frac{v}{\gamma c^2}$ and $a=\pm\frac{1}{\gamma}$. Using our determinant equation **Error! Reference source not found.** we find the following.

$$A'=\begin{pmatrix}\pm\frac{1}{\gamma} & 0 & \frac{-v}{\gamma}\\ 0 & \pm 1 & 0\\ \mp\frac{v}{\gamma c^2} & 0 & \frac{1}{\gamma}\end{pmatrix}$$

Multiplying by a rotation matrix in $SO_3(\mathbb{R})$ we get

$$\begin{pmatrix}\pm\frac{1}{\gamma} & 0 & \frac{-v}{\gamma}\\ 0 & \pm 1 & 0\\ \mp\frac{v}{\gamma c^2} & 0 & \frac{1}{\gamma}\end{pmatrix}\begin{pmatrix}\pm 1 & 0 & 0\\ 0 & \pm 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{\gamma} & 0 & \frac{-v}{\gamma}\\ 0 & 1 & 0\\ \frac{-v}{\gamma c^2} & 0 & \frac{1}{\gamma}\end{pmatrix}=\mathcal{L}_{v\hat{i}}.$$

Now for the case when $d=0$. We know from **Error! Reference source not found.** that $h=\pm 1$ and then from **Error! Reference source not found.** we see $g=0$. But, by definition, $h\geq 0$ so $h=1$. If

$$\begin{pmatrix} a & e \\ b & f \end{pmatrix}$$

is a rotation, then we should have the following equations.

$$a^2 + e^2 = 1$$

$$ab + ef = 0$$

$$b^2 + f^2 = 1$$

$$af - be = 1$$

These correspond to **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, and **Error! Reference source not found.** respectively. This gives a matrix that is a spatial rotation matrix. □

Corollary 1. If $A \in L_3$, then $A = \mathcal{L}_{\vec{v}} R$ for some $\vec{v} \in \mathbb{R}^2, R \in SO_3$.

Proof. Let $A \in L_3$. Then

$$A = R_1 \mathcal{L}_{v\hat{i}} R_2 \text{ for some } v \in \mathbb{R}, R_1, R_2 \in SO_3(\mathbb{R}) \text{ by Theorem 6}$$

$$= (R_1 \mathcal{L}_{v\hat{i}} R_1^{-1})(R_1 R_2).$$

Let $\vec{v} = vR_1^{-1}\hat{i}$. Then we can see that $R_1(\vec{v}) = |\vec{v}|\hat{i}$ so $R_1 = R_{\vec{v}}$. So,

$$A = (R_1 \mathcal{L}_{v\hat{i}} R_1^{-1})(R_1 R_2)$$

$$= \mathcal{L}_{\vec{v}} R \text{ where } R = R_1 R_2.$$

□

Theorem 7. The decomposition $A = \mathcal{L}_{\vec{v}} R$ in Corollary 1 is unique.

Proof. Assume $\exists \vec{v}, \vec{w} \in \mathbb{R}^2, R_1, R_2 \in SO_3$ such that $\mathcal{L}_{\vec{v}} R_1 = \mathcal{L}_{\vec{w}} R_2$.
Then $\mathcal{L}_{\vec{v}} = \mathcal{L}_{\vec{w}} R_2 R_1^{-1}$. Let $R_3 = R_2 R_1^{-1}$. Then $\mathcal{L}_{\vec{v}} = \mathcal{L}_{\vec{w}} R_3$. So, we know

$$\begin{aligned}
\mathcal{L}_{\vec{v}}\begin{pmatrix}0\\0\\t\end{pmatrix} &= R_{\vec{v}}\mathcal{L}_{|\vec{v}|\hat{i}}R_{\vec{v}}^{-1}\begin{pmatrix}0\\0\\t\end{pmatrix}\\
&= R_{\vec{v}}\mathcal{L}_{|\vec{v}|\hat{i}}\begin{pmatrix}0\\0\\t\end{pmatrix}\\
&= R_{\vec{v}}\begin{pmatrix}\dfrac{-vt}{\gamma_{|\vec{v}|}}\\0\\\dfrac{t}{\gamma_{|\vec{v}|}}\end{pmatrix}\\
&= \begin{pmatrix}\dfrac{-\vec{v}t}{\gamma_{|\vec{v}|}}\\\dfrac{t}{\gamma_{|\vec{v}|}}\end{pmatrix}\\
&= \frac{t}{\gamma_{|\vec{v}|}}\begin{pmatrix}\vec{v}\\1\end{pmatrix}
\end{aligned}$$

