Polygonal Wheel Graphs and Links

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Definition of Polygonal Wheel Link $K(W_{a,b})$

A polygonal Wheel graph, denoted $W_{a,b}$, is defined by one central vertex encompassed by b polygons composed of a edges. The most shared edges between any two polygons is two.

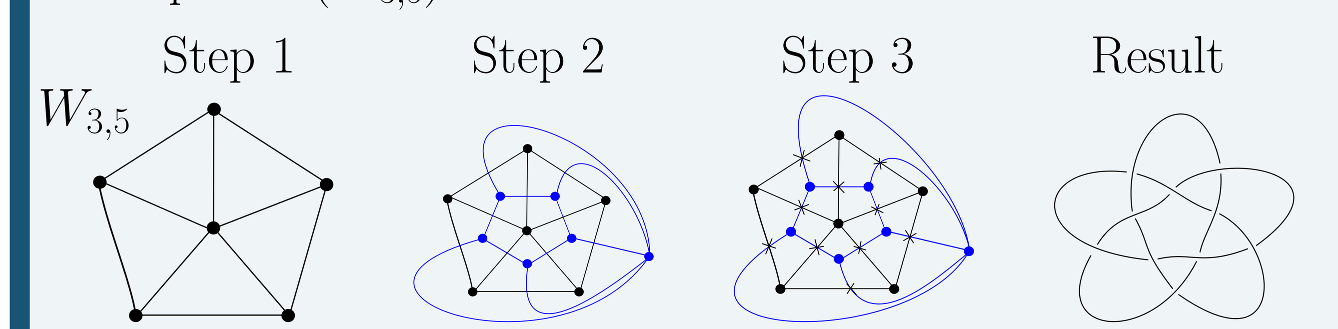
How to construct a Wheel Knot/Link

Step 1: Begin with $(W_{a,b})$ and view it as a Tait Diagram.

Step 2: Place a vertex inside and outside every closed region. Connect each vertex to its adjacent vertices with edges such that every edge in the Tait Diagram is intersected by one edge.

Step 3: Mark each intersection with an X. Connect every X to its adjacent X with arcs. Finish by removing the original graph and assign crossing to each X such that we have an alternating projection.

Example: $K(W_{3,5})$



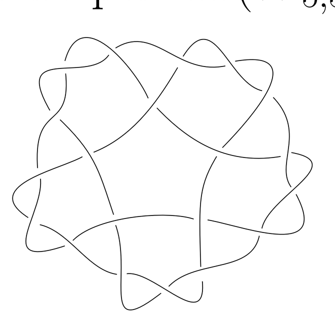
Crossing Number

Theorem: Crossing Number of $K(W_{a,b})$

The number of crossings for $K(W_{a,b})$, is given by $ab - b$.

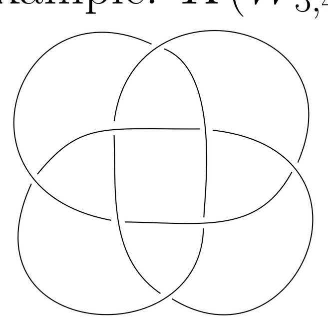
Proof: Each edge of the wheel graph will contribute a crossing to the wheel link. We know that each polygon has a edges, so to represent all polygons, we then multiply by b. We then subtract b from (ab) to correct for double counting, resulting in the simplified equation $ab - b$.

Example: $K(W_{5,5})$



Crossings: $5(5) - 5 = 20$

Example: $K(W_{3,4})$



Crossings: $3(4) - 4 = 8$

Component number

Theorem: Component Number of $K(W_{a,b})$

The component number of $K(W_{a,b})$

	b mod 6						
		0	**1**	**2**	**3**	**4**	**5**
a mod 2	**0**	3	2	3	2	3	2
	1	3	1	1	3	1	1

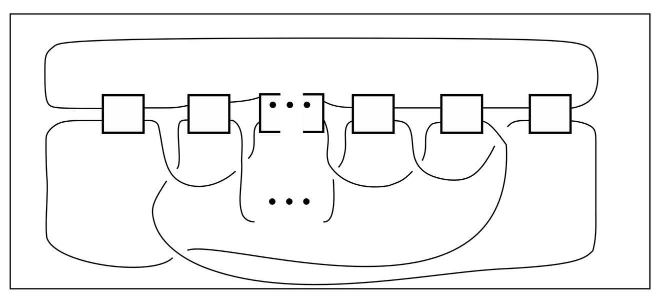
Examples:

