Arrowgrams: The Next Pencil and Paper Phenomenon

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Dr. Kenneth Price earned his undergraduate degree at Western Illinois University and went on to earn his M.S. and Ph.D. in mathematics (specializing in algebra) from UW–Milwaukee. Price became a professor at UW Oshkosh in fall 1999, where he is today. At Wisconsin conferences he gets attendees interested in mathematics with the Face-Off game.

Abstract

Puzzles are a fascinating part of everyday life, and newspapers everywhere feature crosswords, word jumbles, and Sudokus as a way to test and intrigue the human mind. Arrowgrams are a new type of pencil and paper puzzle created by Dr. Kenneth Price of the University of Wisconsin Oshkosh. In these puzzles, directed graphs, which are a widely used physical element in both mathematics and computer science, are partially labeled, and the goal of the puzzle solver is to complete the rest of the labeling using the transitivity rule, which is a rule that resembles the Pythagorean Theorem. After completing the rest of the labeling, the solver may then find a secret message encoded in the puzzle. By analyzing arrowgrams using linear algebra, a puzzle creator can determine exactly what arrows need to be labeled for the puzzle to be uniquely solvable. A theorem is presented that relates which arrows need to be labeled for a special kind of directed graph called a tournament directed graph.

Arrowgrams: An Introduction

Arrowgrams are a type of pencil and paper puzzle developed by Dr. Kenneth Price of the University of Wisconsin Oshkosh. Arrowgrams arose from Dr. Price's research with colleague Dr. Stephen Szydlik, as a way to promote mathematical concepts to those who are not math experts. Dr. Price's first attempt at creating an arrowgram came in December 2009. In spring 2010, he changed the design to its current appearance. After more exploration of his arrowgrams, Dr. Price presented arrowgrams at the Wisconsin regional meeting of the Mathematical Association of America in April 2011. This presentation drew enough excitement and interest to encourage him to submit "Take Aim at an Arrowgram," which was accepted for publication in the puzzle section of *MAA Focus*, and to give another presentation at a national conference in Boston in January 2012. According to Price, the response to his presentation was positive. Dr. Sarah-Marie Belcastro, a research associate at Smith College and a lecturer at the University of Massachusetts–Amherst, decided to teach a unit on arrowgrams to high school students taking her summer honors program mathematics course. Dr. Price is currently studying the mathematical properties of arrowgrams and has his linear algebra students create their own arrowgrams using skills they develop in his class. I started working on arrowgrams in September 2011, when Dr. Price invited me to explore these puzzles with him. Between creating arrowgrams, looking at other puzzles, and analyzing arrowgrams using linear algebra, I have discovered that both math majors and nonmajors can enjoy arrowgrams. In this paper, we will explore what arrowgrams are, how to create them, and a specific kind of arrowgram—an arrowgram on a tournament directed graph—which will hopefully be our key to eventually mass-producing these puzzles.

Arrowgrams are built on vertices connected by arrows, which form mathematical objects called *directed graphs*. Some of these arrows have a designated value already labeled. The value of an arrow is called its grade. The goal of the puzzle is to determine the grade of every arrow. An additional property that every arrowgram possesses is the *transitivity rule*. This rule allows us to determine the grades of the unlabeled arrows.

The transitivity rule involves relating arrows determined by three vertices (Price, "Arrowgrams"). To use the transitivity rule, one looks for three vertices, say *x*, *y*, and *z*, that are connected by a sequence of arrows. One arrow must go from *x* to *y*, another arrow from *y* to *z*, and a third arrow from *x* to *z*. In this situation, we say *x*, *y*, and *z* form a *transitive triple*.

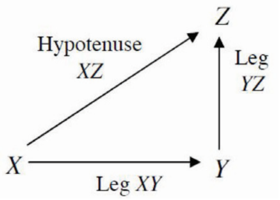


Figure 1. A transitive triple.

The arrows going from *x* to *y* and *y* to *z* are called the legs of the transitive triple, and the arrow going from *x* to *z* is called the *hypotenuse* of the transitive triple. Each arrowgram is associated with a specific mathematical operation, such as ordinary addition. The operation combines the two numbers to get one number. In the case of the addition operation, this outcome would be the sum of the two numbers.

