Analysis of Combining Musical Scale Vectors

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Musical Examples

1. Rebecca Clarke, Sonata for Viola & Piano, 2nd movement

- The octatonic scale as a combination of two major triads a tritone apart
- Missing two of the eight notes of the octatonic scale
  - The missing notes are the tonics of the subsequent triads
  - The missing notes are a tritone apart from each other

The combination according to the mathematical methods:

$\frac{1}{9}[-1\ 2\ 5\ -1\ 2\ 5\ -1\ 2\ 5\ -1\ 2\ 5]'$

*the octatonic is a weighted combination of all 12 major triads

The combination according to this piece of music:

$[1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0]'$

*the octatonic is a combination of two major triads a tritone apart

Measures 1-3:

	AM triad in the right hand	[0 1 0 0 1 0 0 0 0 1 0 0]′
+	E♭M triad in the left hand	+[0 0 0 1 0 0 0 1 0 0 1 0]′
=	Octatonic missing C and F♯	=[~~1~~ 1 0 1 1 0 ~~1~~ 1 0 1 1 0]′

Measure 4:

	F♯M triad in the right hand	[0 1 0 0 0 0 1 0 0 0 1 0]′
+	CM triad in the left hand	+[1 0 0 0 1 0 0 1 0 0 0 0]′
=	Octatonic missing E♭ and A	=[1 1 0 ~~1~~ 1 0 1 1 0 ~~1~~ 1 0]′

2. Béla Bartók, "From the Isle of Bali", No. 109

- The octatonic scale as a combination of four minor seconds
- Emphasis on the half steps

The combination according to the mathematical methods is the same as in this piece of music:

$[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0]'$

*the octatonic is a combination of four major thirds three semitones apart

Measures 1-11:

	top line: BC FG♭	[1 0 0 0 0 1 1 0 0 0 0 1]′
+	bottom line: G♯A DE♭	+[0 0 1 1 0 0 0 0 1 1 0 0]′
=	Octatonic	=[1 0 1 1 0 1 1 0 1 1 0 1]′

Abstract

This interdisciplinary project uses mathematics to represent musically significant collections of notes such as scales, chords, and intervals as combinations of each other. These note collections were converted into one-dimensional vectors so that matrices could be used to solve the systems of linear equations which create the combinations, a method used in one of Sethares' papers [1]. In some cases, this method produces combinations which are too complex to have musical significance, so a new iterative method has been devised to create simpler combinations. All the mathematics used has been coded into MATLAB, resulting in combinations being quickly computed. In particular, we focused on the diatonic scale, interval of a major third, and the octatonic scale. Applications have been made to existing musical literature. We have found 15 examples of one type of note collection being used to evoke the structure of a different collection.

Motivation

What is a scale vector?

A 12×1 vector or column matrix which consists of 1's and 0's, where each entry corresponds to one of the 12 notes of the chromatic scale. A "1" symbolizes the inclusion of a note, while a "0" indicates the exclusion of a note.

For example, the C major scale can be written as the scale vector

$$[1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1]',$$

where the numbers in the scale vector coincide with the ordering of the note names given by

[C C♯ D D♯ E F F♯ G G♯ A A♯ B]′.

Previous Work

Sethares devised a method for creating combinations using these scale vectors, and provides a couple examples of this method in action. We used MATLAB to produce many more examples and to see when this method fails. We developed a new method of finding combinations using iteration of sparse matrices. This new method also has its flaw, though it tends to yield simpler combinations than that of Sethares.

Goals

- To find real musical examples of the combinations found mathematically
- To compare musical combinations to mathematical results

References

[1] Sethares, W.A. and E. Amoit. (2011). An algebra for periodic rhythms and scales. *Journal of Mathematics and Music*, 5(3), pp. 149-169
[2] MathWorks. (2015). tfqmr: Transpose-free quasi-minimal residual method. *MATLAB*.
[3] Clarke, R. (1919). Sonata for Viola and Piano. *Chester*, 1921.
[4] Bartók, B. (1940). From the Isle of Bali, for piano. From Mikrokosmos Vol. 4/109 Sz. 107. *Hawkes & Son London Ltd.*
[5] Debussy, C. (1910). Voiles. From Préludes Book 1.
[6] Rimsky-Korsakov, N. (1867). Sadko, Op. 5.

