The Genus of Tangle Closures

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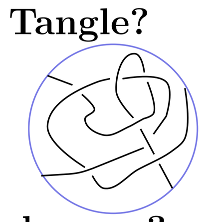
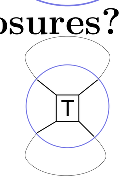
1. Background/History

What is a Tangle?

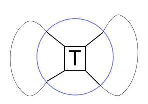


In John Conway's definition, an n-tangle is a proper embedding of the disjoint union of n arcs into a 3-ball; the embedding must send the endpoints of the arcs to $2n$ marked points on the ball's boundary.

What are closures?



A Numerator closure as shown to the left is noted as $N(\text{tangle})$



A Denominator closure as shown left and is noted by $D(\text{tangle})$.

What is a Knot? What is a Link?
A knot is a two dimensional object that can be thought of a taking two ends of a string and twisting them and connecting the ends. A link is a collection of knots which do not intersect, but which may be linked (or knotted) together.

What is a Seifert Surface?
A Seifert surface is a surface whose boundary is a given knot or link.

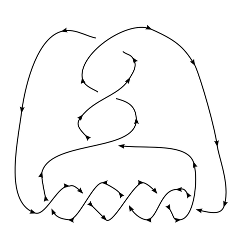
What is Genus?
The genus of a knot K is the knot invariant defined by the minimal genus g of a Seifert surface for K.
Initially I found genus using Seifert's Algorithm which is an upper bound for genus.
A fundamental problem in knot theory is determining when two descriptions represent the same knot. The genus of a knot is a knot invariant. Thus if we know the genus of two knots are different we know the knots are different as well.

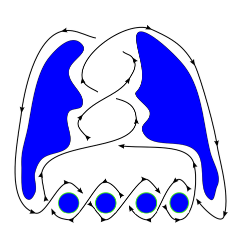
What is Seifert's Algorithm?

Seiferts Algorithm is one way to find an upper bound on genus.
$g = 1 - \frac{s+n-c}{2}$, where
n= number of components
s= number of Seifert surfaces
c= number of crossings

Example

Finding the Denominator closure of the link (3,5):





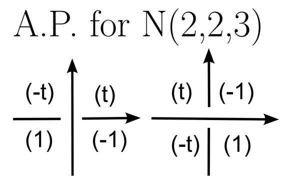
$g = 1 - \frac{6+2-8}{2} = 1$

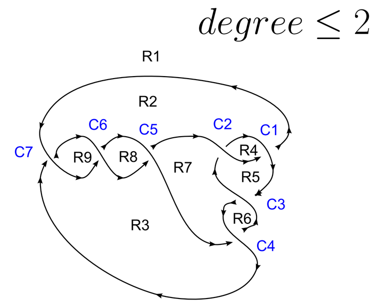
Alexander Polynomial

The Alexander Polynomial can be used to find a lower bound for genus of knots.
This Polynomial can be found for links as well, however we cannot look at genus in the same way.
The Polynomial can be found in many ways, one way is to classify the crossings in a matrix and take the determinant of this matrix.
When the lower bound of genus found by the Alexander Polynomial matched the upper bound found by Seifert's Algorithm the genus is highlighted in green.
Once the polynomial has been obtained only the degree of it is needed, the lower bound for genus is simply:

$$degree \leq 2 \cdot genus$$

A.P. for N(2,2,3)





$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & t & 0 & 0 & 0 & 0 \\ t & -1 & 1 & -t & 0 & 0 & 0 \\ 1 & 0 & t & -t & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & t & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & t \\ t & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

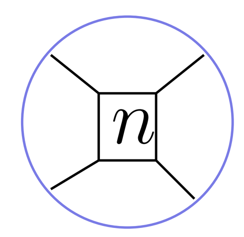
The Alexander Polynomial is
$2t^4 - 4t^3 + 5t^2 - 4t + 2$
$\therefore genus \geq 2$

2. Tangles of One Component Twists

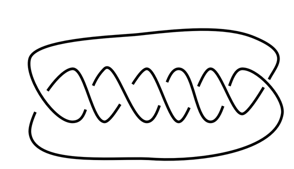
n	$N(T)$	Genus	$D(T)$	Genus
0	link	0	knot	0
1	knot	0	knot	0
2	link	0	knot	0
3	knot	1	knot	0
4	link	0	knot	0
5	knot	2	knot	0
6	link	0	knot	0
7	knot	3	knot	0

If you perform a denominator closure of a one component tangle you always form the unknot (a single loop with no crossings) and can use Reidemeister moves to show this.
If n is even, genus is 0 and the closure is a link.

If n is odd, genus is $\frac{n-1}{2}$ and the closure is a knot.