The transitivity rule states that the results of the operation on the grades of the legs should be equal to the grade of the hypotenuse in every transitive triple (Price 2011). For example, if your operation is addition and two legs have grades 3 and 4, the transitivity rule requires that the hypotenuse has a grade of 7.

Although this may sound like the Pythagorean Theorem (the sum of the squares of the legs equals the square of the hypotenuse in every right triangle), it is not. The transitivity rule features the terms of a right triangle, and our above example (fig. 1) is arranged in the shape of a right triangle, but the Pythagorean Theorem and transitivity rule are different by definition. The transitivity rule deals with a specified operation, which may not be addition. For example, if two legs were graded 3 and 4, and the operation was multiplication, we would have a hypotenuse of grade 12. Furthermore, the Pythagorean Theorem features the squares of the lengths, while the transitivity rule does not square the terms or rely on lengths. The grades on a transitive triple potentially have any values, including negative numbers. Additionally, the hypotenuse, in terms of a transitive triple, need not be the hypotenuse of a right triangle. For example, if you switched the direction on arrow XY, the hypotenuse would become YZ, and XZ would become a leg. Also, the three vertices do not need to be arranged in a right triangle, as in the example in figure 2, which features addition as its grading condition.

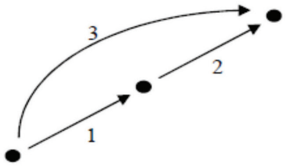


Figure 2. Oddly shaped transitive triple.

While grading the arrows in such a graph is a mathematically interesting challenge, the puzzles become more universally appealing when we modify the objective to determining a secret message. We change the vertices to letters, then, from the grading of the arrowgram, the grades of the arrows are matched with a key to determine the letters of the secret message. For example, a puzzle could consist of two blanks labeled 34 and -5. After solving the puzzle, a person might discover that the arrow going from vertex M to vertex A has grade 34 and the arrow going from vertex T to vertex H has grade -5. Thus, the secret message would be MATH.

At this point, it may be useful to walk through an example arrowgram. Figure 3 is a puzzle created by Dr. Price and is in his notes titled “Arrowgrams.”

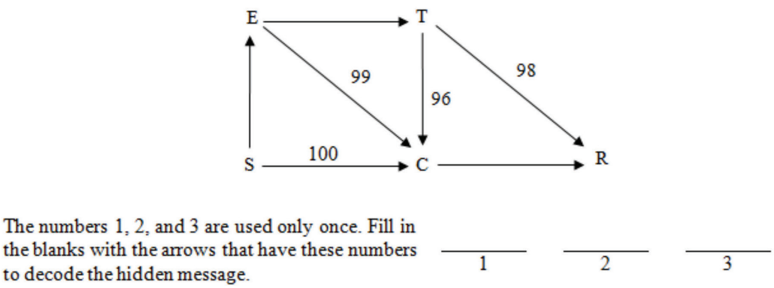


Figure 3. SECRET arrowgram.

As we can see, there are arrows going from S to E, E to C, and then S to C, so (S, E, C) is a transitive triple. By our definition of legs and hypotenuse, we see that arrow SE and arrow EC are considered the legs of this transitive triple and arrow SC is considered the hypotenuse. Another way to see this is that if we start at S and follow the arrow to E, we can continue to follow the path naturally to vertex C. However, if we tried to head back to vertex S, we would be going against the direction of the arrow. Thus, when we have three vertices where a puzzle solver can follow two and then get stopped by the third, we have two legs (the two arrows that flow together, tip to tail) and a hypotenuse (the arrow that opposes the direction). Now that we have identified the legs and hypotenuse, we will label the grade of arrow SE to be the variable X, and we get the equation $X + 99 = 100$. Simple algebra tells us that X is 1, and therefore arrow SE has a grade 1 and “SE” goes in the blank associated with 1.

