Analysis of Stability Regions of Numeric Methods Using the Time Scale Calculus

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1. The Time Scale Calculus

Definition. A *time scale*, denoted $\mathbb{T}$, is a nonempty, closed subset of $\mathbb{R}$.
Examples. A time scale can be $\mathbb{Z}$, $\mathbb{R}$, Cantor sets, $\mathbb{T} = [1, 2] \cup [3, 5]$ or $\mathbb{T} = \{1, 3, 5, 10, 11\}$.
Definition. If $\mathbb{T}$ is a time scale and $t \in \mathbb{T}$, then the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and the *backward jump operator* $\varrho : \mathbb{T} \to \mathbb{T}$ are defined to be

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ and } \varrho(t) := \sup\{s \in \mathbb{T} : s < t\}, \text{ respectively.}$$

Definition. Let $\mu : \mathbb{T} \to [0, \infty)$ be defined by $\mu(t) := \sigma(t) - t$. This is the *graininess function*.
Definition. An *isolated time scale* is a time scale where $\sigma(t) > t$ and $\varrho(t) < t$ for all $t \in \mathbb{T}$. One important attribute of an isolated time scale is that it can be ordered so that $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$ where $t_0 < t_1 < t_2 < \cdots$.
Definition. The *forward difference operator* on a function $f : \mathbb{Z} \to \mathbb{R}$ is defined by $\Delta f(t) = f(t + 1) - f(t)$.
Notation. Since $\mathbb{T}$ is isolated, we use the notation $f(t_n) = f_n$, $\mu_n = \mu(t_n)$ and $p_n = p(t_n)$.
Definition. Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$, then we define the *delta-derivative* as follows

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(s) - f(t)}{s - t}, & \mu(t) = 0, \\ \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \mu(t) > 0. \end{cases}$$

Definition. When working on an isolated time scale, the *delta-derivative* can be defined by $y^{\Delta} = \frac{y_{n+1} - y_n}{\mu_n}$.
Definition. A function is defined to be *regressive* provided that $1 + \mu(t)p(t) \neq 0 \; \forall \, t \in \mathbb{T}$. The set of all regressive functions is defined to be $\mathcal{R}$.
Definition. For $p \in \mathcal{R}$, the *generalized exponential function* $e_p : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $s, t \in \mathbb{T}$. Since $\mathbb{T}$ is isolated, we end up with the following equation for the generalized exponential:

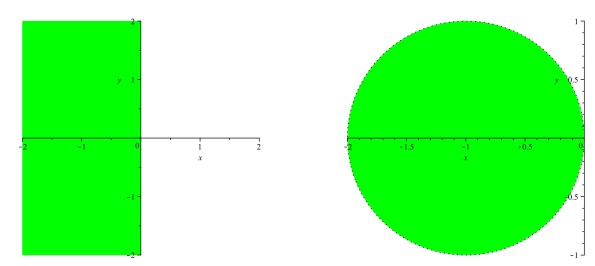
$$e_p(t_n, t_0) = \prod_{k=0}^{n-1}(1 + \mu(t_k)p(t_k)).$$

Definition. We will also define the *circle-minus operator*, $\ominus$, to be $(\ominus p)(t) = \frac{-p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}$.

2. The Euler Method

Consider the initial value problem, which we will call the test equation, for a fixed $z \in \mathbb{C}$, $y' = zy$, $y(t_0) = y_0$. Looking at the particular solution, $y(t) = y_0 e^{z(t-t_0)}$, we see that when $z < 0$, y is decaying and because of that the solution is bounded; however, when $z > 0$, y is growing without bound.

For a fixed step size $h > 0$, the recurrence relation for the explicit Euler method is given by $y_{n+1} = y_n + hf(x_n, y_n)$, while implicit Euler is $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$. Using the explicit Euler method on the test equation, we solve the recurrence relation to obtain $y_{n+1} = y_0(1+hz)^n$. To find the stability region of the method, we find for which $z \in \mathbb{C}$ $y_n = y_0(1 + hz)^n$ converges. To do this, note that $\lim_{n\to\infty}(1 + hz)^n$ converges to 0 iff $|1 + hz| < 1$.



The region in the complex plane where the solution to the test equation converges to 0 as $t \to \infty$ vs. the region of the complex plane where explicit Euler produces an approximate solution that converges to 0 as $n \to \infty$; here $h = 1$.