We also know

$$\begin{aligned}
\mathcal{L}_{\vec{w}}R_3\begin{pmatrix}0\\0\\t\end{pmatrix} &= \mathcal{L}_{\vec{w}}\begin{pmatrix}0\\0\\t\end{pmatrix}\\
&= \begin{pmatrix}\dfrac{-\vec{w}t}{\gamma_{|\vec{w}|}}\\\dfrac{t}{\gamma_{|\vec{w}|}}\end{pmatrix}\\
&= \frac{t}{\gamma_{|\vec{w}|}}\begin{pmatrix}-\vec{w}\\1\end{pmatrix}
\end{aligned}$$

So, we can see that

$$\mathcal{L}_{\vec{v}}\begin{pmatrix}0\\0\\t\end{pmatrix} = \mathcal{L}_{\vec{w}}R_3\begin{pmatrix}0\\0\\t\end{pmatrix}$$

$$\begin{pmatrix}\dfrac{-\vec{v}t}{\gamma_{|\vec{v}|}}\\ \dfrac{t}{\gamma_{|\vec{v}|}}\end{pmatrix} = \begin{pmatrix}\dfrac{-\vec{w}t}{\gamma_{|\vec{w}|}}\\ \dfrac{t}{\gamma_{|\vec{w}|}}\end{pmatrix}$$

If two matrices are equal, they must have the same dimensions and the corresponding entries must be equal. This gives us $\frac{t}{\gamma_{|\vec{v}|}} = \frac{t}{\gamma_{|\vec{w}|}}$. We also know that $\frac{-\vec{v}t}{\gamma_{|\vec{v}|}} = \frac{-\vec{w}t}{\gamma_{|\vec{w}|}}$. So we know $\vec{v} = \vec{w}$. Now, we can say

$$\mathcal{L}_{\vec{v}}R_1 = \mathcal{L}_{\vec{w}}R_2$$

$$\mathcal{L}_{\vec{v}}R_1 = \mathcal{L}_{\vec{v}}R_2$$

So, since $\mathcal{L}_{\vec{v}}$ is invertible, $R_1 = R_2$. □

Definition 12. A *groupoid* is a set with a binary operation.

Definition 13. The groupoid $(G_n, \oplus)$ is defined as $G_n = \left\{\vec{v} \in \mathbb{R}^n \mid \ |\vec{v}| < c\right\}$ and $\vec{v} \oplus \vec{w} = \vec{z}$ where $\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{w}} = \mathcal{L}_{\vec{z}}R$. We also denote this unique *R* by $R_{v,w}$.

Theorem 8. If we have $(G_n, \oplus)$ as in Definition 13 and $a, b, c \in G_n$ and $a \oplus b = a \oplus c$ then $b = c$. This is called the *Left Cancellation Law*.

Proof. Since $a \oplus b = a \oplus c$,

$$\mathcal{L}_{a\oplus b} = \mathcal{L}_{a\oplus c} \tag{10}$$

$$\mathcal{L}_a\mathcal{L}_b = \mathcal{L}_{a\oplus b}R_{a,b} \tag{11}$$

$$\mathcal{L}_a\mathcal{L}_c = \mathcal{L}_{a\oplus c}R_{a,c} \tag{12}$$

Substituting (10) into (11) gives $\mathcal{L}_a\mathcal{L}_b = \mathcal{L}_{a\oplus c}R_{a,b}$. We can now solve for $\mathcal{L}_{a\oplus c}$ and substitute the solution into (12). This gives

$$\begin{aligned}\mathcal{L}_a\mathcal{L}_c &= \mathcal{L}_a\mathcal{L}_b R^{-1}_{a,b} R_{a,c}\\ \mathcal{L}_c &= \mathcal{L}_b R^{-1}_{a,b} R_{a,c}\\ \mathcal{L}_c R^{-1}_{a,c} &= \mathcal{L}_b R^{-1}_{a,b}\end{aligned}$$

By Theorem 7 we know $b = c$. □

The following definition is from [3.]

Definition 14. A groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

1. In *G*, there is at least one element, 0, called a left identity, satisfying
$$0 \oplus a = a$$
for all $a \in G$.
2. There is an element $0 \in G$ satisfying axiom 1 such that for each $a \in G$, there is an element $-a \in G$, called a left inverse of a, satisfying
$$-a \oplus a = 0$$
3. Moreover, for any $a, b, z \in G$, there exists a unique element $gyr[a,b]z \in G$ such that
$$a \oplus (b \oplus z) = (a \oplus b) \oplus gyr[a,b]z$$
This is called the *Left Gyroassociative Law*.
4. If $gyr[a,b]$ denotes the map $gyr[a,b] : G \to G$ given by $z \mapsto gyr[a,b]z$ then
$$gyr[a,b] \in Aut(G, \oplus)$$
And $gyr[a,b]$ is called the *Thomas gyration*, or the gyroautomorphism of *G*, generated by $a, b \in G$. The operation $gyr : G \times G \to Aut(G, \oplus)$ is called the gyroöperation of *G*.
5. The gyroautomorphism $gyr[a,b]$ generated by any $a, b \in G$ satisfies
$$gyr[a,b] = gyr[a \oplus b, b]$$
This is called the *Left Loop Property*.