Next, we turn our attention to the vertices E, T, and C. Once again, if we start at E, we can follow the ungraded arrow to vertex T. Then, we can follow the arrow graded 96 to vertex C. However, if we try to go from vertex C to vertex E, we would be going against the direction of the arrow. Therefore, we know (E, T, C) is a transitive triple. Assigning the variable Y to the arrow going from E to T, we get the equation Y + 96 = 99. Once again, simple algebra finds that Y is 3, and thus ET goes in the blank labeled 3.

Finally, we look at vertices T, C, and R. We have an arrow going from T to C and C to R, and from T to R. Once again, by definition, arrows going from T to C and C to R are legs and the arrow from T to R is the hypotenuse. Replacing the arrow from C to R with variable Z, we get 96 + Z = 98, which gives us that Z = 2. Because of this, CR goes in the blank labeled 2 and our puzzle is solved with the message "SECRET."

This is a basic secret message, but there are many other ways to design a more complicated secret message. For example, a puzzle creator could specify that the grade X appears exactly three times and that the rest of the secret message is determined by the unknown X. The blanks may be labeled X + 3, 2X, 3 - X, or something to that effect. After solving the puzzle for X, a puzzle solver could easily find X + 3, 2X, 3 - X, etc., and then match those grades with the arrows to form the secret message.

The triangle arrowgram in figure 4 appears in the UW Oshkosh Mathematics Department's 2011–2012 newsletter. It is more difficult than example 1, and has some extra stipulations in the directions, but has a unique solution nevertheless.

Triangle Arrowgram

This puzzle uses the grading group of integers under addition, but only the numbers 1–18 are grades of arrows. The grading uses the numbers 1, 6, and 8 exactly twice.

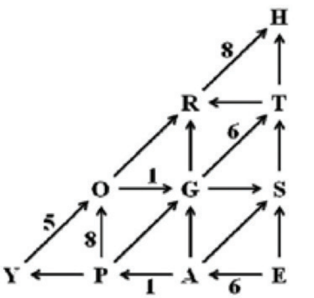
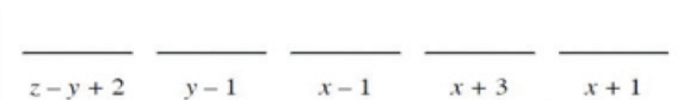


Figure 4. Triangle Arrowgram.
Source: Kenneth Price, "Arrowgrams."

Every other number 1–18 is used exactly once, except for three numbers, *x*, *y*, and *z*, where *x* is the smallest and *z* is the largest of the three numbers.



If an arrowgram is constructed well, someone who knows the transitivity rule should be able to take the arrows whose grades are given and solve for the rest of the arrows. If an arrow does not have a uniquely determined grade, the secret message will not be effectively communicated. This begs two questions. First, how many arrows need to be filled in so that a unique solution is guaranteed? Second, which arrows should be given grades first? I provide context for these questions by summarizing some of the mathematical literature on these types of problems for similar paper and pencil puzzles.

Literature Review: Mathematics of Puzzles

Sudokus have been a growing phenomenon during the past 10 years. As stated by Dan Vergano from *USA Today*, "Since the introduction of the numerical puzzle in London's *The Times* in 2004, Sudoku has taken quiz fans by storm. It has appeared on websites, cell phones, and in newspapers, including *USA Today*" (2013). More than that, we see puzzle books that feature Sudokus in stores everywhere. What is this interesting puzzle that has taken the world by storm? Agnes M. Herzberg and M. Ram Murty describe the nature of Sudokus:

> The puzzle consists of a 9 x 9 grid in which some of the entries of the grid have a number from 1 to 9. One is then required to complete the grid in such a way that every row, every column and every one of the nine 3 x 3 sub-grids contain the digits from 1 to 9 exactly once. (2007, 708)

Herzberg and Murty continue to inform us that Sudokus are similar to a Latin square:

> Recall that a Latin square of rank n is an n x n array consisting of the numbers such that each row and column has all the numbers from 1 to n. In particular, every Sudoku square is a Latin square of rank 9, but not conversely because of the condition on the nine 3 x 3 sub-grids. (2007, 708)