Using the equation for the explicit Euler method, the test equation and consider $\mathbb{T} = h\mathbb{Z}$, we have $\mu_n \equiv h$. We can rewrite the Euler equation as, $\frac{y^{\sigma}(t) - y(t)}{\mu(t)} = zy(t)$. From this we can obtain the dynamic equation, $y^{\Delta} = zy$, $y(t_0) = y_0$, which has unique solution $y(t) = y_0 e_z(t, t_0)$. Now, using properties of the generalized exponential we can determine the region where $e_z(t, t_0)$ converges to 0. Note that $p_n \equiv z$ for all n because of the chosen dynamic equation. Therefore, we have

$$e_z(t_n, t_0) = \prod_{k=0}^{n-1}(1 + \mu_k p_k) = \prod_{k=0}^{n-1}(1 + hz) = (1 + hz)^n,$$

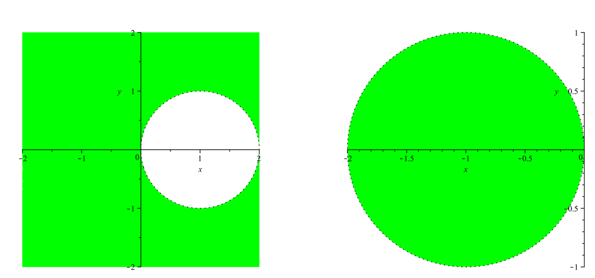
which will converge to 0 when $\lim_{n\to\infty}(1 + hz)^n = 0$. Hence, when $|1 + hz| < 1$, $e_z(t_n, t_0) \to 0$ on the time scale $\mathbb{T} = h\mathbb{Z}$ in the same region as above.

Theorem
The region in $\mathbb{C}$ for which explicit Euler produces approximate solutions that converge to 0 and the region in $\mathbb{C}$ for which $e_z(t_n, t_0) \to 0$ on $\mathbb{T} = h\mathbb{Z}$ are both given by $\{z \in \mathbb{C} : |1 + hz| < 1\}$.

Now consider $e_{\ominus -p}(t_n, t_0) = e_{\frac{p}{1-\mu p}}(t_n, t_0)$. We again use the properties of the generalized exponential to determine the region where $e_{\ominus -p}(t_n, t_0)$ converges to 0. Assuming $\mathbb{T} = h\mathbb{Z}$, we can use similar steps to explicit Euler to obtain

$$e_{\ominus -z}(t_n, t_0) = \prod_{k=0}^{n-1}\left(1 + \frac{hz}{1 - hz}\right) = \left(\frac{1}{1 - hz}\right)^n.$$

This will converge to 0 when $\lim_{n\to\infty}(\frac{1}{1-hz})^n = 0$. This produces the region that implicit Euler converges to 0 as $n \to \infty$.



The region in the complex plane where implicit Euler produces an approximate solution that converges to 0 as $n \to \infty$ vs. the region of the complex plane where explicit Euler produces an approximate solution that converges to 0 as $n \to \infty$; here $h = 1$.

Theorem
The region in $\mathbb{C}$ for which implicit Euler produces approximate solutions that converge to 0 and the region in $\mathbb{C}$ for which $e_{\ominus -z}(t_n, t_0) \to 0$ on $\mathbb{T} = h\mathbb{Z}$ are both given by

$$\left\{z \in \mathbb{C} : \left|\frac{1}{1 - hz}\right| < 1\right\}.$$

3. The Runge-Kutta Method

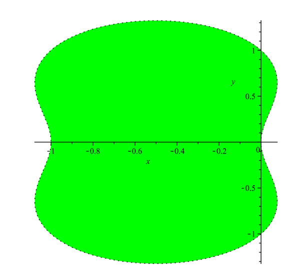
Now we will apply this analysis of the stability region to the well-known Runge-Kutta method. For simplification we will be looking at an explicit Runge-Kutta method of order 2, which has the following equation $y_{n+1} = y_n + hz(y_n + hzy_n)$. One of the ways that Runge-Kutta methods are organized is by a Butcher tableau. Given the tableau, $\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$, the corresponding Runge-Kutta equation is $y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2)$, with $k_1 = f(x_n + c_1 h, y_n + h(a_{11}k_1 + a_{12}k_2))$ and $k_2 = f(x_n + c_2 h, y_n + h(a_{21}k_1 + a_{22}k_2))$. The tableau that represents the Runge-Kutta 2 method we are using is $\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 0 & 1 \end{array}$. With the chosen test equation, we see that $\frac{y_{n+1} - y_n}{h} = z(1+hz)y_n$. Using the same approach as explicit Euler, we can obtain the dynamic equation, $y^{\Delta} = z(1+\mu z)y$, $y(t_0) = 1$. Which has a unique solution given by $e_{z(1+\mu z)}(t_n, t_0)$. Similarly,