Theorem 9. The groupoid $(G_n, \oplus)$ from Definition 13 forms a gyrogroup.

Proof. Let $\vec{u}, \vec{v}, \vec{w} \in G_n$.

(axiom 1)

We know $\mathcal{L}_{\vec{0}} = I$ so $\mathcal{L}_{\vec{0}}\mathcal{L}_{\vec{w}} = \mathcal{L}_{\vec{w}}$ so $\vec{0} \oplus \vec{w} = \vec{w}$.

(axiom 2)

Let $R_{-\hat{i}}$ be a rotation such that $-\hat{i} \mapsto \hat{i}$ and $-\vec{w} \mapsto \vec{w}$. Then

$$\begin{aligned}
\mathcal{L}_{-\vec{w}}\mathcal{L}_{\vec{w}} &= R_{-\vec{w}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{-\vec{w}}^{-1}R_{\vec{w}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{\vec{w}}^{-1} \\
&= R_{\vec{w}}R_{-\hat{i}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{-\hat{i}}^{-1}R_{\vec{w}}^{-1}R_{\vec{w}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{\vec{w}}^{-1} \\
&= R_{\vec{w}}R_{-\hat{i}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{-\hat{i}}^{-1}\mathcal{L}_{|\vec{w}|\hat{i}}R_{\vec{w}}^{-1} \\
&= R_{\vec{w}}\mathcal{L}_{-|\vec{w}|\hat{i}}\mathcal{L}_{|\vec{w}|\hat{i}}R_{\vec{w}}^{-1} \\
&= R_{\vec{w}}R_{\vec{w}}^{-1} \\
&= I
\end{aligned}$$

(axiom 3)

We know that

$$\begin{aligned}
\mathcal{L}_{\vec{a}}\left(\mathcal{L}_{\vec{b}}\mathcal{L}_{\vec{z}}\right) &= \mathcal{L}_{\vec{a}}\left(\mathcal{L}_{\vec{b}\oplus\vec{z}}R_{\vec{b},\vec{z}}\right) \\
&= \mathcal{L}_{\vec{a}}\mathcal{L}_{\vec{b}\oplus\vec{z}}R_{\vec{b},\vec{z}} \\
&= \mathcal{L}_{\vec{a}\oplus(\vec{b}\oplus\vec{z})}R_{\vec{a},\vec{b}\oplus\vec{z}}R_{\vec{b},\vec{z}}
\end{aligned}$$

We also know

$$\begin{aligned}
\left(\mathcal{L}_{\vec{a}}\mathcal{L}_{\vec{b}}\right)\mathcal{L}_{\vec{z}} &= \left(\mathcal{L}_{\vec{a}\oplus\vec{b}}R_{\vec{a},\vec{b}}\right)\mathcal{L}_{\vec{z}} \\
&= \mathcal{L}_{\vec{a}\oplus\vec{b}}R_{\vec{a},\vec{b}}\mathcal{L}_{\vec{z}}
\end{aligned}$$

Then we know

$$\begin{aligned}
\mathcal{L}_{\vec{a}\oplus(\vec{b}\oplus\vec{z})}R_{\vec{a},\vec{b}\oplus\vec{z}}R_{\vec{b},\vec{z}} &= \mathcal{L}_{\vec{a}\oplus\vec{b}}R_{\vec{a},\vec{b}}\mathcal{L}_{\vec{z}} \\
&= \mathcal{L}_{\vec{a}\oplus\vec{b}}(R_{\vec{a},\vec{b}}\mathcal{L}_{\vec{z}}R_{\vec{a},\vec{b}}^{-1})R_{\vec{a},\vec{b}} \\
&= \mathcal{L}_{\vec{a}\oplus\vec{b}}(R_{\vec{a},\vec{b}}R_{\vec{z}}\mathcal{L}_{|\vec{z}|\hat{i}}R_{\vec{z}}^{-1}R_{\vec{a},\vec{b}}^{-1})R_{\vec{a},\vec{b}}
\end{aligned}$$

