Duality in Linear Fractional Programming

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Abstract

In this presentation a dual of a linear fractional functionals programming problem is formulated as another linear fractional functionals programming problem. Duality in linear programming is used to establish the duality results for a linear fractional functionals programming problem.

Introduction

Computational possibilities for a linear fractional functionals programming problem have been studied by many [3, 4, 6 , 7]. Dual programs for this class of problem have been proposed by many including Chadha [1, 3], Kaska [5], and Swarup [8]. To prove the duality results Chadhas have assumed the differentiability of the objective function; Kaska's dual is constrained as a variable of the primal problem; Swarup has used the optimality conditions to prove the desired duality results. Present work considers a minimization linear fractional functionals programming problem as the primal problem (P-P). A maximization linear fractional functionals programming problem is proposed its dual problem (D-P).

Duality results are proved without assuming the differentiability of the objective function; dual has no primal variable in its formulation; and finally optimality conditions are not used to prove duality theorems. To establish our results we have only assumed the duality results of linear programming.

Primal Problem

The following problem with assumptions is taken as the primal problem (P-P):

minimize $f(x) = \frac{px}{qx}$

subject to (P-P)

$$x \in S,$$

where $S = [x: Ax \geq b, x \geq 0]$

A is an m by n matrix, x and b are column vectors with n and m components respectively, and p and q are row vectors with n components. It is assumed that (i) qx > 0 for all x ϵ S and q>0; (ii) set S is regular (i.e. non-empty and bounded).

Dual Problem

The proposed dual problem (D-P) is:

maximize $g(y,z) = \frac{py}{qy}$

subject to

$$A^{'}z - p^{'}(qy) + q^{'}(py) \leq 0$$

$$b'z = 0 \qquad \text{(D-P)}$$

$$y \geq 0; z \geq 0$$

The constraint set of the dual problem is denoted by *T*, i.e. $T = [y, z : A'z - p'(qy) + q^{'}(py) \leq 0; b'z = 0; y \geq 0; z \geq 0]$.

' over a matrix is used to denote its transpose; *y* and *z* are column vectors with *n* and *m* components respectively. Clearly, *py* and *qy* do not vanish simultaneously over the set *T*.

Duality Theorems

Theorem 1.

For all *x* in *S* and for all (*y*,*z*) in *T*, $g(y,z) \leq f(x)$.

Proof: To prove this, we observe that

$$Ax \geq b \Longrightarrow x'A' \geq b'$$

or

$$x'A'z \geq b'z. \tag{1}$$

$$\text{Again } A'z - p'(qy) + q'(py) \leq 0$$

$$\Longrightarrow z'A - p(qy) + q(py) \leq 0$$

or

$$z'Ax - px(qy) + qx(py) \leq 0. \tag{2}$$

(1) and (2) together with the fact that $b'z = 0$ yield

$$-px \cdot qy + qx \cdot py \leq 0. \tag{3}$$

$$\Longrightarrow qx \cdot py \leq px \cdot qy$$

$$\Longrightarrow \frac{py}{qy} \leq \frac{px}{qx}, \tag{4}$$

Assuming $qy > 0$. In case $qy = 0$ then from (3) and from the fact that *qy* and *py* cannot vanish simultaneously, it follows that

$$\frac{py}{qy} \rightarrow -\infty \leq \frac{px}{qx}. \tag{5}$$

Result follows from (4) and (5).

Theorem 2.

If $\hat{x}$ is a feasible solution for the (P-P) and $(\hat{y}, \hat{z})$ is a feasable solution for the (D-P) such that $f(\hat{x}) = g(\hat{y}, \hat{z})$, then $\hat{x}$ solves the (P-P), and $(\hat{y}, \hat{z})$ solves the (D-P).

Theorem 3.

If $\hat{x}$ solves the (P-P), then there exists $(\hat{y}, \hat{z})$ which solves the (D-P) and their objective function values are equal.

Theorem 4.

Let *u* and *v* be surplus and slack column vectors associated with the (P-P) and (D-P) respectively. $(\hat{x}, \hat{u})$ solves the (P-P) and $(\hat{y}, \hat{z}, \hat{v})$ solves the (D-P) if and only if

$$\hat{v}'\hat{x} + \hat{u}'\hat{z} = 0$$

or

$$\hat{x}_j \hat{v}_j = 0,\ j = 1, 2, \cdots, n,$$

$$\hat{z}_i \hat{u}_i = 0, i = 1, 2, \cdots, m.$$

Results

$\hat{x}_j \hat{v}_j = 0$ for $j = 1, 2, \cdots, n$ and $\hat{z}_i \hat{u}_i = 0$ for $i = 1, 2, \cdots, m$ follow from (14) together with the fact that $\hat{x}_j \geq 0, \hat{v}_j \geq 0, \hat{z}_i \geq 0$, and $\hat{u}_i \geq 0$ for $j = 1, 2, \cdots, n$ and for $i = 1, 2, \cdots, m$.

References

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Showing Duality of a Linear Programming Problem Using MATLAB

Primal Problem

Max Z = cx

Subject to Ax $\leq$ b

x $\geq$ 0

Dual Problem

Min Z = $b^T w$

Subject to $A^T w \geq c^T$

w $\geq$ 0

EXAMPLE

Minimize Z = $-5x_1 - 4x_2 - 6x_3$

Subject to:
$x_1 - x_2 + x_3 \leq 20$
$3x_1 + 2x_2 + 4x_3 \leq 42$
$3x_1 + 2x_2 \leq 30$
$x_1, x_2, x_3 \geq 0$

DUAL →

Maximize Z = $20w_1 + 42w_2 + 30w_3$

Subject to:
$w_1 + 3w_2 + 3w_3 \leq -5$
$-w_1 + 2w_2 + 2w_3 \leq -4$
$w_1 + 4w_2 \leq -6$
$w_1, w_2, w_3 \leq 0$

MATLAB Commands:

First enter coefficients:

```
f = [ -5 -4 -6]
C= [1 -1 1; 3 2 4; 3 2 0; -1 0 0; 0 -1 0; 0 0 -1]
b=[20; 42; 30; 0; 0; 0]
```

← take Minimum of (Z)

← switch inequalities→ to less than or equal to

Enter the linear programming command:

```
linprog(f,A,b)
```

Solution:

```
ans=
    x1 = 0.0000
    x2 = 15.0000
    x3 = 3.0000
```

Plug in the Min (Z):

-5(0.0000) - 4(15.0000) – 6(3.0000)= -78.000

MATLAB Commands:

```
f 1= [ -20 -42 -30]
C1= [1 3 3; -1 2 2; 1 4 0; 1 0 0; 0 1 0; 0 0 1]
b1=[-4; -5; -6; 0; 0; 0]
```

← take Minimum of (Z)

Enter the linear programming command:

```
linprog(f1,C1,b1)
```

Solution:

```
ans =
    w1 = 0.0000
    w2 = -1.5000
    w3 = -.5000
```

Plug in the Max (Z):

20(0.0000) + 42(-1.5000) + 30(-.5000) =-78.0000