Musical Examples

3. Claude Debussy, Preludes II: Voiles

- The whole tone scale as a combination of four major thirds
- The entire piece uses the whole tone scale, with the exception of one portion using the pentatonic scale, though the beginning emphasizes the decomposition into major thirds

The combination according to the mathematical methods:

$\frac{1}{2}[1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0]'$

*the whole tone scale is a weighted combination of six major thirds two semitones apart

The combination according to this piece of music:

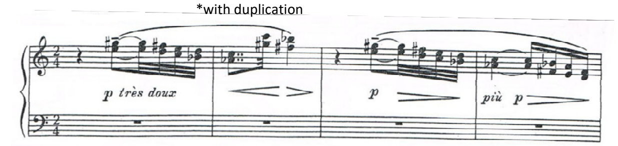
$[1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0]'$

*the whole tone scale is a combination of four major triads two semitones apart

Measures 1-2:

	E G♯	[0 0 0 0 1 0 0 0 1 0 0 0]′
+	D F♯	+[0 0 1 0 0 0 1 0 0 0 0 0]′
+	C E	+[1 0 0 0 1 0 0 0 0 0 0 0]′
+	B♭ D	+[0 0 1 0 0 0 0 0 0 0 1 0]′
=	Whole tone scale	=[1 0 2 0 2 0 1 0 1 0 1 0]′

*with duplication



4. Rimsky-Korsakov, Sadko

- The octatonic scale as a combination of four dominant seventh chords
- The roots of the four dominant seventh chords form a diminished seventh chord

The combination according to the mathematical methods:

$\frac{1}{14}[-1\ 3\ 5\ -1\ 3\ 5\ -1\ 3\ 5\ -1\ 3\ 5]'$

*the octatonic is a weighted combination of all 12 dominant seventh chords

The combination according to this piece of music:

$[1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0]'$

*the octatonic is a combination of four dominant seventh chords three semitones apart

	C^7	[1 0 0 0 1 0 0 1 0 0 1 0]′
+	$E\flat^7$	+[0 1 0 1 0 0 0 1 0 0 1 0]′
+	$F\sharp^7$	+[0 1 0 0 1 0 1 0 0 0 1 0]′
+	A^7	+[0 1 0 0 1 0 0 1 0 1 0 0]′
=	Octatonic	=[1 3 0 1 3 0 1 3 0 1 3 0]′

*with duplication

Methodology

Step 1:

Create a circulant matrix

- Take the scale vector for the note collection which will make up the linear combination
- Set the scale vector as the first column of a 12×12 matrix
- Each successive column is the scale vector rotated one slot
- Example: interval of a major third

$$A = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&1\\ 1&0&0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0&0&1 \end{bmatrix}$$

Step 2:

Set up the matrix equation

- Let b be the scale vector for a different note collection than the circulant matrix.
- Let x be the coefficients necessary to represent b in terms of the columns of A.
- Then $Ax = b$

The circulant matrix is invertible

Step 3:

Solve for x

- Since A is invertible, $$Ax = b \Rightarrow x = A^{-1}b.$$

Example: A major triad chord as a combination of major thirds:

$x = \frac{1}{2}[2\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ -1]'$

*A major triad is a weighted combination of four major thirds three semitones apart

The circulant matrix is singular

Step 3:

Use the Pseudoinverse

- $Ax = b \Rightarrow x = A^{+}b$, where A^{+} is the pseudoinverse of A.

Example: An octatonic scale as a combination of diminished seventh chords:

$$x = \frac{1}{4}[1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1]'$$

Iterative method: tfqmr

- Concatenate A and b into a 12×13 matrix: $[A \mid b]$
- Put the new matrix into reduced row echelon form
- Deconcatenate into a 12×12 matrix A' and a 12×1 matrix b'.
- Solve $A'x = b'$ using the Transpose-free quasi-minimal residual iterative method

Example: An octatonic scale as a combination of diminished seventh chords:

$$x = [1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]'$$

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