In other words, every Sudoku is a Latin square, but not every Latin square is a Sudoku (in the same way that every square is a rectangle, but not every rectangle is a square). The nine 3 x 3 subgrids, where each subgrid must contain numbers 1 to 9, are what make a Sudoku unique. An interesting question to explore is: How many boxes must be filled in to ensure a unique solution? The answer was recently shown to be 17. In her 2007 paper, "Puzzling Over Sudoku," James Madison University faculty Laura Taalman states: "Mathematicians and computer scientists have conjectured that at least 17 clues are always needed. Although there are many known 17-clue puzzles and no known 16-clue Sudoku puzzles, it is still an open problem to prove that no 16-clue puzzles exist" (2007, 57). However, more recently, in "There Is No 16-Clue Sudoku: Solving the Sudoku Minimum Number of Clues Problem," Gary McGuire et al. state, "We are not saying that all completed sudoku grids contain a 17-clue puzzle (in fact, only a few do). We are saying that no completed sudoku grid contains a 16-clue puzzle" (2013, 5). Furthermore, Felgenhauer and Jarvis, in their paper "Mathematics of Sudoku I," count the number of different Sudoku labelings to be approximately 6.71×10^{21} (this is 671 followed by 19 zeros). Although this is interesting, we understand that Sudokus are different from arrrowgrams in that Sudokus do not require any arithmetic. There is no need to worry about the numerical properties of the numbers; one simply uses logic and reasoning to solve the puzzle. In other words, in a Sudoku, one could replace the numbers 1–9 with any nine other symbols and still find a solution. For example, one could change each number to a color and still use reason and logic to find a solution to the puzzle. This could not be done on an arrowgram because solving the puzzle relies on the mathematical operation and numbers that are given. It is also worth noting that, due to their logical necessities, J. F. Crook has claimed to have found an algorithm for solving any Sudoku. If you follow his process, which involves using preemptive sets, you will be guaranteed to find the answer to any given Sudoku, which may or may not take the fun out of solving the puzzle.

Another similar puzzle is a KenKen. In 2004, a Japanese mathematics teacher, Tetsuya Miyamoto, invented a Sudoku-like puzzle called a KenKen (loosely translated, this means "wisdom squared"). A KenKen is similar to a Sudoku in that it is an n x n puzzle where each row and column must have one of each of the numbers from 1 to n.

Instead of giving puzzle solvers values, a KenKen features cages that require certain mathematical operations. Changing the operation is also a feature of arrowgrams. Figure 5 shows a 5 x 5 KenKen puzzle.

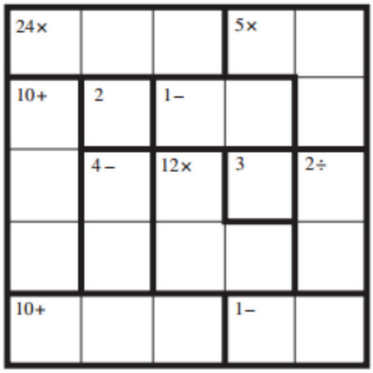


Figure 5. An example of a KenKen.
Source: John Watkins, "Triangular Numbers, Gaussian Integers, and KenKen."

We see that the cage in the bottom right-hand corner contains a 1- in the top left corner. This means that the two numbers in that cage should have a difference of 1. It does not matter which number goes in which of the two squares of the cage, but they must have a difference of 1. Furthermore, in the bottom left corner, we see a cage with a 10+. This means that the three boxes in the cage add up to 10. In a Sudoku, the numbers have no specific relevance aside from their job as placeholders; this is not the case in a KenKen.

Some mathematical aspects of KenKens include prime factorization and the partition of an integer. A prime factorization is the unique way in which a number can be written as a product of prime numbers. For example, the prime factorization of 24 is 2 x 2 x 2 x 3. There is no other way to write 24 as a product of prime numbers other than changing the order. Furthermore, a partition of a number is a sum of positive integers equal to that specific number. For example, 4 can be partitioned in five different ways: 4, 3 + 1, 1 + 1 + 2, 1 + 1 + 1 + 1, 2 + 2.