Now, we want to know what Lorentz Transformation $R_{\vec{a},\vec{b}}\mathcal{L}_{\vec{z}}R_{\vec{a},\vec{b}}^{-1}$ gives us. So, we solve $R_{\vec{z}}^{-1}R_{\vec{a},\vec{b}}^{-1}\vec{w} = |\vec{z}|\hat{i}$ and we find that $\vec{w} = R_{\vec{a},\vec{b}}\vec{z}$. So,

$$\begin{aligned}
\mathcal{L}_{\vec{a}\oplus\vec{b}}(R_{\vec{a},\vec{b}}\mathcal{L}_{\vec{z}}R_{\vec{a},\vec{b}}^{-1})R_{\vec{a},\vec{b}} &= \mathcal{L}_{\vec{a}\oplus\vec{b}}\mathcal{L}_{R_{\vec{a},\vec{b}}\vec{z}}R_{\vec{a},\vec{b}} \\
&= \mathcal{L}_{(\vec{a}\oplus b)\oplus R_{\vec{a},\vec{b}}\vec{z}}(R_{(\vec{a}\oplus b),R_{\vec{a},\vec{b}}\vec{z}}R_{\vec{a},\vec{b}})
\end{aligned}$$

So we know that $\vec{a} \oplus (\vec{b} \oplus \vec{z}) = (\vec{a} \oplus b) \oplus R_{\vec{a},\vec{b}}\vec{z}$ and $R_{\vec{a},\vec{b}\oplus\vec{z}}R_{\vec{b},\vec{z}} = R_{(\vec{a}\oplus b),R_{\vec{a},\vec{b}}\vec{z}}R_{\vec{a},\vec{b}}$ by Theorem 7, and we see that we should take $gyr[a,b] = R_{\vec{a},\vec{b}}$.

To show that $gyr[a,b]z$ is unique, we will show that if $\exists S \in G$ such that $a \oplus (b \oplus z) = (a \oplus b) \oplus S = (a \oplus b) \oplus gyr[a,b]z$ then $S = gyr[a,b]z$. Since $(a \oplus b) \oplus S = (a \oplus b) \oplus gyr[a,b]z$ we can use the Left Cancellation Law of Theorem 8 to say $S = gyr[a,b]z$.

(axiom 4)

To show that $gyr[a,b] \in Aut(G,\oplus)$ we must show $gyr[a,b]$ is a bijection and satisfies $gyr[a,b](x \oplus y) = gyr[a,b]x \oplus gyr[a,b]y$.

Clearly, $gyr[a,b]$ is a bijection because it is a spatial rotation. Let $x, y \in G$. We can see that $\mathcal{L}_{R_{a,b}x}\mathcal{L}_{R_{a,b}y} = \mathcal{L}_{R_{a,b}x\oplus R_{a,b}y}R_{R_{a,b}x,R_{a,b}y}$. Noting that $R_{R_{a,b}x} = R_{a,b}R_x$, we can also see that

$$\begin{aligned}
\mathcal{L}_{R_{a,b}x}\mathcal{L}_{R_{a,b}y} &= R_{R_{a,b}x}\mathcal{L}_{|x|\hat{i}}R^{-1}_{R_{a,b}x}R_{R_{a,b}y}\mathcal{L}_{|y|\hat{i}}R^{-1}_{R_{a,b}y} \\
&= R_{a,b}R_x\mathcal{L}_{|x|\hat{i}}R_x^{-1}R_{a,b}^{-1}R_{a,b}R_y\mathcal{L}_{|y|\hat{i}}R_y^{-1}R_{a,b}^{-1} \\
&= R_{a,b}\mathcal{L}_xR_{a,b}^{-1}R_{a,b}\mathcal{L}_yR_{a,b}^{-1} \\
&= R_{a,b}\mathcal{L}_x\mathcal{L}_yR_{a,b}^{-1} \\
&= R_{a,b}\mathcal{L}_{x\oplus y}R_{x,y}R_{a,b}^{-1} \\
&= R_{a,b}\mathcal{L}_{x\oplus y}R_{a,b}^{-1}R_{a,b}R_{x,y}R_{a,b}^{-1} \\
&= \mathcal{L}_{R_{a,b}(x\oplus y)}R_{a,b}R_{x,y}R_{a,b}^{-1}
\end{aligned}$$

So, by Theorem 7, we see that $R_{a,b}x \oplus R_{a,b}y = R_{a,b}(x \oplus y)$ so $gyr[a,b] \in Aut(G,\oplus)$.

(axiom 5)

This can be proven using the definitions of $\oplus$ and $gyr[a,b]$ and a computer algebra system such as Maple© version 7. □

Works Cited

[1] Gallian, Joseph A. *Contemporary Abstract Algebra*. 5th Edition. New York, NY: Houghton Mifflin Company, 2002.

[2] Smith, Jonathan and Abraham Ungar. "Abstract space-times and their Lorentz groups." *Journal of Mathematics and Physics*. 37(6), June 1996: (3073-3098).

[3] Ungar, Abraham A. *Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2001.

[4] Ungar, Abraham A. "Midpoints in gyrogroups." *Foundations of Physics*. 26 (10) October 1996: (1277-1328).