$K(W_{5,5})$
a mod $2 = 1$ b mod $6 = 5$
Component Number $= 1$

$K(W_{4,7})$
a mod $2 = 0$ b mod $6 = 1$
Component Number $= 2$

We present the outline of a proof that if $a = 1$ mod 2 (equivalent to $a = 1, 3, 5$ mod 6) and $b = 0$ mod 2 then $K(W_{a,b})$ is a link of two components.

We begin by redrawing the link in a linear way so that we can proceed by induction.



Each square block represents $a-2$ crossings. There are b of these blocks. When a is even, $a - 2$ is even, so the strands end on the same side that they began on.

Next we check our b base case of $b = 1$ and then show that for some odd b, $b + 2$ is also two components. This completes our proof that when a is even and b is odd then $K(W_{a,b})$ is a two component link. The other four cases follow similarly.

Determinant

Using a result about the number of spanning trees of the Tait graph in terms of a knot, we are able to come up with formulas for the determinant for specific values of b.

By setting b as a constant we break down the various cases the spanning tree can take, adding the results of each case to get the following results:

Theorem: Determinant of $K(W_{a,b})$

$b = 2: \; det = a^2 - 4$

$b = 3: \; det = a^3 - 3a - 2$

$b = 4: \; det = a^4 - 4a^2$

$b = 5: \; det = a^5 - 5a^3 + 5a - 2$

Genus

Theorem: Genus of $K(W_{a,b})$

- 1-component: $g = 1 - \dfrac{4 - ab + b}{2}$
- 2-component (Same/Opposite orientation)

$$g = 1 - \frac{(5 - ab + b)}{2} \qquad g = 1 - \frac{(3 - b)}{2}$$

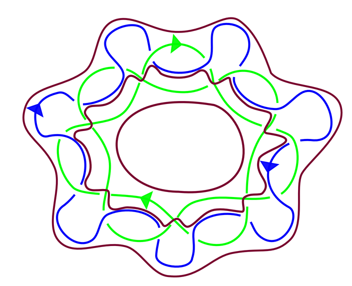
- 3-component
  - Case 1 (Same/Opposite orientation)

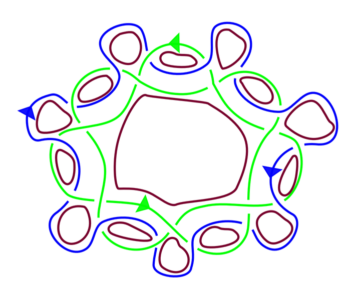
$$g = 1 - \frac{(4 + \frac{2ab}{3} - 2b + \frac{b}{3} - ab + b)}{2} \qquad g = 1 - \frac{b}{2}$$

  - Case 2 (Same/Opposite 1/Opposite 2 orientation)

$$g = 1 - \frac{(4 - b)}{2} \qquad g = 1 - \frac{\frac{ab-2b}{2} + 1}{2} \qquad g = 1 - \frac{(6 - ab + b)}{2}$$

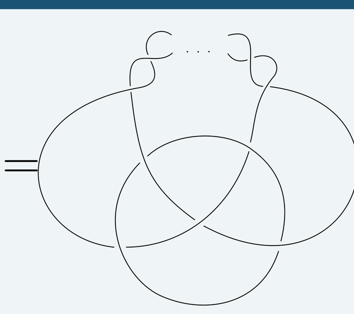
Example: $K(W_{4,7})$ with same and opposite orientation





Alexander Polynomial

Theorem and Future Work

$K =$ 

$$\Delta(K) = t^{n+4} - 4t^{n+3} + 7t^{n+2} + 7(-1)^n t^2 - 4(-1)^n t + (-1)^n + \left(\sum_{i=1}^{n+1} 8(-1)^i t^{n+2-i}\right)$$

We are trying to prove a general formula for $\Delta K(W_{a,3})$. We are proving this via triple induction. Our first induction is on just one of the outside crossing blocks, as seen in the above knot.

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