Another interesting concept that arises in the solution of a KenKen is the Greek idea of triangular numbers. The first few triangular numbers are 1, 3, 6, and 10. The relationship with triangles can be described in terms of bowling pins. One pin is in the front. There are two pins behind it, which gives three pins in a triangle. Next, three more pins give a six-pin triangle. Finally, there are four pins in the back, which gives a ten-pin triangle. You could get more triangular numbers, such as 15, 21, 28, and 36, by continuing to add pins to your triangle. John Watkins, a mathematician at Colorado College, writes specifically about these triangular numbers in his paper "Triangular Numbers, Gaussian Integers, and KenKen":

> What is quite surprising is that another important notion in number theory, the ancient Greek concept of triangular numbers, can also often be used to solve a KenKen because in any solution to a KenKen the sum of the numbers in any row or column of the grid is 1 + 2 + … + n; that is, the sum is the nth triangular number. In Figure 1 [figure 5 in this paper], the sum of the numbers in any row or column must be 15 [that is, the triangular number 15 = 1 + 2 + 3 + 4 + 5]. For example, since in the bottom row the sum in the first cage is 10, the sum in the

> second cage must be 5. Hence, that cage contains 2 and 3. But, there is already a 3 in the fourth column, so we can immediately place the 2 and 3 in their correct positions in the bottom row. (2012, 38)

KenKens are similar to Sudokus, but may be even more similar to the puzzle we will be exploring in this paper: the arrowgram.

Selecting Arrows to Label

When creating an arrowgram, one of the most important questions to be answered is: What arrows do I need to label to ensure that there is a unique solution and a solution that does not contradict itself? We do not want to label too few or too many arrows. For example, what if our secret arrowgram from example 1 was changed to the labeling in figure 6?

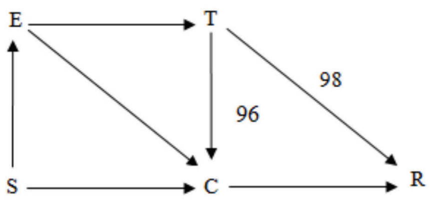


Figure 6. An incomplete example of an arrowgram.

One quickly sees, after finding the label on the arrow going from C to R, that we cannot solve any more of the puzzle. Thus, we clearly do not have a well-constructed arrowgram.

Now let us look at the flip side—what happens when we label too many arrows—by looking at the example in figure 7.

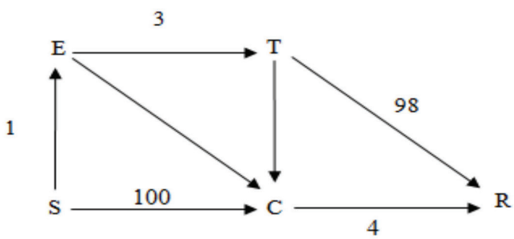


Figure 7. An arrowgram with too many labels.

Trying to solve this puzzle leads to a contradiction. By using the transitivity rule, one could find that the arrow from E to C is 99. Now let us label the arrow from T to C as X. We would thus have the equation X + 3 = 99 from the transitive triple E, T, and C, meaning that X would be equal to 96. However, we would then look at the transitive triple between vertices T, C, and R. The equation that would follow would be 96 + 4 = 98. It does not take a math major to see that something is wrong here. Thus, we need to be careful that our labeling does not contain any mathematical contradictions. Let us look at a new example that we will call T_4, shown in figure 8.

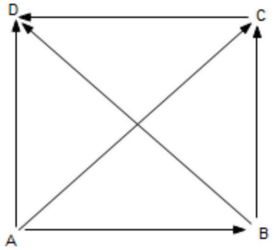


Figure 8. A tournament directed graph with four vertices.

We would like to construct an arrowgram based on this graph. To ensure that our arrowgram is well-constructed, we start by setting up equations for each of the transitive triples. For ease of notation, let us represent the grade on the arrow going from some vertex p to some vertex q using variables $X_{p,q}$. For example, the arrow going from A to B would be represented $X_{A,B}$. There are a total of six arrows, so there are six unknowns. By considering the four transitive triples in the arrowgram, we obtain the following equations:

