When do Links Admit Homeomorphic C-complexes?

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Introduction

Given any knot, it is interesting to study a surface bounded by the knot called a **Seifert surface**.

Figure 1: Seifert surface

Given any link there is a generalization of a Seifert surface called a **C-complex**. A C-complex is a collection of embedded surfaces in S^3 which might intersect each other in clasps.

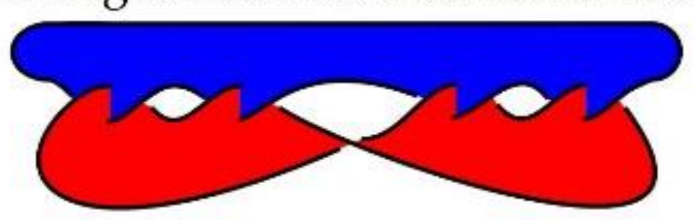

Figure 2: C-complex

For the knots of Figure 3, the surface can't tell that there exist twists in the bands. Thus, the knots bound topologically equivalent Seifert surface. Any two knots bound topologically equivalent Seifert surfaces.

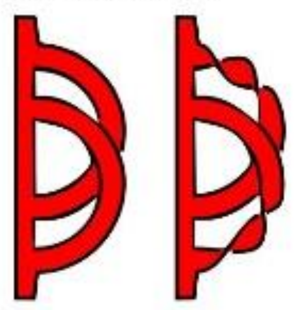

Figure 3: Unknot and trefoil

The goal of this poster is to ask when the same is true of C-complexes. Given two links L and J, when do there exist C-complexes F and G for these links which are topologically equivalent? To what extent is the homeomorphism type of a C-complex an invariant of the link?

Results

Linking Number is an obstruction to links admitting topologically equivalent C-complexes.

Theorem 1 *Let L and J be links. If L and J admit equivalent C-complexes, then L and J have the same pairwise linking numbers.*

In the case of **2-component links**, linking number is the only obstruction.

Theorem 2 *If $L = L_1, L_2$ and $J = J_1, J_2$ are 2-component links and $\text{lnk}(L_1, L_2) = \text{lnk}(J_1, J_2)$, then L and J admit equivalent C-complexes.*

Triple linking is another obstruction in n-component links admitting topologically equivalent C-complexes.

Theorem 3 *Let L and J be n-component links. If L and J admit equivalent C-complexes, then L and J have the same triple linking number.*

Indeed in the case of vanishing linking numbers the converse holds.

Theorem 4 *L and J are n-component links with vanishing pairwise linking numbers and the same triple linking number then L and J admit equivalent C-complexes.*

Application

It is not obvious that the links in Figure 4 do not admit topologically equivalent C-complexes. However since the links have different linking number, then by Theorem 1 they do not admit topologically equivalent C-complexes.

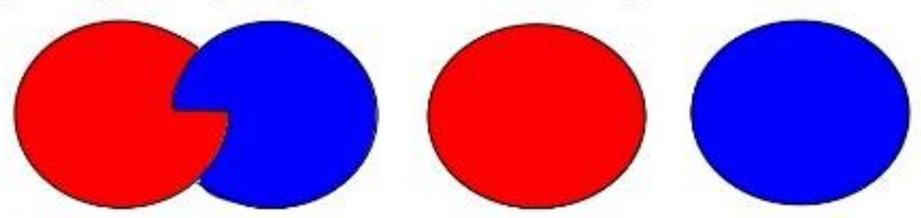

Figure 4: Hopf link and two component unlink.

Since the links in Figure 5 have two component and the same linking number, then by Theorem 2 they admit topologically equivalent C-complexes.

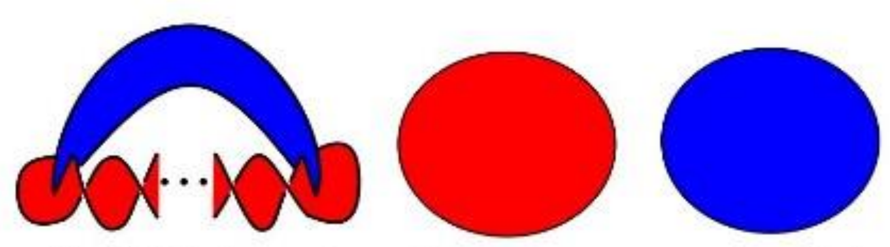

Figure 5: Whitehead and two component unlink

As shown in Figure 6, the Boromean rings and three component unlink have the same number of components but different triple linking number. Therefore by Theorem 4, they do not admit topologically equivalent C-complexes.