$$e_{z(1+\mu z)}(t_n, t_0) = \prod_{k=0}^{n-1}(1 + h(z(1 + hz))) = (1 + hz + h^2z^2)^n,$$

which converges to 0 when $\lim_{n\to\infty}(1 + hz + h^2z^2)^n = 0$, or when $|1 + hz + h^2z^2| < 1$.

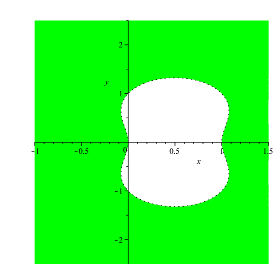
Theorem
The region in $\mathbb{C}$ for which explicit Runge-Kutta 2 given by the equation $y_{n+1} = y_n + hz(y_n + hzy_n)$. will produces approximate solutions that converge to 0 and the region in $\mathbb{C}$ for which $e_{z(1+\mu z)}(t_n, t_0) \to 0$ on $\mathbb{T} = h\mathbb{Z}$ are both given by $\{z \in \mathbb{C} : |1 + hz + h^2z^2| < 1\}$.

By applying the same circle-minus operator, we are able to obtain a implicit Runge-Kutta 2 method related to the chosen explicit Runge-Kutta 2 method.



$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 0 & 1 \end{array}$$

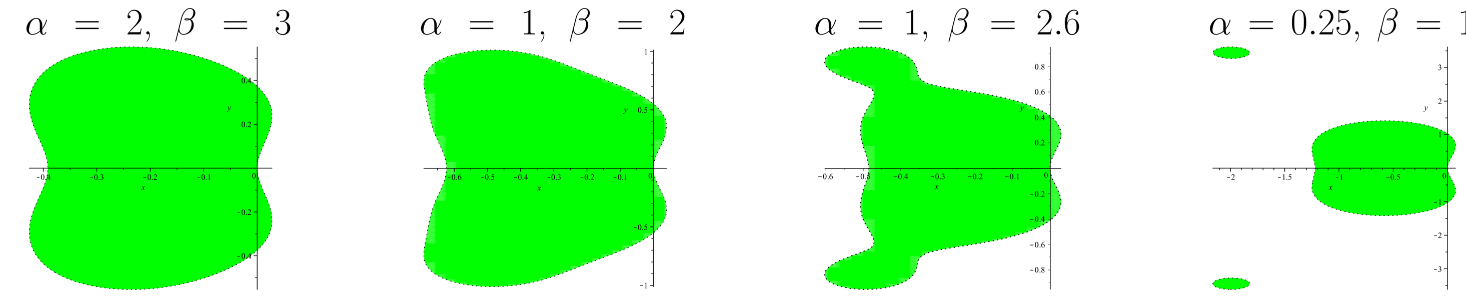
The convergence region of the **explicit** Runge-Kutta 2 method and its tableau next to a related **implicit** Runge-Kutta 2 method and its tableau.



$$\begin{array}{c|cc} 2 & 1 & 1 \\ -1 & -1 & 0 \\ \hline & \frac{1}{3} & \frac{2}{3} \end{array}$$

We will now see what happens when we vary the time scale for Runge-Kutta. Consider the time scale $\mathbb{T}_0 = \{t_0, t_1, \dots\}$ where $\mu(t_n) := \begin{cases} \alpha, & \text{when } n \text{ is even,} \\ \beta, & \text{when } n \text{ is odd.} \end{cases}$ Inductively, we notice that $e_z(1 + \mu z)(t_n, t_0)$ will converge to 0 when $\lim_{n\to\infty}(1 + \alpha z + \alpha^2 z^2)^n(1 + \beta z + \beta^2 z^2)^n = 0$. This will converge to 0 when $|1 + \alpha z + \alpha^2 z^2||1 + \beta z + \beta^2 z^2| < 1$. Different ratios of α and β produce different regions.

$\alpha = 2,\ \beta = 3$ | $\alpha = 1,\ \beta = 2$ | $\alpha = 1,\ \beta = 2.6$ | $\alpha = 0.25,\ \beta = 1$



References

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