$$X_{A,B} + X_{B,C} = X_{A,C}$$

$$X_{A,B} + X_{B,D} = X_{A,D}$$

$$X_{A,C} + X_{C,D} = X_{A,D}$$

$$X_{B,C} + X_{C,D} = X_{B,D}$$

Our goal with these equations is to figure out which variables are the free variables, which are a set of variables whose values can be chosen arbitrarily. However, after being chosen, they uniquely determine the values of all the other variables in the system. In other words, by declaring values for our free variables, we will have a uniquely solvable arrowgram. In our next step in determining the free variables, we can move every term to the left side by subtraction. This leaves us with:

$$X_{A,B} + X_{B,C} - X_{A,C} = 0$$

$$X_{A,B} + X_{B,D} - X_{A,D} = 0$$

$$X_{A,C} + X_{C,D} - X_{A,D} = 0$$

$$X_{B,C} + X_{C,D} - X_{B,D} = 0$$

This corresponds to the matrix equation $A_4X = 0$, where A_4 is the coefficient matrix of the above equations, which is shown below:

$$A_4 = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

Each row corresponds to an equation and each column corresponds to a variable. Also, X is the column vector associated with the variables, shown below:

$$X = \begin{bmatrix} X_{A,B} \\ X_{A,C} \\ X_{B,C} \\ X_{A,D} \\ X_{B,D} \\ X_{C,D} \end{bmatrix}$$

Thus, if we multiplied A_4 by X using matrix multiplication, we would be left with a set of equations in which the right sides are all equal to 0. If we look at A_4, we can see that each row corresponds to a transitive triple equation, and each column corresponds to an arrow on the graph.

The standard way to find the free variables of a linear system is by finding the reduced row echelon form of the matrix. However, this form is not especially helpful to our arrowgram problem because it produces fractions. The reduced row echelon form may reveal, for example, that the value of a variable is half that of another variable. This is fine if our underlying number system is the real numbers, but if we want to work with the integers, we may encounter problems. Trying to take half of an odd number will give us a fraction.

The *Smith form* of a matrix is the appropriate tool for solving equations based on abelian group operations, which is a group where operations are commutative ($2 + 4 = 4 + 2$). The Smith form of the matrix is a transformed version of the original matrix, where the new matrix is a matrix filled with zeros, except on the diagonal, starting from the left-hand corner (Rotman 2010). On the diagonal, we have positive integer entries, which are listed in nondescending order. The number of zero columns in the matrix is of considerable importance because it is equal to the number of free variables in our system. To change the matrix to the Smith form, there is a change of variables that takes place. Thus, the columns of the Smith form still relate to variables, but these variables are no longer the originals. They are new variables that we will label Y_1, Y_2, Y_3, Y_4, Y_5, and Y_6. By looking at the column operations that were enacted on the original matrix, we find that our original variables relate to these new variables in the following fashion.

$$X_{A,B} = Y_1 + Y_2 + Y_4$$

$$X_{A,C} = Y_2 + Y_3 + Y_4 + Y_5$$

$$X_{B,C} = Y_3 + Y_5$$

$$X_{A,D} = Y_4 + Y_5 + Y_6$$

$$X_{B,D} = Y_5 + Y_6$$

$$X_{C,D} = Y_6$$

From the Smith form of the matrix, we can determine that the variables that correspond with a column with a nonzero entry in it are equal to 0. Thus, we had columns of zeros in the columns corresponding to Y_4, Y_5, Y_6, so we know that Y_1, Y_2, and Y_3 are all zeros. Therefore, we are left with:

$$X_{A,B} = Y_4$$

$$X_{A,C} = Y_4 + Y_5$$

$$X_{B,C} = Y_5$$

$$X_{A,D} = Y_5 + Y_6$$

$$X_{B,D} = Y_5 + Y_6$$

$$X_{C,D} = Y_6$$

Thus, we can see that if we simply label $X_{A,B}$, $X_{B,C}$, and $X_{C,D}$, we will have a unique and solvable puzzle. Therefore, by finding the Smith form of the matrix and determining the column operations that were imposed on the original matrix, we can find our free variables and the arrows that need to be labeled in our puzzles.