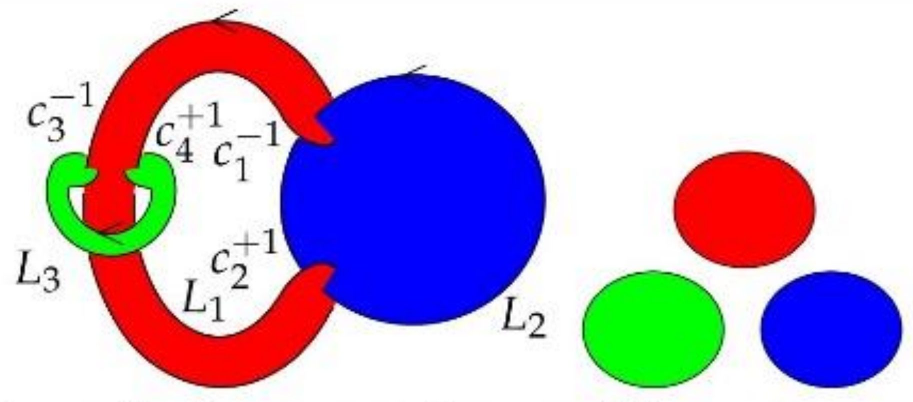

Figure 6: Boromean rings and three component unlink.

Techniques

By picking a basepoint not on a clasp, one can build a word ω_k by following the boundary of the link and recording c_i^{ϵ} based on the clasps past through. i represents the clasp number which is passed through, and ϵ is either $+1$ or -1 depending on the sign of the clasp, see Figure 7. We call these **claspwords**. For Figure 6 by following L_1, $\omega_1 = c_4^{+1} c_1^{-1} c_3^{-1} c_2^{+1}$.

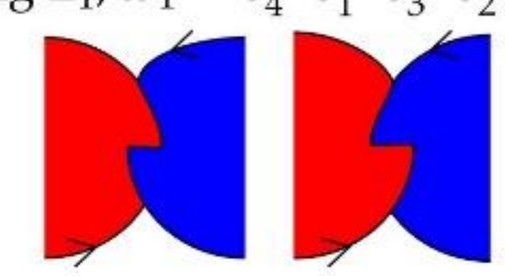

Figure 7: Positive and negative clasps

By adding the number of positive clasps and subtracting the number of negative clasps, we get the **linking numbers** for a link. Since this invariant depended only on ω_1, which in turn depended only on the C-complex, we see that any links with equivalent C-complexes have the same linking number. This proves Theorem 1.

The following technical result give a complete combinatorial means of detecting equivalent C-complexes.

Proposition 5 *Let $F = F_1 \cup \cdots \cup F_n$ and $G = G_1 \cup \cdots \cup G_n$ be n-component C-complexes. Then F is equivalent to G if and only if for all k, $g(F_k) = g(G_k)$ and $\omega_k(F) = \omega_k(G)$.*

According to this proposition, if two links admit C-complexes with identical claspwords, then by stabilizing to increase the genus we can detect the presence of identical C-complexes.

The following figures reveal that we can insert additional cancelling clasps into a C-complex for a link (Figure 8) and that we can transpose any two adjacent clasps between two components of a C-complex (Figure 9). Between these two moves any two component C-complex can be transformed into one with the same claspwords as any other. This proves Theorem 2.

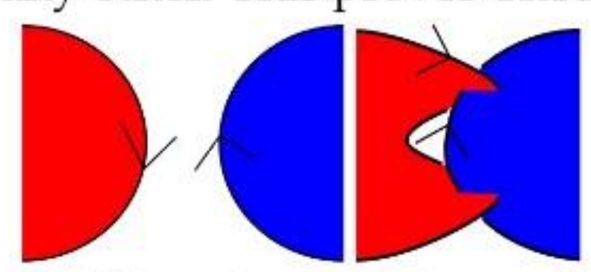

Figure 8: Addition of a pair of canceling clasps.

By transposing two consecutive clasps, we modify ω_1.

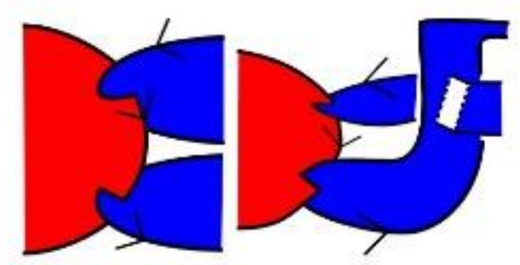

Figure 9: The transposition of two consecutive clasps.

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Advisor: Dr. Christopher Davis We gratefully acknowledge support by the Blugold Commitment Differential Tuition Grants Program and the Trends in Undergraduate Math Research Symposium.