The Moduli Space of 3|2-dimensional Complex Associative Algebras

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The Power of AND

What is a Complex Associative Algebra?

A complex associative algebra is a vector space over the complex numbers equipped with a multiplication operation which satisfies the following:

- $a * (b + c) = a * b + a * c$, (Left Distributive Law)
- $(a + b) * c = a * c + b * c$, (Right Distributive Law)
- $r(a * b) = (ra) * b = a * (rb)$ for any r in $\mathbb{C}$ (Homogeneity)
- $a * (b * c) = (a * b) * c$. (Associativity)

There are 5 basis elements of each 3|2 dimensional algebra, three even and two odd elements. Let the basis of our algebras be $\langle v_1, v_2, v_3, v_4, v_5 \rangle$, where v_1 and v_2 are odd elements and v_3, v_4 and v_5 are even elements.

It is often useful to encode the multiplication structure of an algebra into a ***codifferential***. An algebra is a map $V \otimes V \to V$, and this gives rise to a coderivation of the tensor coalgebra of the parity reversion $W = \Pi(V)$.

Constructing the Algebras

The Fundamental Theorem of Finite Dimensional Associative Algebras says that if an associative algebra structure on a finite dimensional vector space V is non nilpotent, then there is an exact sequence of algebras

$$0 \to M \to V \to W \to 0$$

where M is the *maximal nilpotent ideal* in V, and W is a *semisimple* algebra. Over the complex numbers, we have a stronger result, namely that $V = M \rtimes W$, that is, V is a semidirect product of the maximal nilpotent ideal M and a semisimple subalgebra W.

Wedderburn's Theorem classifies all simple algebras which allows us to construct all semisimple algebras. Previous constructions of moduli spaces of lower dimension give us all possible nilpotent ideals. Putting these together we can construct all non nilpotent 3|2-dimensional complex associative algebras.

For $\mathbb{Z}_2$-graded algebras, the classical theorem above still holds, but the definition of a division algebra, which arises in Wedderburn's Theorem, has to be modified as follows:

Definition. A *unital $\mathbb{Z}_2$-graded complex algebra* is a *division algebra* if every nonzero homogeneous element is invertible.

Example of a 3 | 2-Dimensional Algebra

We have constructed all 617 3 | 2-Dimensional Complex Associative

Algebras in the Moduli Space. Here is an example of one of them:

This is the multiplication structure for the 6th 3|2-dimensional algebra:

$$v_5 \cdot v_2 = v_2, \quad v_2 \cdot v_5 = v_2, \quad v_2 \cdot v_2 = v_5,$$
$$v_5 \cdot v_5 = v_5, \quad v_4 \cdot v_4 = v_4, \quad v_3 \cdot v_3 = v_3,$$
$$v_3 \cdot v_1 = v_1$$

We represent this algebra structure as the codifferential d_6.

$$d_6 = -\psi_2^{5,2} + \psi_2^{2,5} + \psi_5^{2,2} - \psi_5^{5,5} + \psi_4^{4,4} + \psi_3^{3,3} + \psi_1^{3,1}$$

The terms of the algebra correspond to the terms in the codifferential. Notice some of the terms have minus signs, which do not appear in the original multiplication table. This occurs because of some subtleties that arise because when we go to the codifferential representation of an algebra, we reverse the parity of the elements, and some signs of the terms become reversed.

Deforming the Algebras

Suppose that d is an algebra, and

$$d_t = d + \psi_1 t + \psi_2 t^2 + \text{ho}$$

is a 1-parameter deformation of d, where ho stands for higher order terms in the variable t. This means that d_t is a (possibly formal) power series in t, and the associativity condition is that

$$[d_t, d_t] = 0,$$

where [,] is the *Gerstenhaber* bracket.

We say that d_t is a jump deformation of d if there is some algebra given by d' such that $d_t \sim d'$ for all $t \neq 0$ in some neighborhood of $t = 0$.

We say that d_t is a smooth deformation if $d_t \nsim d_{t'}$ for $t \neq t'$ in some neighborhood of $t = 0$.

If $\langle \delta^i \rangle$ is a basis for H^2, then there is a deformation with multiple parameters t_i, called a *versal* deformation, which encodes all of the deformations of d. The computability of a versal deformation makes it possible to determine all jumps and smooth deformations of an algebra at once. The versal deformation d^∞ is of the form

$$d^\infty = d + \delta^i t_i + \text{ho}$$

where we use the Einstein summation convention for repeated indices.

Example of a Versal Deformation

The algebra d_{285} is given by the formula $d_{285} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_1^{4,1} + \psi_2^{4,2} + \psi_3^{4,3} - \psi_1^{1,5} + \psi_3^{3,4}$. Its versal deformation is given by:

$$d^\infty = \psi_4^{4,4} + \psi_5^{5,5} + \psi_1^{4,1} + \psi_2^{4,2} + \psi_3^{4,3} - \psi_1^{1,5} + \psi_3^{3,4} + t_1\psi_1^{3,1} + t_2\psi_2^{3,2}$$

We find there is a relation on the parameters t_1 and t_2 that must be satisfied in order for the expression above to give an associative algebra:

$$-2t_2^2 + 2t_1^2 = 0$$

- One solution occurs when $t_1 = t_2$. In this case, as long as $t_2 \neq 0$, we find the deformation is isomorphic to

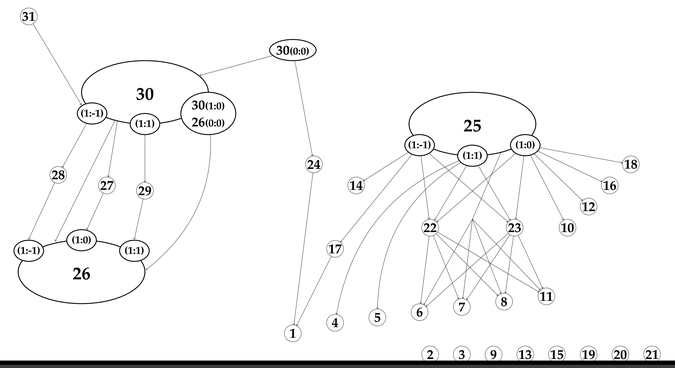
$$d_{36} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1} + \psi_2^{3,2} - \psi_1^{1,4}$$

- Another solution occurs when $t_1 = -t_2$. In this case, again as long as $t_2 \neq 0$, we find the deformation is isomorphic to

$$d_{41} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1} + \psi_2^{4,2} - \psi_1^{1,5}$$

Deformations of 1|3-Dimensional Algebras

This is an example of the end goal of our project. The figure below illustrates how the moduli space is glued together by deformations for 1|3-Dimensional Algebras. For our current project, we have deformed roughly 596 out of 617 Algebras in the moduli space of 3|2-dimensional Algebras.



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