Tournament Directed Graphs

One particular type of directed graph is a tournament directed graph. A *tournament directed graph* T_k consists of vertices numbered from 1 to k and arrow set $\{xy : 1 \leq x < y \leq k\}$. That is, the graph consists precisely of those arrows connecting a vertex to another vertex of greater value. For example, T_5 consists of five vertices, and arrows from 1 to 2, 1 to 3, 1 to 4, 1 to 5, 2 to 3, 2 to 4, 2 to 5, 3 to 4, 3 to 5, and 4 to 5. A *good-grading* of a tournament directed graph is any grading for which the transitivity rule is satisfied. A slightly different statement of the following theorem, which originally had algebraic applications in mind, appears in the paper "Good Gradings of Full Matrix Rings" by S. Dascalescu et al.

Theorem. *Let G be a finite group. There are* $|G|^{k-1}$ *good G-gradings of* T_k *for every* $k \geq 3$.

A *group* is a set of numbers that needs to be associative (a + b = b + a), closed under the operation, and closed when you have inverses. For our purposes, a group is a set of numbers along with an operation for combining them, such as addition or multiplication. We use the operation in the transitivity rule for an arrowgram. Sometimes the label is missing on a leg. We then need to subtract, divide, or somehow solve the equation given by the transitivity rule. This requires an inverse for the number under the operation. If we use addition, we need every number's opposite to belong to the group. For example, the opposite of 3 is -3. If we use multiplication, then we need every number to have an inverse. For example, the inverse of 3 is $3^{-1} = 1/3$. We also want the order of operations to hold, so the binary operation must be associative.

A finite group means that there are only a limited number of elements we could grade each arrow. For example, {0, 1, 2, 3, 4} would be a finite group, where we could only label each arrow 0, 1, 2, 3, or 4. The symbol $|G|$ stands for the number of elements in the group G. So if we could only label each arrow 0, 1, 2, 3, or 4, we would have five elements in a finite group G and $|G| = 5$. For example, if we were using T_5, and we could only label each arrow 0, 1, 2, 3, or 4, then there would be $5^4 = 625$ different ways to label the tournament directed graph. Also, if we simply changed the vertices to letters, we would have an arrowgram, so there would be 625 different arrowgrams to make from that group. This would be like needing to label four squares for designing a Sudoku, but you could only use 1, 2, 3, 4, or 5. If no two of the four squares are in the same row or column, then there are 625 different possible choices.

Clockwork arithmetic provides examples of finite groups. How do you assign an operation on the set? For example, for the set {0, 1, 2, 3, 4}, what happens if we add 4 and 3? We get 7, but 7 does not belong to the set, so we must find some way to make sure the number we get is an element of the group for the closure property to hold. This is where clockwork, or modular, arithmetic comes in handy. It works like a clock, where if you are adding, you move clockwise, and if you are subtracting, you move counterclockwise. So in figure 9, we have a clock with elements {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 0} (we will identify the 12 with a 0).

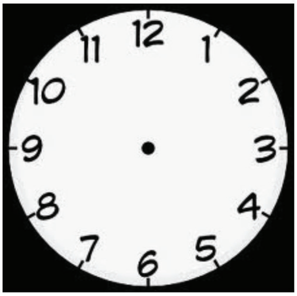


Figure 9. Clockwork arithmetic may be extremely useful as we look to mass-produce arrowgrams.
Source: Iyo, "Purzen Clock Face."

Thus, if we were adding 8 + 7, we would move seven elements clockwise, and end up at 3. Thus, (7 + 8) mod12 (mod12 because we have 12 elements in our set) is equal to 3. Subtraction would work in the opposite direction, counterclockwise. Thus, (3 - 8) mod12 would be like moving eight elements counterclockwise, which would bring us to 7. With this new clockwork arithmetic, we ensure that all the grades on our arrowgram are elements of the group. Note that the set also contains the inverse of every element in the set. For example, because 2 + 10 mod12 is 0, we have that 10 is the inverse of 2 in this operation.