The arbitrary one component tangle, n represents the number of horizontal twists

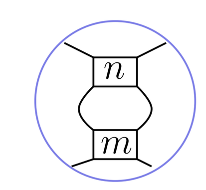


This is the numerator closure of a 7 twists tangle. This knot has the Alexander Polynomial:
$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$

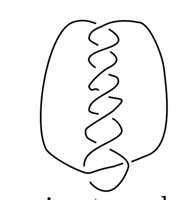
3. Tangles of Two Component Twists

n	m	$N(T)$	Genus	$D(T)$	Genus
1	1	unknot	0	link	0
1	2	link	0	knot	0
1	3	knot	0	link	0
2	1	unknot	0	knot	1
2	2	link	0	knot	1
2	3	knot	0	knot	1
3	1	knot	0	link	0
3	2	link	0	knot	1
3	3	knot	0	link	1

Findings:
$N(1,n) \equiv N(n)$
$N(n,m) \equiv N(m)$
$N(n,1) \equiv N(n+1,0)$
$D(1,n) \equiv N(n+1)$
$D(n,1) \equiv N(-(n-1))$
$n, m \geq 2$ and both even the $g(D(n,m)) = 1$
$n, m \geq 2$ and m is odd and n even the $g(D(n,m))=\frac{n}{2}$
$n, m \geq 2$ and m is even and n is odd the $g(D(n,m))=\frac{m}{2}$
$n, m \geq 3$ and both odd the $g(D(n,m))=1 - \frac{3-m}{2}$



The arbitrary two component tangle,
n represents the number of vertical twists
m represents the number of horizontal twists



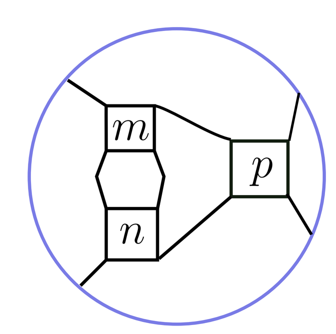
This is the denominator closure of the $(6,2)$ tangle. Its Alexander Polynomial is $3t^2 - 7t + 3$.

4. Tangles of Three Component Twists

Findings:

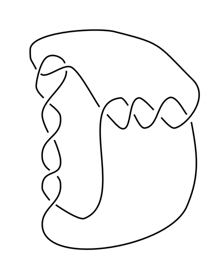
Let o= an odd number of twists
Let e= an even number of twists

m	n	p	$N(T)$	Genus
o	o	o	knot	$1 - \frac{-\lvert n\rvert+1}{2}$
e	e	e	link	1
o	e	e	knot	$1 - \frac{1-m-p}{2}$
o	o	e	link	$1 - \frac{2-m-p}{2}$
e	o	o	knot	$1 - \frac{-1-p}{2}$
e	o	e	link	$1 - \frac{-m-2}{2}$
o	e	o	knot	$min\{1 - \frac{-m-p}{2}, 1 - \frac{-3-n}{2}\}$



The arbitrary three component tangle,
m represents the number of horizontal twists
n represents the number of vertical twists
p represents the number of horizontal twists

$D(m,n,p)$ will always be equivalent to $D(n,m)$
$N(1,n,p)$ will always be equivalent to $D(-p,-(n+1))$
$N(m,1,p)$ will always be equivalent to $D(-m,-(p+1))$
$N(m,n,1)$ will always be equivalent to $D(m,(n+1))$

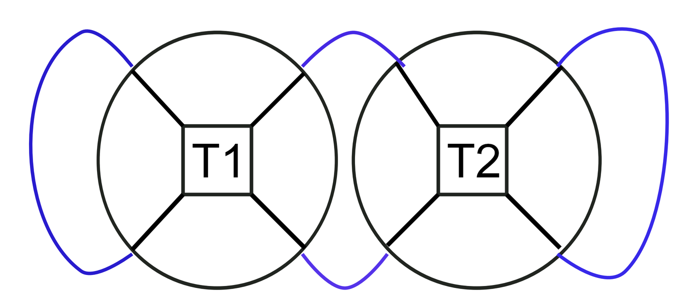


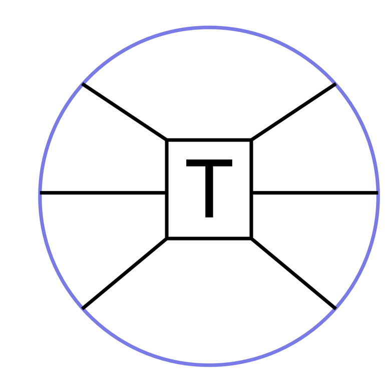
The above the link with numerator closure $N(2,4,4)$

5. Further Studies

When I continue my knot theory research there are a few topics I would like to expand on:

♦ Breaking complicated tangles up into simpler tangles is a very common process, but can we deduce the original genus by knowing the broken down genus?
♦ There is debate on how the genus of a link can be obtained from the Alexander polynomial, I would like to know if the rest of my information about links matches up as nicely as the knot information did.
♦ I would also like to study tangles with more than four outputs and how the closures will effect the genus of these more complicated tangles.





Acknowledgments

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Acknowledgments

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