Going back to the theorem, Dr. Price and I provided a new proof of this theorem by analyzing the different tournament directed graphs and their coefficient matrices. We found that there was a pattern in the way each matrix was composed, and we proved, via induction, that the arrows that need to be labeled for graph T_k are 1 to 2, 2 to 3, 3 to 4, and so on until k - 1 to k. Thus, for T_5, we must label the arrow from 1 to 2, 2 to 3, 3 to 4, and 4 to 5. Because each arrow has five possibilities for its value, and we have four arrows that need to be labeled, one can see how we determined that there are 5^4 ways we can label T_5. However, not all 625 arrowgrams are useful for encoding secret messages. For example, if two arrows are labeled with a 4, and the puzzler is asked to put the arrow labeled 4 in the blank, how are puzzlers supposed to know which set of letters fills the blank? Factors like this must be taken into account when creating an arrowgram.

In the case of finite groups, it is possible to count the number of different ways to grade a tournament directed graph. For example, if we were using T_5 and we could only label each arrow 0, 1, 2, 3, 4, there would be $5^4 = 625$ different ways to label the tournament directed graph. Also, if we changed the vertices to letters, we would have an arrowgram, so there would be 625 different arrowgrams to make from that group.

Figures 10 and 11 feature arrowgrams designed on tournament directed graphs:

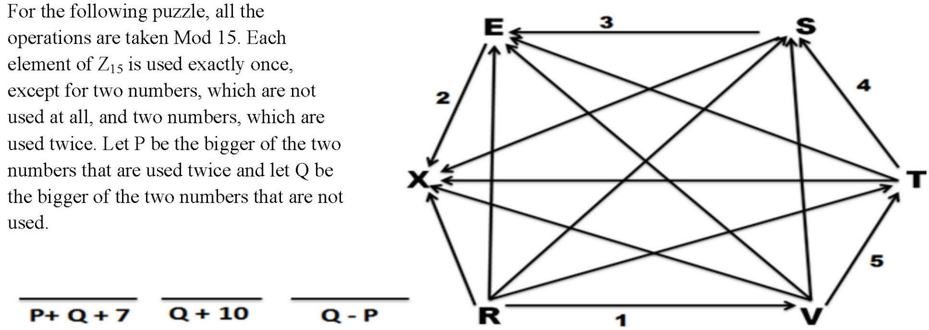


Figure 10. A six-vertex example of an arrowgram on a tournament directed graph. *Source*: Kenneth Price, "Arrowgrams."

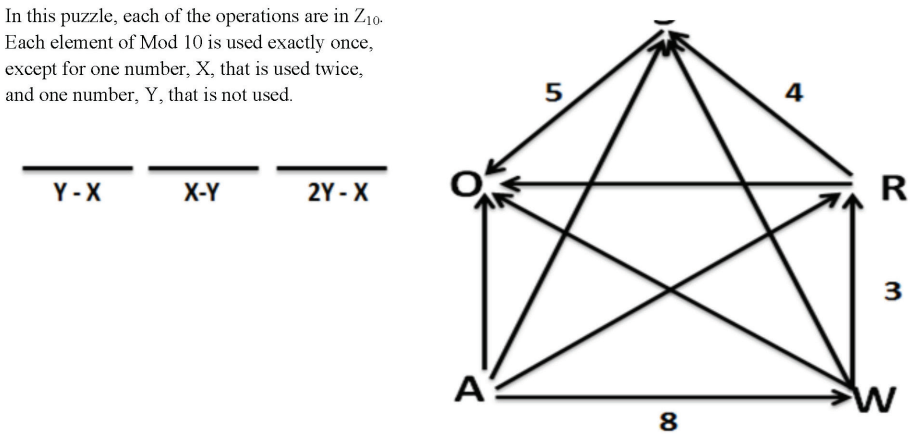


Figure 11. A five-vertex example of an arrowgram on a tournament directed graph. *Source*: Kenneth Price, "Arrowgrams."

Conclusion

Arrowgrams are an exciting new puzzle that require the puzzler to use math in solving them. Creating the puzzles is currently difficult and tedious because one has to consider many factors when encoding the secret message, including which arrows need to be labeled. However, by looking at tournament directed graphs, we have a way to quickly formulate puzzles and to know which arrows need to be labeled. Although we are just beginning to create them, arrowgrams may eventually be as popular as KenKens or Sudokus, and expose the world to some interesting mathematics.

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